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Two-Warehouse Inventory Model for Non-Instantaneous Deteriorating Items with Exponential Demand Rate

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Abstract
A two-warehouse inventory problem for non-instantaneous deteriorating items with exponential demand rate under different dispatching policies is planned. While formulating the inventory model for deteriorating items, usually it is assumed that the items start deteriorating as soon as they enter into the warehouse. However, there are numerous products like dry fruits, food grains etc. that have a shelf-life and start deteriorating after a time lag that is termed as non-instantaneous deterioration. Moreover, the price discount, low cost storage, huge demand etc. and under such a situation one may decide to procure large quantity of the items which would arise the problem of storing. As the capacity of own warehouse is limited, therefore one has to hire another rented warehouse to store the excess quantity. A numerical example is given to illustrate the model. Sensitivity analysis of the optimal solution is carried out.

Keywords: Inventory, non-instantaneous deterioration, two-warehouse, exponential demand rate.

1. INTRODUCTION

In recent years, inventory problems for deteriorating items have been widely studied. Deterioration is defined as decay, change or spoilage such that the items are not in a condition of being used for its original purpose. Electronic goods, radioactive substances, grains, blood, alcohol, gasoline, turpentine are examples of deterioration items. For any business organization, it is major concern to control and maintain the inventories of deteriorating items.

The problems on classical inventory models which are found in the existing literature generally deal with single storage facility. But when the optimal lot size dictated by the EOQ model becomes more than the total amount that can be stored in the existing storage facility (Warehouse owned by the management OW) the question of acquiring some extra storage facility to store these excess quantity arises. This additional storage facility may be a rented warehouse (RW) with sophisticated preservation facility and abundant space.

A model with exponentially decaying inventory was initially proposed by Ghare and Schrader (1963). Covert and Philip (1973) formulated model with variable deteriorating rate of two-parameter Weibull distribution. Philip (1974) generalized this model by taking three-parameter Weibull distribution. After that many researchers such as Goyal(1987), Raafat et al. (1991), Wee (1993), and others developed models on deteriorating items. A detailed review of deteriorating inventory literatures is given by Goyaland Giri(2001). There is a vast inventory literature on deteriorating items under different conditions, the outline which can be found in articles (2001),(2003),(2006),(2008),(2011) and their references.

However, in many inventory systems, the deterioration of goods is a realistic phenomenon. It is well known that certain products such as refrigerated food, fruit and vegetable, fresh seafood and many others have a high deterioration rate. Mandal and Phaujdar (1989) developed a production inventory model for deteriorating items with uniform rate of production and linearly stock-dependent demand. Giri, Pal, Goswami, and Chaudhuri (1996) studied the model of Datta and Pal (1990) for deteriorating items. Meanwhile, Giri and Chaudhuri (1998) extended Goh's model (1994) to consider the inventory model with a constant deterioration rate. Padmanabhan and Vrat (1995) considered an EOQ model for perishable items with a constant selling price and linearly stock dependent demand.

Inventory model with double storage facility OW and RW was first developed by Hartley (1976).Sahu and Bishi (2017) extended the inventory deteriorating Items under permissible delay in payments. After his pioneering contribution, several other researchers have attempted to extend his work to various other realistic situations. In this connection, mention may be made of the studies undertaken by Sarma(1983, 1990), Murdeshwar and Sathe (1985), Pakkala and Achary (1992), Dave (1988), Bhunia and Maity (1997) Yang (2004), Singh and Sahu (2012), Lee (2006), Yang (2006), Dey et al.(2008) to name only a few.

In this paper we consider a two-warehouse inventory problem for non-instantaneous deteriorating items with exponential demand rate under different dispatching policies is planned. While formulating the inventory model for deteriorating items, usually it is assumed that the items start deteriorating as soon as they enter into the warehouse. However, there are numerous products like dry fruits, food grains etc. that have a shelf-life and start deteriorating after a time lag that is termed as non-instantaneous deterioration. Moreover, the price discount, low cost storage, huge demand etc. and under such a situation one may decide to procure large quantity of the items which would arise the problem of storing. As the capacity of own warehouse is limited, therefore one has to hire another rented warehouse to store the excess quantity.

2. ASSUMPTION AND NOTATIONS

The following assumptions and notations have been used in the entire paper.

Assumption

(i) Replenishment rate is instantaneous.
(ii) Lead-time is negligible.
(iii) The planning horizon of the inventory system is infinite.
(iv) t_d is the length of time during which the product has no deterioration.
(v) The OW has a fixed capacity of W units; the RW has unlimited capacity.
(vi) The unit inventory holding cost per unit time in RW is higher than that in OW and the deterioration rate in RW is less than that in OW.
(vii) Unsatisfied demand/shortages are allowed. Unsatisfied demand is partially backlogged and the fraction of shortages backlogged is a differentiable and decreasing function of time t, denoted by $g(t)$, where t is the waiting time up to the next replenishment. We have defined the partial backlogging rate $g(t) = e^{-\delta t}$, where δ is a positive constant.

Notations

In addition, the following notations are used throughout this paper.

A	ordering cost per order
c	purchasing cost per unit
W	capacity of the owned warehouse
$\lambda e^{\lambda t}$	demand rate per unit time
Q_f	order quantity per cycle
S_f	maximum inventory level per cycle
H	holding cost per unit per unit time in OW
F	holding cost per unit per unit time in RW, where $F > H$
s	the backlogging cost per unit per unit time, if shortage is backlogged
c_1	unit opportunity cost due to lost sale, if the shortage is lost
α	deterioration rate in OW, where $0 \le \alpha < 1$
β	deterioration rate in RW, where $0 \le \beta < 1; \beta < \alpha$.
t_d	time period during which no deterioration occurs
t_r	time at which the inventory level reaches zero in RW
t_w	time at which the inventory level reaches zero in OW
T	the length of the replenishment cycle in year
$I_0(t)$	inventory level in the OW at any time t where $0 \le t \le T$
$I_r(t)$	inventory level in the RW at any time t where $0 \le t \le T$
TCF_i	total relevant cost per unit time for case $i = 1, 2$
TCL_i	total relevant cost per unit time for case $i = 1, 2$
$B(t)$	backlogged level at any time t where $t_w \le t \le T$
$L(t)$	number of lost sales at any time t where $t_w \le t \le T$

3. MATHEMATICAL MODEL FORMULATION

In the present study a two warehouse inventory model has been developed, where the OW has a fixedcapacity of *W* units and the RW has unlimited capacity. The units in RW are stored only when the capacity of OW has been utilized completely. Demand is assumed to be constant. Shortages are allowed but are partially backlogged. The goods are stored in Owned Warehouse (OW) initially after satisfying the OW; remaining goods are stored in Rented Warehouse (RW) but uses the goods of RW prior to the goods of OW to satisfy the demand in order to reduce the inventory carrying charge (holding cost). Whereas those goods are sold that are stored first in order to maintain the freshness of product which results in greater customer satisfaction. Which ultimately boost the sales and increase the value of the organization in the long term. The following sections discuss the model formulation for both the policies.

Case 1: When $t_d < t_w$

During the time interval $[0, t_d]$, there is no deterioration. So, the inventory in OW $I_0(t)$ is depleted only due to demand whereas in RW, inventory level remains the same. Further, during the time interval $[t_d, t_w]$ the inventory level in OW $I_0(t)$ is dropping to zero due to the combined effect of demand and deterioration and the inventory in RW $I_r(t)$ gets depleted due to deterioration only. Now, during the time interval $[t_w, t_r]$ depletion of inventory $I_r(t)$ occurs in RW due to the combined effect of demand and deterioration and it reaches to zero at time t_r . Moreover, during the interval $[t_r, T]$ the demand is backlogged. So, $B(t)$ represents the level of negative inventory at time *t* during the interval $[t_r, T]$ .The behavior of the model over the time interval $[0, T]$ has been represented graphically in figure 1.

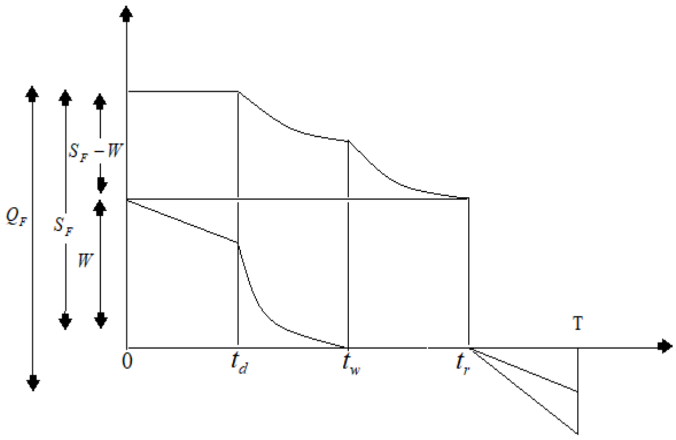

Figure1: Two-warehouse inventory system,

when $t_d < t_w$.

Therefore, the differential equations that describe the inventory level in the RW and OW at time t over the Period $(0,T)$ are given by:

$$\frac{dI_0(t)}{dt} = -\lambda e^{\lambda t} \ , \ 0 \le t \le t_d \tag{1}$$

$$\frac{dI_0(t)}{dt} + \alpha I_0(t) = -\lambda e^{\lambda t} \ , \ t_d \le t \le t_w \tag{2}$$

$$\frac{dI_r(t)}{dt} = 0 \ , \ 0 \le t \le t_d \tag{3}$$

$$\frac{dI_r(t)}{dt} + \beta I_r(t) = 0 \ , \ t_d \le t \le t_w \tag{4}$$

$$\frac{dI_r(t)}{dt} + \beta I_r(t) = -\lambda e^{\lambda t} \ , \ t_w \le t \le t_r \tag{5}$$

$$\frac{dB(t)}{dt} = De^{-\delta(T-t)} \ , \ t_r \le t \le T \tag{6}$$

The solution of the above Withthe boundary condition $I_0(0) = W$, $I_0(t_w) = 0$, $I_r(t_d) = S_f - W$, $I_r(t_r) = 0$, $B(t_r) = 0$

$$I_0(t) = (w+1) - e^{\lambda t} \ , \ 0 \le t \le t_d \tag{7}$$

$$I_0(t) = \frac{\lambda}{\lambda+\alpha}\left[e^{(\lambda+\alpha)t_W - \alpha t} - e^{\lambda t}\right], \ t_d \le t \le t_w \tag{8}$$

$$I_r(t) = (S_f - W)e^{\beta(t_d - t)}, \ t_d \le t \le t_w \tag{9}$$

$$I_r(t) = \frac{\lambda}{\lambda+\beta}\left[e^{(\lambda+\beta)t_r - \beta t} - e^{\lambda t}\right], \ t_w \le t \le t_r \tag{10}$$

$$B(t) = \frac{D}{\delta}\left\{e^{-\delta(T-t)} - e^{-\delta(T-t_r)}\right\}, \ t_r \le t \le T \tag{11}$$

The numbers of lost sales at time t is $L(t)$ given by

$$L(t) = \int_{t_r}^{T} D\left\{1 - e^{-\delta(T-t)}\right\}dt \qquad t_r < t < T$$

$$= D\left[(t - t_r) - \frac{1}{\delta}\left\{e^{-\delta(T-t)} - e^{-\delta(T-t_r)}\right\}\right], t_r \le t \le T \tag{12}$$

$$I_r(t) = S_f \ \text{with} \ I_r(0) = S_f \ , \ 0 \le t \le t_d \tag{13}$$

Considering the continuity of $I_0(t)$ at $t = t_d$, it follows from equation (7) and (8), we get

$$W + 1 - e^{\lambda t_d} = \frac{\lambda}{\lambda+\alpha}\left[e^{(\lambda+\alpha)t_w - \alpha t_d} - e^{\lambda t_d}\right]$$

$$t_w = \frac{\alpha}{\lambda+\alpha}t_d + \frac{1}{\lambda+\alpha}\ln\left|e^{\lambda t_d} + \frac{\lambda+\alpha}{\lambda}(1 + W - e^{\lambda t_d})\right| \tag{14}$$

Considering the continuity of $I_r(t)$ at $t = t_w$, it follows from equation (9) and (10), we get

$$(S_f - W)e^{\beta(t_d - t_w)} = \frac{\lambda}{\lambda+\beta}\left[e^{(\lambda+\beta)t_r - \beta t_w} - e^{\lambda t_w}\right]$$

$$S_f = W + \frac{\lambda}{\lambda+\beta}\left[e^{(\lambda+\beta)t_r - \beta t_d} - e^{\beta(t_w - t_d) - \lambda t_w}\right] \tag{15}$$

Putting $t = T$ in equation (11), The maximum amount of demand backlogged per cycle is

$$B(T) = \frac{D}{\delta}\left(1 - e^{-\delta(T-t_r)}\right) \tag{16}$$

Therefore the order quantity over the replenishment cycle can be determined as eqn. (14) & (16), we get

$$Q_f = S_f + B(T)$$

$$Q_f = W + \frac{\lambda}{\lambda+\beta}\left[e^{(\lambda+\beta)t_r-\beta t_d} - e^{\beta(t_w-t_d)-\lambda t_w}\right]$$

$$+\frac{D}{\delta}\left(1-e^{-\delta(T-t_r)}\right) \quad (17)$$

Hence, $(0.T)$ the costs during the cycle are evaluated as follows

(a) Ordering cost per cycle $= A$

(b) The Inventory holding cost per cycle in RW

$$HC_{RW} = F\left[\int_0^{t_d} I_r(t)dt + \int_{t_d}^{t_w} I_r(t)dt + \int_{t_d}^{t_r} I_r(t)dt\right]$$

By using the value of $S_f - w$ in equation (9) we get

$$I_r(t) = \frac{\lambda}{\lambda+\beta}\left[e^{(\lambda+\beta)t_r-\beta t_d} - e^{\beta(t_w-t_d)-\lambda t_w}\right]e^{\beta(t_d-t)}$$

(18)

Then

$$HC_{RW} = FS_f t_d\left[\frac{\lambda}{\lambda+\beta}\left\{e^{(\lambda+\beta)t_r} - e^{(\beta-\lambda)t_w}\right\}\right.$$
$$\left(\frac{-1}{\beta}\right)e^{\beta(t_w-t_d)} + \frac{\lambda}{\lambda+\beta}\left\{\frac{-1}{\beta}e^{(\lambda+\beta)t_r-\beta(t_r-t_w)}\right.$$
$$\left.\left.-\frac{1}{\lambda}e^{\lambda(t_r-t_w)}\right\}\right]$$

(c) The Inventory holding cost per cycle in OW

$$HC_{OW} = H\left[\int_0^{t_d} I_0(t)dt + \int_{t_d}^{t_w} I_0(t)dt\right]$$

$$HC_{OW} = H\left[(W+1)t_d - \frac{1}{\lambda}(e^{\lambda t_d} - 1)\right.$$
$$\left.-\frac{\lambda}{\lambda+\alpha}\left\{\frac{1}{\alpha}e^{(\lambda+\alpha)t_w-\alpha(t_w-t_d)} + \frac{1}{\lambda}e^{\lambda(t_w-t_d)}\right\}\right]$$

(d) The backlogged cost per cycle is $= s\int_{t_r}^{T} B(t)dt$

$$SC = \frac{sD}{\delta}\left[\frac{1}{\delta} - \left\{\frac{1}{\delta} + T - t_r\right\}e^{-\delta(T-t_r)}\right]$$

(e) The opportunity cost due to lost sale is

$$= c_1 D\int_{t_r}^{T}\left\{1-e^{-\delta(T-t)}\right\}dt$$

$$= c_1 D\left[T - t_r - \frac{1}{\delta}\left\{1-e^{-\delta(T-t_r)}\right\}\right]$$

(f) The deterioration cost per cycle

$$= c\left[\beta\int_{t_d}^{t_r} I_r(t)\,dt + \alpha\int_{t_d}^{t_w} I_0(t)\,dt\right]$$

$$= c\beta\left[\frac{\lambda}{\lambda+\beta}\left\{\left(e^{(\beta-\lambda)t_w} - e^{(\lambda+\beta)t_r}\right)\frac{e^{-\beta(t_w-t_d)}}{\beta}\right.\right.$$
$$\left.-\left(\frac{1}{\beta}e^{(\lambda+\beta)t_r-\beta(t_r-t_w)} + \frac{1}{\lambda}e^{\lambda(t_r-t_w)}\right)\right\}$$
$$\left.+\frac{\alpha\lambda}{\lambda+\alpha}\left\{\frac{1}{\alpha}e^{(\lambda+\alpha)t_w-\alpha(t_w-t_d)} + \frac{1}{\lambda}e^{\lambda(t_w-t_d)}\right\}\right]$$

Now, the Total relevant cost per time unit time during the Cycle $(0,T)$ using equation is given by

$$TCF1(t_r,T) = \frac{1}{T}\left[OC + HC + BC + OC + DC\right]$$

$$= \frac{1}{T}\left[A + FS_f t_d\left[\frac{\lambda}{\lambda+\beta}\left\{e^{(\lambda+\beta)t_r} - e^{(\beta-\lambda)t_w}\right\}\right.\right.$$
$$\left(\frac{-1}{\beta}\right)e^{\beta(t_w-t_d)} + \frac{\lambda}{\lambda+\beta}\left\{\frac{-1}{\beta}e^{(\lambda+\beta)t_r-\beta(t_r-t_w)}\right.$$
$$\left.\left.-\frac{1}{\lambda}e^{\lambda(t_r-t_w)}\right\}\right] + H\left[(W+1)t_d - \frac{1}{\lambda}(e^{\lambda t_d} - 1)\right.$$

$$-\frac{\lambda}{\lambda+\alpha}\left\{\frac{1}{\alpha}e^{(\lambda+\alpha)t_w-\alpha(t_w-t_d)}+\frac{1}{\lambda}e^{\lambda(t_w-t_d)}\right\}\Bigg]\Bigg]$$

$$+\frac{SD}{\delta}\left[\frac{1}{\delta}-\left\{\frac{1}{\delta}+T-t_r\right\}e^{-\delta(T-t_r)}\right]$$

$$+c_1D\left[T-t_r-\frac{1}{\delta}\left\{1-e^{-\delta(T-t_r)}\right\}\right]$$

$$+c\beta\Bigg[\frac{\lambda}{\lambda+\beta}\Bigg\{\left(e^{(\beta-\lambda)t_w}-e^{(\lambda+\beta)t_r}\right)\frac{e^{-\beta(t_w-t_d)}}{\beta}$$

$$-\left(\frac{1}{\beta}e^{(\lambda+\beta)t_r-\beta(t_r-t_w)}+\frac{1}{\lambda}e^{\lambda(t_r-t_w)}\right)\Bigg\}$$

$$-\frac{\alpha\lambda}{\lambda+\alpha}\left\{\frac{1}{\alpha}e^{(\lambda+\alpha)t_w-\alpha(t_w-t_d)}+\frac{1}{\lambda}e^{\lambda(t_w-t_d)}\right\}\Bigg]\Bigg]\Bigg] \quad (19)$$

Case 2: When $t_d > t_w$

In this case, time during which no deterioration occurs is greater than the time during which inventory in OW becomes zero and the behavior of the model over the time interval $[0,T]$ has been graphically represented below in Figure 2.

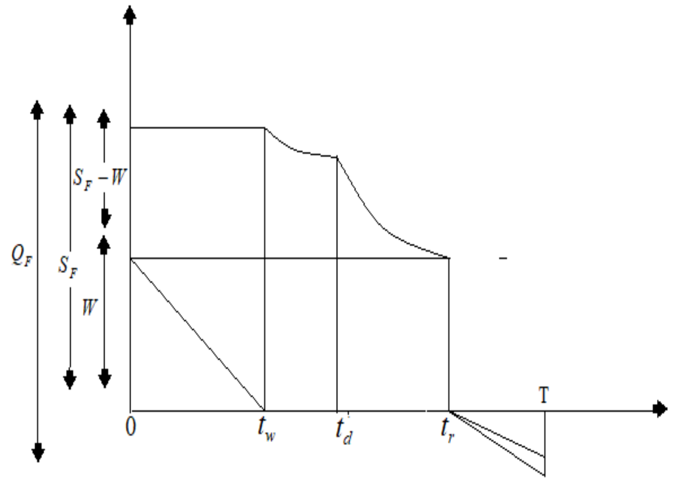

Figure2: Two-warehouse inventory system, when $t_d > t_w$.

Therefore, the differential equations that describe the inventory level in the RW and OW at time t over the period $(0,T)$ are given by:

$$\frac{dI_0(t)}{dt}=-\lambda e^{\lambda t} \quad ,\ 0\le t\le t_w \quad (20)$$

$$\frac{dI_r(t)}{dt}=0\ ,\ 0\le t\le t_d \quad (21)$$

$$\frac{dI_r(t)}{dt}=-\lambda e^{\lambda t} \quad ,\ t_w\le t\le t_d \quad (22)$$

$$\frac{dI_r(t)}{dt}+\beta I_r(t)=-\lambda e^{\lambda t} \quad ,\ t_d\le t\le t_r \quad (23)$$

$$\frac{dB(t)}{dt}=De^{-\delta(T-t)} \quad ,\ t_r\le t\le T \quad (24)$$

The solution of the above differential equation With the boundary condition $I_0(0)=W$, $I_r(t_w)=S_f-W$, $I_r(t_r)=0, B(t_r)=0$

$$I_0(t)=(W+1)-e^{\lambda t} \quad ,\ 0\le t\le t_w \quad (25)$$

$$I_r(t)=S_f-W-e^{\lambda t}+e^{\lambda t_w},\ t_w\le t\le t_d \quad (26)$$

$$I_r(t)=\left(\frac{-\lambda}{\lambda+\beta}\right)e^{\lambda t}+\left(\frac{-\lambda}{\lambda+\beta}\right)e^{(\lambda+\beta)t_r-\beta t},$$

$$t_d\le t\le t_r \quad (27)$$

$$B(t)=\frac{D}{\delta}\left\{e^{-\delta(T-t)}-e^{-\delta(T-t_r)}\right\},\ t_r\le t\le T \quad (28)$$

$$I_r(t)=W,\ 0\le t\le t_d \quad (29)$$

The Numbers of lost sales at time t is $L(t)$ given by

$$L(t)=\int_{t_r}^{T}D\left\{1-e^{-\delta(T-t)}\right\}dt\ ,t_r<t<T$$

$$=D\left[(t-t_r)-\frac{1}{\delta}\left\{e^{-\delta(T-t)}-e^{-\delta(T-t_r)}\right\}\right],$$

$$t_r\le t\le T \quad (30)$$

Now at $t=t_w$ when $I_0(t)=0$ we get $t_w=\frac{W}{D}$

Considering the continuity of $I_r(t)$ at $t=t_d$, it follows from equation (26) and (27), we get

$$S_f=W+\frac{\lambda}{\lambda+\beta}e^{\lambda t_d}+e^{\lambda t_w}+\frac{\lambda}{\lambda+\beta}e^{(\lambda+\beta)t_r-\beta t_d} \quad (31)$$

Putting $t=T$ in equation (28),The maximum amount of demand backlogged per cycle is

$$B(T)=\frac{D}{\delta}\left(1-e^{-\delta(T-t_r)}\right) \quad (32)$$

Therefore the order quantity over the replenishment cycle can be determined as

$$Q_f = S_f + B(T)$$

Using equation (31) & (32) we get

$$Q_f = W + \frac{\beta}{\beta+\lambda} e^{\lambda t_d} + e^{\lambda t_w} + \frac{\lambda}{\lambda+\beta} e^{(\lambda+\beta)t_r-\beta t_d} + \frac{D}{\delta}\left(1-e^{-\delta(T-t_r)}\right) \qquad (33)$$

The total costsper cycle cost of the following

(a) Ordering cost per cycle $= A$

(b) Holding cost per cycle in RW

$$HC_{RW} = F\left[\int_0^{t_w} I_r(t)dt + \int_{t_w}^{t_d} I_r(t)dt + \int_{t_d}^{t_r} I_r(t)dt\right]$$

$$= Wt_d + (S_f - W + e^{\lambda t_w})(t_d - t_w) - \frac{1}{\lambda} e^{\lambda(t_d - t_w)} - \frac{e^{\lambda(t_r - t_d)}}{\lambda+\beta} - \frac{\lambda}{\beta(\lambda+\beta)} e^{\lambda t_r + \beta t_d}$$

(c)Holding cost per cycle in OW

$$= HC_{OW} = H\int_{t_d}^{t_w} I_0(t)dt$$

$$HC_{OW} = H\left[(W+1)t_w - \frac{1}{\lambda}(e^{\lambda t_w} - 1)\right]$$

(d) The backlogged cost per cycle is $= s\int_{t_r}^{T} B(t)dt$

$$= \frac{sD}{\delta}\left[\frac{1}{\delta} - \left\{\frac{1}{\delta} + T - t_r\right\} e^{-\delta(T-t_r)}\right]$$

(e) The opportunity cost due to Sales is

$$c_1 D\int_{t_r}^{T}\left\{1 - e^{-\delta(T-t)}\right\} D\,dt$$

$$= c_1 D\left[T - t_r - \frac{1}{\delta}\left\{1 - e^{-\delta(T-t_r)}\right\}\right]$$

(f) The deterioration cost per cycle $= c\beta\int_{t_d}^{t_r} I_r(t)\,dt$

$$= \frac{-c\beta}{\lambda+\beta}\left[e^{\lambda(t_r-t_d)} + \frac{\lambda}{\beta} e^{\lambda t_r - \beta t_d}\right]$$

Now, the Total relevant cost per time unit time is given by

$$TCF\,2(t_r, T) = \frac{1}{T}\left[OC + HC + BC + OC + DC\right]$$

$$= \frac{1}{T}\Bigg[A + \Bigg[Wt_d + (S_f - W + e^{\lambda t_w})(t_d - t_w) - \frac{1}{\lambda} e^{\lambda(t_d - t_w)} - \frac{e^{\lambda(t_r - t_d)}}{\lambda+\beta} - \frac{\lambda}{\beta(\lambda+\beta)} e^{\lambda t_r + \beta t_d}\Bigg] + H\left[(w+1)t_w - \frac{1}{\lambda}(e^{\lambda t_w} - 1)\right]$$
$$+ \frac{SD}{\delta}\left[\frac{1}{\delta} - \left\{\frac{1}{\delta} + T - t_r\right\} e^{-\delta(T-t_r)}\right] + c_1 D\left[T - t_r - \frac{1}{\delta}\left\{1 - e^{-\delta(T-t_r)}\right\}\right]$$
$$+ \left(\frac{-c\beta}{\lambda+\beta}\right)\left[e^{\lambda(t_r-t_d)} + \frac{\lambda}{\beta} e^{\lambda t_r - \beta t_d}\right]\Bigg] \qquad (34)$$

Case 3: When $t_d < t_r$

During the time interval $[0, t_d]$ there is no deterioration so the inventory in RW is depleted only due to demand whereas in OW inventory level remains the same. Further, during the time interval $[t_d, t_r]$the inventory level in RW is dropping to zero due to the combined effect of demand and deterioration and the inventory in OW gets depleted due to deterioration alone. Now, during the time interval $[t_r, t_w]$ depletion of inventory occurs in OW due to the combined effect of demand and deterioration and it reaches to zero at time t_w . Moreover, during the interval $[t_w, T]$ the demand is backlogged. The behavior of the model over the time interval $[0, T]$ has been graphically represented below in Figure 3.

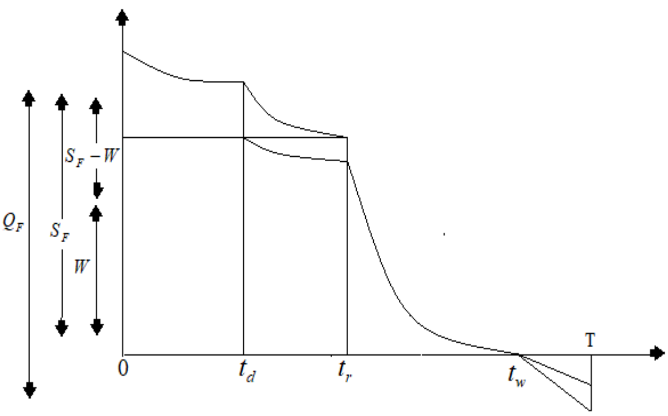

Figure3: Two-warehouse inventory system, when $t_d < t_r$.

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Therefore, the differential equations that describe the inventory level in the RW and OW at time t over the period $(0,T)$ are given by:

$$\frac{dI_r(t)}{dt} = -\lambda e^{\lambda t} \quad , 0 \le t \le t_d \tag{35}$$

$$\frac{dI_r(t)}{dt} + \beta I_r(t) = -\lambda e^{\lambda t} \quad , t_d \le t \le t_r \tag{36}$$

$$\frac{dI_0(t)}{dt} + \alpha I_0(t) = 0 \quad , t_d \le t \le t_r \tag{37}$$

$$\frac{dI_0(t)}{dt} + \alpha I_0(t) = -\lambda e^{\lambda t} , \quad t_r \le t \le t_w \tag{38}$$

$$\frac{dB(t)}{dt} = De^{-\delta(T-t)} \quad , t_w \le t \le T \tag{39}$$

The solution of the above differential equation With the boundary condition $I_r(0) = S_L - W$, $I_r(t_r) = 0$, $I_0(t_d) = W$, $I_0(t_w) = 0$, $B(t_w) = 0$

$$I_r(t) = S_L - W + 1 - e^{\lambda t} \quad , 0 \le t \le t_d \tag{40}$$

$$I_r(t) = \left(\frac{-\lambda}{\lambda+\beta}\right)e^{\lambda t} + \left(\frac{\lambda}{\lambda+\beta}\right)e^{(\lambda+\beta)t_r - \beta t} ,$$
$$t_d \le t \le t_r \tag{41}$$

$$I_0(t) = W e^{-\alpha(t-t_d)} , t_d \le t \le t_r \tag{42}$$

$$I_0(t) = \left(\frac{-\lambda}{\lambda+\alpha}\right)e^{\lambda t} + \left(\frac{\lambda}{\lambda+\alpha}\right)e^{(\lambda+\alpha)t_w - \alpha t} ,$$
$$t_r \le t \le t_w \tag{43}$$

$$B(t) = \frac{D}{\delta}\left\{e^{-\delta(T-t)} - e^{-\delta(T-t_w)}\right\} , t_w \le t \le T \tag{44}$$

The numbers of cost sales at time t is $t = t_d$ it follows from equation (40) & (41)

$$S_f - W + 1 - e^{\lambda t} = I_r(t) = \left(\frac{-\lambda}{\lambda+\beta}\right)e^{-\lambda t_d}$$
$$+\left(\frac{\lambda}{\lambda+\beta}\right)e^{(\lambda+\beta)t_r - \beta t_d} \tag{45}$$

The number of lost sales at time t is

$$L(t) = \int_{t_w}^{T} D\left(1 - e^{-\delta(T-t)}\right)dt , \quad t_w \le t \le T$$

$$= D\left[(t - t_w) - \frac{1}{\delta}\left\{e^{-\delta(T-t)} - e^{-\delta(T-t_w)}\right\}\right] \tag{46}$$

$$S_f = W - 1 + \frac{\beta}{\lambda+\beta}e^{\lambda t_d} + e^{\lambda t_w} + \frac{\lambda}{\lambda+\beta}e^{(\lambda+\beta)t_r - \beta t_d} \tag{47}$$

Considering the continuity of $I_0(t)$ at $t = t_r$, it follows from equation (42) and (43), we get

$$W e^{-\alpha(t_r - t_d)} = \frac{-\lambda}{\lambda+\alpha}e^{\lambda t_r} + \frac{\lambda}{\lambda+\alpha}e^{(\lambda+\alpha)t_w - \alpha t_r}$$

$$t_w = \frac{1}{\lambda+\alpha}\ln\left[\frac{(\lambda+\alpha)W}{\lambda}e^{\alpha t_d} + e^{(\lambda+\alpha)t_r}\right] \tag{48}$$

Putting $t = T$ in equation (44),We will get the maximum amount of demand backlogged per cycle is

$$B(T) = \frac{D}{\delta}\left(1 - e^{-\delta(T-t_w)}\right) \tag{49}$$

Therefore the order quantity over the replenishment cycle can be determined as

$$Q_f = S_f + B(T) = W - 1 + \frac{\beta}{\lambda+\beta}e^{\lambda t_d} + e^{\lambda t_w}$$
$$+ \frac{\lambda}{\lambda+\beta}e^{(\lambda+\beta)t_r - \beta t_d} + \frac{D}{\delta}\left(1 - e^{-\delta(T-t_w)}\right) \tag{50}$$

The total costs per cycle of the following elements

(a) Ordering cost per cycle $= A$

(b) Holding cost per cycle in RW

$$HC_{RW} = F\left[\int_0^{t_d} I_r(t)dt + \int_{t_d}^{t_r} I_r(t)dt\right]$$
$$= F\left[(S_L - W + 1)t_d - \frac{1}{\lambda}\left(e^{\lambda t_d} - 1\right)\right]$$

$$-\left[\frac{1}{\lambda+\beta}e^{\lambda(t_r-t_d)}+\frac{\lambda}{\beta(\lambda+\beta)}e^{\lambda t_r+\beta t_d}\right]$$

(c) Inventory holding cost in OW

$$HC_{OW}=H\left[\int_0^{t_d} w\,dt+\int_{t_d}^{t_r} I_0(t)dt+\int_{t_r}^{t_w} I_0(t)dt\right]$$

$$HC_{OW}=H\left[Wt_d-\frac{1}{\lambda+\alpha}\left\{e^{\lambda(t_w-t_r)}+\frac{\lambda}{\alpha}e^{\lambda t_w+\alpha t_r)}\right\}-\frac{W}{\alpha}e^{\alpha(t_d-t_r)}\right]$$

(d) The Backlogged Cost per cycle is $=s\int_{t_w}^{T}B(t)dt$

$$SC=\frac{sD}{\delta}\left[\frac{1}{\delta}-\left\{\frac{1}{\delta}+T-t_w\right\}e^{-\delta(T-t_w)}\right]$$

(e) The Opportunity Cost due to Sales per cycle is

$$=c_1\int_{t_w}^{T} D\left\{1-e^{-\delta(T-t)}\right\}dt$$

$$=c_1D\left[T-t_w-\frac{1}{\delta}\left\{1-e^{-\delta(T-t_w)}\right\}\right]$$

(f) The Deterioration Cost per cycle

$$=c\left[\beta\int_{t_d}^{t_r} I_r(t)\,dt+\alpha\int_{t_r}^{t_w} I_0(t)\,dt\right]$$

$$=-c\left[\frac{\beta}{\lambda+\beta}\left\{e^{\lambda(t_r-t_d)}+\frac{\lambda}{\beta}e^{\lambda t_r-\beta t_d)}\right\}+\frac{\alpha}{\lambda+\alpha}\left\{e^{\lambda(t_w-t_r)}+\frac{\lambda}{\alpha}e^{\lambda t_w-\beta t_r)}\right\}\right]$$

Now, the Total relevant cost is given by

$$TCL1(t_r,T)=\frac{1}{T}\left[OC+HC+OC+DC+BC\right]$$

$$=\frac{1}{T}\left[A+F\left[(S_L-W+1)t_d-\frac{1}{\lambda}\left(e^{\lambda t_d}-1\right)\right]-\left[\frac{1}{\lambda+\beta}e^{\lambda(t_r-t_d)}+\frac{\lambda}{\beta(\lambda+\beta)}e^{\lambda t_r+\beta t_d}\right]\right.$$

$$+H\left[Wt_d-\frac{1}{\lambda+\alpha}\left\{e^{\lambda(t_w-t_r)}+\frac{\lambda}{\alpha}e^{\lambda t_w+\alpha t_r)}\right\}-\frac{W}{\alpha}e^{\alpha(t_d-t_r)}\right]+\frac{SD}{\delta}\left[\frac{1}{\delta}-\left\{\frac{1}{\delta}+T-t_w\right\}e^{-\delta(T-t_w)}\right]$$

$$+c_1D\left[T-t_w-\frac{1}{\delta}\left\{1-e^{-\delta(T-t_w)}\right\}\right]$$

$$\left.+(-c)\left[\frac{\beta}{\lambda+\beta}\left\{e^{\lambda(t_r-t_d)}+\frac{\lambda}{\beta}e^{\lambda t_r-\beta t_d)}\right\}+\frac{\alpha}{\lambda+\alpha}\left\{e^{\lambda(t_w-t_r)}+\frac{\lambda}{\alpha}e^{\lambda t_w-\beta t_r)}\right\}\right]\right] \quad (51)$$

Case 4: When $t_d > t_r$

In this case, time during which no deterioration occurs is greater than the time during which inventory in RW becomes zero and the behavior of the model over the whole cycle $[0,T]$ has been graphically represented as in Figure 4.

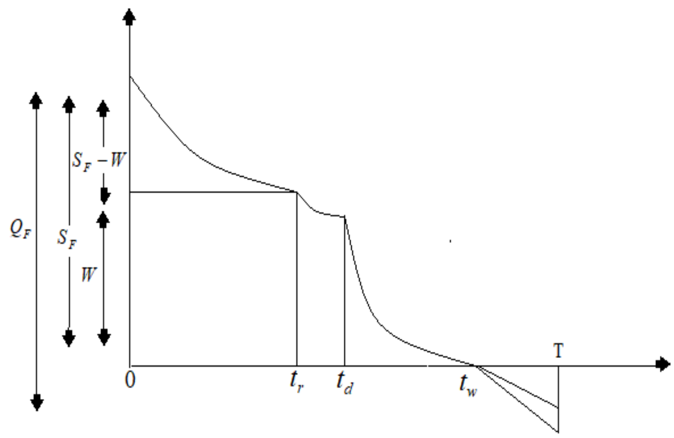

Figure-4: Two-warehouse inventory system, when $t_d > t_r$.

Therefore, the differential equations that describe the inventory level in the RW and OW at time t over the period $(0,T)$ are given by:

$$\frac{dI_r(t)}{dt}=-\lambda e^{\lambda t} \quad ,\ 0\le t\le t_r \quad (52)$$

$$\frac{dI_0(t)}{dt}=-\lambda e^{\lambda t} \quad ,\ t_r\le t\le t_d \quad (53)$$

$$\frac{dI_0(t)}{dt}+\alpha I_0(t)=-\lambda e^{\lambda t} \quad ,\ t_d\le t\le t_w \quad (54)$$

$$\frac{dB(t)}{dt} = De^{-\delta(T-t)} \text{ , } t_w \le t \le T \quad (55)$$

The solution of the above differential equation With the boundary condition $I_r(t_r)=0$, $I_r(t_r)=0$, $I_0(t_r)=W$, $I_0(t_w)=0$, $B(t_w)=0$

$$I_r(t) = e^{\lambda t_r} - e^{\lambda t}, \ 0 \le t \le t_r \quad (56)$$

$$I_0(t) = W + e^{\lambda t_r} - e^{\lambda t}, \ t_r \le t \le t_d \quad (57)$$

$$I_0(t) = \frac{\lambda}{\lambda+\alpha} e^{(\lambda+\alpha)t_w - \alpha t} - e^{\lambda t}, \ t_d \le t \le t_w \quad (58)$$

$$B(t) = \frac{D}{\delta}\left\{e^{-\delta(T-t)} - e^{-\delta(T-t_w)}\right\}, \ t_w \le t \le T \quad (59)$$

The number of lost sales at time t is

$$L(t) = \int_{t_w}^{T} D\left(1-e^{-\delta(T-t)}\right)dt, \quad t_w \le t \le T$$

$$= D\left[(t-t_w) - \frac{1}{\delta}\left\{e^{-\delta(T-t)} - e^{-\delta(T-t_w)}\right\}\right] \quad (60)$$

Considering the continuity of $I_0(t)$ at $t = t_r$, it follows from equation (57) and (58), we get

$$W + e^{\lambda t_r} - e^{\lambda t_d} = \frac{\lambda}{\lambda+\alpha}\left[e^{(\lambda+\alpha)t_w - \alpha t_d} - e^{\lambda t_d}\right] \quad (61)$$

$$t_w = \frac{\alpha}{\lambda+\alpha}t_d + \frac{1}{\lambda+\alpha}\ln\left[\frac{(\lambda+\alpha)}{\lambda}(W+e^{\lambda t_r}) - \frac{\alpha}{\lambda}e^{\lambda t_d}\right]$$

(62)

Now at $t=0$, $I_r(t) = S_L - w$ by the following equation (56),We will get the maximum inventory level is

$$S_f = W + e^{\lambda t_r} - 1$$

$$S_f = W - 1 + e^{\lambda t_r} \quad (63)$$

Putting $t=T$, in (59),We will get the maximum amount of demand backlogged is

$$B(t) = \frac{D}{\delta}\left\{1-e^{-\delta(T-t_w)}\right\} \quad (64)$$

Therefore the order quantity over the replenishment cycle can be determined as

$$Q_f = S_f + B(T) = W - 1 + e^{\lambda t_r} + \frac{D}{\delta}\left(1-e^{-\delta(T-t_w)}\right)$$

(65)

The total costs per cycle of the following elements

(a) Ordering cost per cycle $= A$

(b) Holding cost per cycle in RW is $HC_{RW} = F\left[\int_0^{t_r} I_r(t)dt\right]$

$$= F\left[e^{\lambda t_r} t_r - \frac{1}{\lambda}e^{\lambda t_r} + 1\right] = F\left[1 + e^{\lambda t_r}\left(t_r - \frac{1}{\lambda}\right)\right]$$

(c) Inventory holding cost in OW =

$$HC_{OW} = H\left[\int_0^{t_r} w\,dt + \int_{t_r}^{t_d} I_0(t)dt + \int_{t_d}^{t_w} I_0(t)dt\right]$$

$$HC_{OW} = H\left[Wt_d + e^{\lambda t_r}(t_d - t_r) - \frac{1}{\lambda}e^{\lambda(t_d - t_r)} - \frac{1}{\lambda+\alpha}\left\{\frac{1}{\alpha}e^{\lambda t_w + \alpha t_d} + \frac{1}{\lambda}e^{\lambda(t_w - t_d)}\right\}\right]$$

(d) The backlogged cost $= s\int_{t_w}^{T} B(t)dt$

$$SC = \frac{sD}{\delta}\left[\frac{1}{\delta} - \left\{\frac{1}{\delta} + T - t_w\right\}e^{-\delta(T-t_w)}\right]$$

(e) The opportunity cost $= c_1\int_{t_w}^{T} D\left\{1-e^{-\delta(T-t)}\right\}dt$

$$= c_1 D\left[T - t_w - \frac{1}{\delta}\left\{1-e^{-\delta(T-t_w)}\right\}\right]$$

(f) The deterioration cost $= c\alpha\int_{t_d}^{t_w} I_0(t)\,dt$

$$= -c\alpha\frac{\lambda}{\lambda+\alpha}\left[\frac{1}{\alpha}e^{\lambda t_w - \alpha t_d} + \frac{1}{\lambda}e^{\lambda(t_w - t_d)}\right]$$

Now, the Total relevant cost is given by

$$TCL2(t_r, T) = \frac{1}{T}\left[OC + HC + OC + DC + BC\right]$$

$$TCL\,2(t_r,T)=\frac{1}{T}\left[A+F\left[1+e^{\lambda t_r}\left(t_r-\frac{1}{\lambda}\right)\right]\right.$$
$$+H\left[Wt_d+e^{\lambda t_r}(t_d-t_r)-\frac{1}{\lambda}e^{\lambda(t_d-t_r)}\right.$$
$$\left.-\frac{1}{\lambda+\alpha}\left\{\frac{1}{\alpha}e^{\lambda t_w+\alpha t_d}+\frac{1}{\lambda}e^{\lambda(t_w-t_d)}\right\}\right]$$
$$+\frac{SD}{\delta}\left[\frac{1}{\delta}-\left\{\frac{1}{\delta}+T-t_w\right\}e^{-\delta(T-t_w)}\right]$$
$$+c_1D\left[T-t_w-\frac{1}{\delta}\left\{1-e^{-\delta(T-t_w)}\right\}\right]$$
$$\left.-c\alpha\frac{\lambda}{\lambda+\alpha}\left[\frac{1}{\alpha}e^{\lambda t_w-\alpha t_d}+\frac{1}{\lambda}e^{\lambda(t_w-t_d)}\right]\right] \quad (66)$$

So the Total relevant cost is

$$TCF(t_r,T)=\begin{cases}TCF\,1(t_r,T)\,; & if\ \ t_d\le t_w\\ TCF\,2(t_r,T)\,; & if\ \ t_d\ge t_w\end{cases} \quad (67)$$

$$TCL(t_r,T)=\begin{cases}TCL\,1(t_r,T)\,; & if\ \ t_d\le t_r\\ TCL\,2(t_r,T)\,; & if\ \ t_d\ge t_r\end{cases} \quad (68)$$

Where

$$\frac{\partial TCF\,1(t_r,T)}{\partial t_r}=\left(\frac{0.5(-3+xy^{0.06})}{y^{1.03}}-\frac{x}{y^{0.97}}\right.$$
$$+0.5\left(\frac{0.06}{y^{1.06}}+(-0.06+0.06x)y^{-1.06+0.06x}\right)$$
$$+0.5\left(-1.(-3+x)y^{-1+0.03(-3+x)}\right)$$
$$-0.0291262\left(-1.(-3+x.)y^{-1+0.03(-3+x)}\right.$$
$$\left.+0.03(3+x)y^{-1+0.03(-3+x)}\right)$$
$$+0.12\left(-1.(-3+x)y^{-1+0.03(-3+x)}-1.xy^{-1.03+0.06x}\right.$$
$$\left.-1.(3+x)y^{-1+0.03(3+x)}-\frac{0.5(1.-y^{0.06x})}{y^{1.03}}\right)$$
$$+0.75\left(\frac{2.}{y^{0.97}}-0.0291262\left(-1.(-3+x)y^{-1+0.03(-3+x)}\right.\right.$$
$$\left.+0.03(3+x)y^{-1+0.03(3+x)}\right)$$
$$+200\left(1+10.y^{-0.1(-x+y)}\left(-\frac{0.1(-x+y)}{y}-0.1Log[y]\right)\right)\Bigg)$$
$$+2000.\left(-y^{-0.1(-x+y)}-y^{-0.1(-x+y)}(10.-x+y)\right)$$
$$\left(-\frac{0.1(-x+y)}{y}-0.1Log[y]\right) \quad (a)$$

$$\frac{\partial TCF\,2(t_r,T)}{\partial T}=\left(-16.6667\,y^{0.03}\right.$$
$$-0.0509709\left(-1.y^{-0.03(-3+x)}Log[y]\right.$$
$$+0.03\,y^{-0.06+0.06x}Log[y]+0.5\left(-1.y^{0.03(-3+x)}Log[y]\right.$$
$$\left.-1.y^{0.03(-3+x)}Log[y]\right)$$
$$+0.12\left(-1.y^{-0.03+0.06x}Log[y]-1.y^{0.03(-3+x)}Log[Y]\right.$$
$$\left.-1.y^{0.03(-3+x)}Log[y]-1.y^{0.03(3+x)}Log[y]\right)$$
$$\left.+0.03y^{0.03(3+x)}Log[Y]\right)+200(-1+1.y^{-0.1(-x+y)}Log[y])$$
$$\left.+2000.\left(y^{-0.1(-x+y)}-0.1y^{-0.1(-x+y)}(10.-x+y)Log[y]\right)\right) \quad (b)$$

$$\frac{\partial TCL\,1(t_r,T)}{\partial t_r}=-cHy^{-1+c\lambda}$$
$$-\frac{m\beta\left((-b+x)y^{-1+(-b+x)\lambda}\lambda+\frac{y^{-1+b\beta+x\lambda}\lambda(b\beta+x\lambda)}{\beta}\right)}{\beta+\lambda}$$
$$+\left(F-(b-c)y^{-1+(b-c)\lambda}-\frac{(b-c)y^{-1+(-b+x)\lambda}\lambda}{\beta+\lambda}\right.$$
$$-\frac{y^{-1+b\beta+x\lambda}\lambda(b\beta+x\lambda)}{\beta(\beta+\lambda)}$$
$$+(b-c)\left(by^{-1+b\lambda}\lambda-cy^{-1+c\lambda}\lambda+\frac{by^{-1+b\lambda}\beta\lambda}{\beta+\lambda}\right)$$
$$\left.+\frac{y^{-1-b\beta+x(\beta+\lambda)}\lambda(-b\beta+x(\beta+\lambda))}{\beta+\lambda}\right)$$
$$+\frac{Ds\left(\begin{array}{c}-y^{-(-x+y)\delta}-y^{-(-x+y)\delta}\left(-x+y+\frac{1}{\delta}\right)\\ \left(-\frac{(-x+y)\delta}{y}-\delta Log[y]\right)\end{array}\right)}{\delta}$$
$$+Dr\left(1+\frac{-y^{-(-x+y)\delta}\left(-\frac{(-x+y)\delta}{y}-\delta Log[y]\right)}{\delta}\right) \quad (c)$$

$$\frac{\partial TCL\,2(t_r,T)}{\partial T}=Dr\left(-1+y^{-(-x+y)\delta}Log[y]\right)$$
$$+\frac{Ds\left(y^{-(-x+y)\delta}-y^{-(-x+y)\delta}\left(-x+y+\frac{1}{\delta}\right)\delta Log[y]\right)}{\delta}$$

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$$-\frac{m\beta\left(y^{(-b+x)\lambda}\lambda Log[y]+\frac{y^{b\beta+x\lambda}\lambda^{2}Log[y]}{\beta}\right)}{\beta+\lambda}$$

$$+F\left((b-c)y^{-b\beta+x(\beta+\lambda)}\lambda Log[y]-\frac{y^{(-b+x)\lambda}\lambda Log[y]}{\beta+\lambda}-\frac{y^{b\beta+x\lambda}\lambda^{2}Log[y]}{\beta(\beta+\lambda)}\right) \quad \text{(d)}$$

$$\frac{\partial^2 TCL1(t_r,T)}{\partial t_r^2}=-0.5(-2+x)y^{-1+0.03(-2+x)}-16.6667(0.06+0.03x)y^{-0.94+0.03x}+4\left(0.0608738y^{1.09}+1.(3-x)y^{-1+0.03(3-x)}+0.015(-2+x)y^{-1+0.03(-2+x)}+1.(0.06+0.03x)y^{-0.94+0.03x}\right)$$

$$+0.5\left(-\frac{2.}{y^{0.94}}+2\left(\frac{0.03}{y^{0.94}}+0.05(-0.06+0.06x)y^{-1.06+0.06x}\right)\right)+0.75\left(-200(4-x)y^{3-x}-0.970874\left(0.03(3-x)y^{-1+0.03(3-x)}+0.03(0.09+x)y^{-0.91+x}\right)\right)$$

$$+200\left(1+10.y^{-0.1(-3+y)}\left(-\frac{0.1(-3+y)}{y}-0.1Log[y]\right)+2000.\left(-y^{-0.1(-3+y)}-y^{-0.1(-3+y)}(7.+y)\left(-\frac{0.1(-3+y)}{y}-0.1Log[y]\right)\right)\right) \quad \text{(e)}$$

$$\frac{\partial^2 TCL2(t_r,T)}{\partial T^2}=-0.5y^{0.06+0.03x}Log[y]+0.03y^{-0.06+0.06x}Log[y]-0.5y^{0.03(-2+x)}Log[y]+4\left(-1.y^{0.03(3-x)}Log[y]+0.03y^{0.06+0.03x}Log[y]+0.015y^{0.03(-2+x)}Log[y]\right)+0.75\left(200y^{4-x}Log[y]-0.970873786407767\left(-0.03y^{0.03(3-x)}Log[y]+0.03y^{0.09+x}Log[y]\right)\right) \quad \text{(f)}$$

$$\frac{\partial^2 TCF1(t_r,T)}{\partial t_r^2}=0.015(-33.3333+x)xy^{-1+0.03x}-0.116505\left(\frac{1.}{y^{0.97}}+2.09y^{1.09}\right)+0.75\left(-1.(2-x)y^{-1+0.03(2-x)}+0.03(2-x)xy^{-1+0.03x}-0.0291262\left(\frac{1.}{y^{0.97}}+2.09y^{1.09}\right)\right)$$

$$+2\left(1+10.y^{-0.1(-3+y)}\left(-\frac{0.1(-3+y)}{y}-0.1Log[y]\right)+2000.\left(-y^{-0.1(-3+y)}-y^{-0.1(-3+y)}(7.+y)\left(-\frac{0.1(-3+y)}{y}-0.1Log[y]\right)\right)\right) \quad \text{(g)}$$

$$\frac{\partial^2 TCF2(t_r,T)}{\partial T^2}=0.75\left(-y^{0.03x}+1.y^{0.03(2-x)}Log[y]+0.03(2-x)y^{0.03x}Log[y]\right)+0.5\left(y^{0.03x}+0.03(-33.3333+x)y^{0.03x}Log[y]\right) \quad \text{(h)}$$

Optimality: The Necessary condition for optimality is

$$\frac{\partial TCL(t_r,T)}{\partial t_r}=0,\ \frac{\partial TCL(t_r,T)}{\partial T}=0$$

$$\frac{\partial^2 TCL(t_r,T)}{\partial t_r^2}>0,\ \frac{\partial^2 TCL(t_r,T)}{\partial T^2}>0$$

and

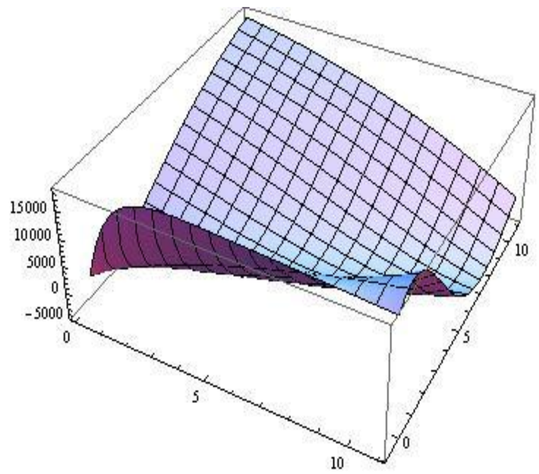
$$\frac{\partial^2 TCL}{\partial t_r^2}\times\frac{\partial^2 TCL}{\partial T^2}-\frac{\partial^2 TCL}{\partial T\partial t_r}\times\frac{\partial^2 TCL}{\partial t_r\partial T}>0$$

$$\frac{\partial TCF(t_r,T)}{\partial t_r}=0,\ \frac{\partial TCF(t_r,T)}{\partial T}=0$$

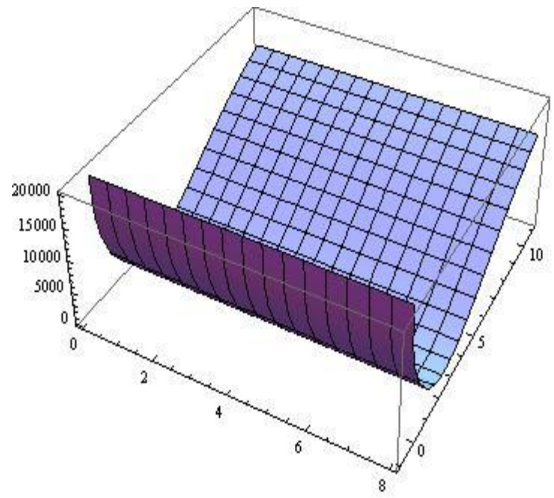
$$\frac{\partial^2 TCF(t_r,T)}{\partial t_r^2}>0,\ \frac{\partial^2 TCF(t_r,T)}{\partial T^2}>0$$

and

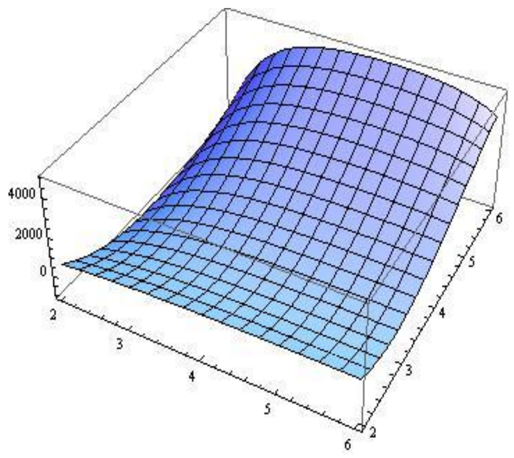
$$\frac{\partial^2 TCF}{\partial t_r^2}\times\frac{\partial^2 TCF}{\partial T^2}-\frac{\partial^2 TCF}{\partial T\partial t_r}\times\frac{\partial^2 TCF}{\partial t_r\partial T}>0$$



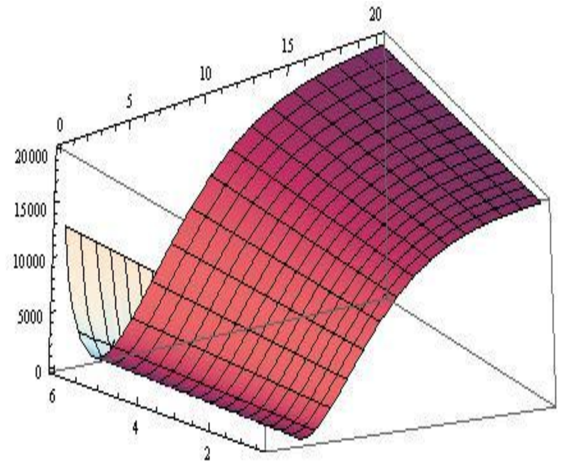
Case-1: $t_d < t_w$



Case-2: $t_d > t_w$



Case-3: $t_d < t_r$



Case-4: $t_d > t_r$

4. NUMERICAL AND SENSITIVITY ANALYSIS

To illustrate the result, let us consider an inventory system with the following data: $A = 100$ Rupee, $c = 4$ Rupee/Unit, $s = 2$ Rupee/Unit, $c_1 = 2$ Rupee/Unit, $H = 0.75$ Rupee/Unit: year, $F = 0.5$ Rupee/Unit/year, $W = 200$ Unit, $D = 100$ Unit/year, $\lambda = 0.3$, $\delta = 0.1$, $\alpha = 1$, $t_d = 2$, $\beta = 0.3$ Unit/year.

$TCF1\,[\lambda = 0.03] = 3383.07$

% Changes	Changes Value	Different from Original Values	Ratio= Different/Original Values	% Changes with Significant
+50	3377.46	-5.61	-0.002	0.2% Down
+25	3380.05	-3.02	-0.001	0.1% Down
-25	3384.31	+1.24	+0.0004	0.04% Up
-50	3385.77	+2.70	+0.0008	0.08% Up

$TCF2\,[\lambda = 0.03] = 3476.73$

% Chang	Changes Va	Different from Original Values	Ratio= Different/Origina alues	% Changes wi Significant
+50	3481.68	+4.95	+0.0015	0.15% Up
+25	3479.76	+3.03	+0.0009	0.09% Up
-25	3475.31	-1042	-0.0004	0.04% Dow
-50	3473.59	-3.14	-0.0008	0.08% Dow

$TCL1\,[\lambda = 0.03] = 2657.22$

% Changes	Changes Value	Different from Original Values	Ratio=Different/Original Values	% Changes with Significant
+50	2613.36	-43.80	-0.016	1.6% Down
+25	2630.80	-26.42	-0.010	1% Down
-25	2669.28	+12.06	+0.0045	0.45% Up
-50	2790.39	+133.17	+0.050	5% Up

$TCL2\,[\lambda = 0.03] = 3197.35$

% Changes	Changes Value	Different from Original Values	Ratio=Different/ Original Values	% Changes with Significant
+50	3210.19	+12.84	+0.004	0.4% Up
+25	3205.16	+7.81	+0.002	0.2% Up
-25	3183.98	-13.37	-0.004	0.4% Down
-50	3156.71	-40.64	-0.013	1.3% Down

$TCF1\,[\beta = 0.03] = 3383.07$

% Changes	Changes Value	Different from Original Values	Ratio=Different Original Values	% Changes with Significant
+50	3379.28	-3.79	-0.001	0.1% Down
+25	3380.74	-2.33	-0.0007	0.07% Down
-25	3387.46	+4.39	+0.0014	0.14% Up
-50	3397.85	+14.78	+0.0043	0.43% Up

$TCF2\,[\beta = 0.03] = 3476.73$

% Changes	Changes Value	Different from Original Values	Ratio=Different Original Values	% Changes with Significant
+50	3482.51	+5.78	+0.0017	0.17% Up
+25	3480.51	+3.47	+0.0009	0.09% Up
-25	3470.94	-5.79	-0.0017	0.17% Down
-50	3459.36	-17.43	-0.005	0.5% Down

$TCL1\,[\beta = 0.03] = 2657.22$

% Changes	Changes Value	Different from Original Values	Ratio=Different Original Values	% Changes with Significant
+50	2668.72	+11.50	+0.0043	0.43% Down
+25	2664.10	+6.88	+0.0022	0.22% Down
-25	2645.84	-11.38	-0.0043	0.43% Up
-50	2623.28	-33.94	-0.0130	1.3% Up

$TCL2\,[\beta = 0.03] = 3197.35$

% Changes	Changes Value	Different from Original Values	Ratio=Different Original Values	% Changes with Significant
+50	3197.35	Not Changed		
+25	3197.35			
-25	3197.35			
-50	3197.35			

$TCF1\,[D = 100] = 3383.07$

% Changes	Changes Value	Different fron Original Values	Ratio=Differen Original Values	% Changes wi Significant
+50	4766.02	+1082.95	+0.320	32% Up
+25	4074.55	+691.148	+0.204	20.4% Up
-25	2691.59	-691.148	-0.204	20.4% Down
-50	2000.12	-1082.95	-0.320	32% Down

$TCF2\,[D = 100] = 3476.73$

% Changes	Changes Value	Different fron Original Values	Ratio=Differen Original Values	% Changes wi Significant
+50	4859.68	+1382.95	+0.3977	39.77% Up
+25	4168.20	+691.47	+0.199	19.9% Up
-25	2785.25	-691.47	-0.199	19.4% Dowr
-50	2093.76	-1382.95	-0.3977	39.77% Dowr

$TCL1\,[D = 100] = 2657.22$

% Change	Changes Value	Different fro Original Values	Ratio=Diffe nt/Original Values	% Changes with Significant
+50	4040.17	+1382.95	+0.520	52.0% Up
+25	3348.69	+690.74	+0.529	25.9% Up
-25	1965.74	-691.48	-0.261	26.1% Down
-50	1274.26	-1382.96	-0.520	52.0% Down

$TCL2\,[D = 100] = 3197.35$

% Changes	Changes Value	Different fron Original Values	Ratio=Differen Original Values	% Changes wi Significant
+50	4620.55	+1423.20	+0.445	44.5% Up
+25	3908.95	+711.60	+0.223	22.3% Up
-25	2485.75	-711.60	-0.223	22.3% Down
-50	1774.15	-1423.20	-0.445	44.5% Down

$TCF1\,[W = 200] = 3383.07$

% Changes	Changes Value	Different from Original Values	Ratio=Different/(iginal Values	% Changes wi Significant
+50	3633.07	+250.00	+0.074	7.4% Up
+25	3508.07	+125.00	+0.037	3.7% Up
-25	3258.07	-125.00	-0.037	3.7% Down
-50	3133.07	-250.00	-0.074	7.4% Down

$TCF2\,[W=200]=3476.73$

% Changes	Changes Value	Different from Original Values	Ratio=Different/Original Values	% Changes with Significant
+50	3801.73	+325.00	+0.093	9.3% Up
+25	3639.23	+162.5	+0.047	4.7% Up
-25	3314.23	-162.5	-0.047	4.7% Down
-50	3151.73	-325.00	-0.093	9.3% Down

$TCL1\,[W=200]=2657.22$

% Changes	Changes Value	Different from Original Values	Ratio=Different/Original Values	% Changes with Significant
+50	2507.22	-150.00	-0.056	5.6% Down
+25	2582.22	-75.00	-0.029	2.9% Down
-25	2732.22	+75.00	+0.029	2.9% Up
-50	2807.22	+150.00	+0.056	5.6% Up

$TCL2\,[W=200]=3197.35$

% Changes	Changes Value	Different from Original Values	Ratio=Different/Original Values	% Changes with Significant
+50	3347.35	+150.00	+0.047	4.7% Up
+25	3272.35	+75.00	+0.024	2.4% Up
-25	3122.35	-75.00	-0.024	2.4% Down
-50	3047.35	-150.00	-0.047	4.7% Down

$TCF1\,[A=100]=3383.07$

% Changes	Changes Value	Different from Original Values	Ratio=Different/Original Values	% Changes with Significant
+50	3433.07	+50	+0.015	1.5% Up
+25	3408.07	+25	+0.008	0.8% Up
-25	3358.07	-25	-0.008	0.8% Down
-50	3333.07	-50	-0.015	1.5% Down

$TCF2\,[A=100]=3476.73$

% Changes	Changes Value	Different from Original Values	Ratio=Different/Original Values	% Changes with Significant
+50	3526.73	+50	+0.014	1.4% Up
+25	3501.73	+25	+0.007	0.7% Up
-25	3451.73	-25	-0.007	0.7% Down
-50	3426.73	-50	-0.014	1.4% Down

$TCL1\,[A=100]=2657.22$

% Changes	Changes Value	Different from Original Values	Ratio=Different/Original Values	% Changes with Significant
+50	2702.22	+50	+0.019	1.9% Up
+25	2682.22	+25	+0.01	1% Up
-25	2632.22	-25	-0.01	1% Down
-50	2607.22	-50	-0.019	1.9% Down

$TCL2\,[A=100]=3197.35$

% Changes	Changes Value	Different from Original Values	Ratio=Different/Original Values	% Changes with Significant
+50	3247.35	+50	+0.0156	1.56% Up
+25	3222.35	+25	+0.0078	0.78% Up
-25	3172.35	-25	-0.0078	0.78% Down
-50	3147.35	-50	-0.0156	1.56% Down

$TCF1\,[F=0.5]=3383.07$

% Changes	Changes Value	Different from Original Values	Ratio=Different/Original Values	% Changes with Significant
+50	5024.60	+1641.53	+0.485	48.5% Up
+25	4203.84	+820.77	+0.243	24.3% Up
-25	2562.30	-820.77	-0.243	24.3% Down
-50	1741.53	-1641.53	-0.485	48.5% Down

$TCF2\,[F=0.5]=3476.73$

% Changes	Changes Value	Different from Original Values	Ratio=Different/Original Values	% Changes with Significant
+50	3559.77	+83.04	+0.024	2.4% Up
+25	3518.25	+41.52	+0.012	1.2% Up
-25	3435.21	-41.52	-0.012	1.2% Down
-50	3393.69	-83.04	-0.024	2.4% Down

$TCL1\,[F=0.5]=2657.22$

% Changes	Changes Value	Different from Original Values	Ratio=Different/Original Values	% Changes with Significant
+50	2657.06	-0.16	-0.00006	0.006% Down
+25	2657.14	-0.08	-0.00003	0.003% Down
-25	2657.29	+0.07	+0.00003	0.003% Up
-50	2657.37	+0.15	+0.00006	0.006% Up

$TCL2\,[F = 0.5] = 3197.35$

% Changes	Changes Value	Different from Original Values	Ratio=Different/ Original Values	% Changes with Significant
+50	3189.01	-8.34	-0.0026	0.26% Down
+25	3193.18	-4.17	-0.0013	0.13% Down
-25	3201.52	+4.17	+0.0013	0.13% Up
-50	3205.69	+8.34	+0.0026	0.26% Up

$TCF1\,[H = 0.75] = 3383.07$

% Changes	Changes Value	Different fron Original Values	Ratio=Differen Original Values	% Changes with Significant
+50	3547.70	+164.63	+0.0486	4.86% Up
+25	3465.39	+82.32	+0.0243	2.43% Up
-25	3300.75	-82.32	-0.0243	2.43% Down
-50	3218.44	-164.63	-0.0486	4.86% Down

$TCF2\,[H = 0.75] = 3476.73$

% Changes	Changes Value	Different fron Original Values	Ratio=Differen Original Values	% Changes with Significant
+50	3701.19	+224.46	+0.0645	6.45% Up
+25	3587.46	+112.23	+0.0322	3.22% Up
-25	3364.50	-112.23	-0.0322	3.22% Down
-50	3252.26	-221.46	-0.0645	6.45% Down

$TCL1\,[H = 0.75] = 2657.22$

% Changes	Changes Value	Different fron Original Values	Ratio=Different Original Values	% Changes with Significant
+50	2506.15	-151.07	-0.0568	5.68% Down
+25	2582.69	-74.53	-0.0284	2.84% Down
-25	2732.75	+74.53	+0.0284	2.84% Up
-50	2808.28	+151.07	+0.0568	5.68% Up

$TCL2\,[H = 0.75] = 3197.35$

% Changes	Changes Value	Different fron Original Values	Ratio=Differen Original Values	% Changes with Significant
+50	3334.40	+137.05	+0.0428	4.28% Up
+25	3264.96	+67.61	+0.0214	2.14% Up
-25	3128.82	-67.61	-0.0214	2.14% Down
-50	3060.29	-137.05	-0.0428	4.28% Down

$TCF1\,[\delta = 0.1] = 3383.07$

% Changes	Changes Value	Different fron Original Values	Ratio=Differen Original Values	% Changes wi Significant
+50	2580.35	-802.82	-0.2408	24.08% Dowr
+25	2903..51	-479.56	-0.1477	14.77% Dowr
-25	4373.63	+790.56	+0.2372	23.72% Up
-50	5738.45	+2355.88	+0.6873	68.73% Up

$TCF2\,[\delta = 0.1] = 3476.73$

% Changes	Changes Value	Different fron Original Values	Ratio=Differen Original Values	% Changes wi Significant
+50	2673.91	-802.82	-0.2308	23.08% Dowr
+25	2997.17	-479.56	-0.1377	13.77% Dowr
-25	4267.29	+790.56	+0.2272	22.72% Up
-50	5832.61	+2355.88	+0.6773	67.73% Up

$TCL1\,[\delta = 0.1] = 2657.22$

% Changes	Changes Value	Different fron Original Values	Ratio=Differen Original Values	% Changes wi Significant
+50	1854.40	-802.82	-0.3021	30.21% Dowr
+25	2177.65	-479.57	-0.1804	18.04% Dowr
-25	3447.78	+790.56	+0.1800	18.00% Up
-50	5013.10	+2355.88	+0.8866	88.66% Up

$TCL2\,[\delta = 0.1] = 3197.35$

% Changes	Changes Value	Different fron Original Values	Ratio=Differen Original Values	% Changes wi Significant
+50	2333.47	-863.58	-0.2701	27.01% Dowr
+25	2686.25	-510.80	-0.1595	15.95% Dowr
-25	4021.63	+824.58	+0.2577	25.77% Up
-50	5623.62	+2426.57	+0.7588	75.88% Up

$TCF1\,[\alpha = 1] = 3383.07$

% Changes	Changes Value	Different from Original Values	Ratio=Different Original Values	% Changes witl Significant
+50	3382.88	-0.19	-0.000056	0.0056% Dowr
+25	3382.96	-0.11	-0.000042	0.0042% Dowr
-25	3383.24	+0.17	+0.000052	0.0052% Up
-50	3383.57	+0.50	+0.000148	0.0148% Up

$TCF2\,[\alpha = 1] = 3476.73$

% Changes	Changes Value	Different from Original Values	Ratio=Different/Original Values	% Changes with Significant
+50	3476.73	No Changes		
+25	3476.73			
-25	3476.73			
-50	3476.73			

$TCL1\,[\alpha = 1] = 2657.22$

% Changes	Changes Value	Different from Original Values	Ratio=Different/Original Values	% Changes with Significant
+50	2456.95	-200.27	-0.0753	7.53% Down
+25	2578.51	-78.71	-0.0296	2.96% Down
-25	2691.21	+33.99	+0.0128	1.28% Up
-50	2656.33	-0.89	-0.0003	0.03% Down

$TCL2\,[\alpha = 1] = 3197.35$

% Changes	Changes Value	Different from Original Values	Ratio=Different/Original Values	% Changes with Significant
+50	3193.46	-3.39	-0.00106	0.106% Down
+25	3196.17	-0.40	-0.00028	0.028% Down
-25	3196.87	-0.48	-0.00015	0.015% Down
-50	3197.80	+0.45	+0.00014	0.014% Up

$TCF1\,[s = 2] = 3383.07$

% Changes	Changes Value	Different from Original Values	Ratio=Different Original Values	% Changes with Significant
+50	4806.27	+1423.20	+0.4207	42.07% Up
+25	4094.67	+711.60	+0.2103	21.03% Up
-25	2671.47	-711.60	-0.2103	21.03% Down
-50	1959.87	-1423.20	-4207	42.07% Down

$TCF2\,[s = 2] = 3476.73$

% Changes	Changes Value	Different from Original Values	Ratio=Different Original Values	% Changes with Significant
+50	4899.93	+1423.20	+0.4093	40.93% Up
+25	4188.33	+711.60	+0.2046	20.46% Up
-25	2765.13	-711.20	-0.2046	20.46% Down
-50	2053.53	-1423.20	-0.4093	40.93% Down

$TCL1\,[s = 2] = 2657.22$

% Changes	Changes Value	Different from Original Values	Ratio=Different Original Values	% Changes with Significant
+50	4080.42	+1423.2	+0.5355	53.55% Up
+25	3368.82	+711.60	+0.2678	26.78% Up
-25	1945.62	-711.60	-0.2678	26.78% Down
-50	1234.02	-1423.2	-0.5355	53.55% Down

$TCL2\,[s = 2] = 3197.35$

% Changes	Changes Value	Different from Original Values	Ratio=Different Original Values	% Changes with Significant
+50	4620.55	+1423.20	+0.4451	44.51% Up
+25	3908.95	+711.60	+0.2225	22.25% Up
-25	2485.75	-711.20	-0.2225	22.25% Down
-50	1774.15	-1423.20	-0.4451	44.51% Down

$TCF1\,[c_1 = 2] = 3383.07$

% Changes	Changes Value	Different from Original Values	Ratio=Different Original Values	% Changes with Significant
+50	3342.82	-40.25	-0.0119	1.19% Down
+25	3362.95	-20.12	-0.0059	0.59% Down
-25	3403.19	+20.12	+0.0059	0.59% Up
-50	3423.32	+40.25	+0.0119	1.19% Up

$TCF2\,[c_1 = 2] = 3476.73$

% Changes	Changes Value	Different from Original Values	Ratio=Different Original Values	% Changes with Significant
+50	3436.48	-40.25	-0.0116	1.16% Down
+25	3456.60	-20.12	-0.0058	0.58% Down
-25	3496.85	+20.12	+0.0058	0.58% Up
-50	3516.97	+40.25	+0.0116	1.16% Up

$TCL1\,[c_1 = 2] = 2657.22$

% Changes	Changes Value	Different from Original Values	Ratio=Different Original Values	% Changes with Significant
+50	2616.97	-40.25	-0.01514	1.51% Down
+25	2637.09	-20.12	-0.00757	0.75% Down
-25	2677.34	+20.12	+0.00757	0.75% Up
-50	2697.46	+40.25	+0.01514	1.51% Up

$TCL2\,[c_1 = 2] = 3197.35$

% Changes	Changes Value	Different from Original Values	Ratio=Different Original Values	% Changes with Significant
+50	3196.95	-0.40	-0.000125	0.0125% Down
+25	3197.15	-0.20	-0.000063	0.0063% Down
-25	3197.55	+0.20	+0.000063	0.0063% Up
-50	3197.75	+0.40	+0.000125	0.0125% Up

$TCF1\,[c = 4] = 3383.07$

% Changes	Changes Value	Different from Original Values	Ratio=Different/Oginal Values	% Changes with Significant
+50	3378.78	-4.29	-0.00126	0.126% Down
+25	3380.93	-2.14	-0.00063	0.063% Down
-25	3385.21	+2.14	+0.00063	0.063% Up
-50	3387.36	+4.29	+0.00126	0.126% Up

$TCF2\,[c = 4] = 3476.73$

% Changes	Changes Value	Different from Original Values	Ratio=Different/Oginal Values	% Changes with Significant
+50	3474.64	-2.09	-0.00060	0.06% Down
+25	3475.68	-1.05	-0.00030	0.03% Down
-25	3477.77	+1.05	+0.00030	0.03% Up
-50	3478.82	+2.09	+0.00060	0.06% Up

$TCL1\,[c = 4] = 2657.22$

% Changes	Changes Value	Different from Original Values	Ratio=Different/Oginal Values	% Changes with Significant
+50	2723.04	+65.82	+0.02477	2.47% Up
+25	2690.13	+32.91	+0.01238	1.23% Up
-25	2629.31	-32.91	-0.01238	1.23% Down
-50	2591.39	-65.82	-0.02477	2.47% Down

$TCL2\,[c = 4] = 3197.35$

% Changes	Changes Value	Different from Original Values	Ratio=Different/Oginal Values	% Changes with Significant
+50	3194.11	-3.24	-0.0010	0.1% Up
+25	3195.53	-1.82	-0.0006	0.06% Up
-25	3198.36	+1.01	+0.0003	0.03% Down
-50	3199.78	+2.43	+0.0008	0.08% Down

$TCF1\,[t_d = 2] = 3383.07$

% Changes	Changes Value	Different from Original Values	Ratio=Different Original Values	% Changes with Significant
+50	5291.76	+1908.69	+0.5642	56.42% Up
+25	4315.14	+932.07	+0.2754	27.54% Up
-25	2495.53	-887.54	-0.2621	26.21% Down
-50	1652.51	-1730.56	-0.5113	51.13% Down

$TCF2\,[t_d = 2] = 3476.73$

% Changes	Changes Value	Different from Original Values	Ratio=Different Original Values	% Changes with Significant
+50	3576.45	+99.72	+0.0288	2.88% Up
+25	3526.59	+50.14	+0.0150	1.50% Up
-25	3426.86	-49.87	-0.0149	1.49% Down
-50	3376.98	-99.75	-0.0289	2.89% Down

$TCL1\,[t_d = 2] = 2657.22$

% Changes	Changes Value	Different from Original Values	Ratio=Different Original Values	% Changes with Significant
+50	-6187.20	-8844.42	-3.328	332.8% Down
+25	934.05	-1723.17	-0.6484	64.84% Down
-25	3031.34	+374.12	+0.1407	14.07% Up
-50	3068.42	+411.20	+0.1546	15.46% Up

$TCL2\,[t_d = 2] = 3197.35$

% Changes	Changes Value	Different from Original Values	Ratio=Different Original Values	% Changes with Significant
+50	3340.44	+143.09	+0.0448	4.48% Up
+25	3270.05	+72.70	+0.0225	2.25% Up
-25	3123.48	-73.87	-0.0226	2.26% Down
-50	3049.02	-148.33	-0.0510	5.10% Down

$TCF1\,[\lambda = 0.03, \beta = 0.03, \delta = 0.1, \alpha = 1] = 3383.07$

% Changes	Changes Value	Different from Original Values	Ratio=Different Original Values	% Changes with Significant
+50	2571.20	-811.87	-0.2399	23.99% Down
+25	2898.13	-484.94	-0.1433	14.33% Down
-25	4182.43	+799.36	+0.2361	23.61% Up
-50	5765.16	+2382.03	+0.7041	70.41% Up

$TCF2\,[\lambda = 0.03, \beta = 0.03, \delta = 0.1, \alpha = 1] = 3476.73$

% Changes	Changes Value	Different from Original Values	Ratio=Different Original Values	% Changes with Significant
+50	2684.76	-791.97	-0.2277	22.77% Down
+25	3003.70	-473.03	-0.1360	13.60% Down
-25	4256.31	+799.58	+0.2298	22.98% Up
-50	5799.52	+2322.79	+0.6678	66.78% Up

$TCL1\,[\lambda = 0.03, \beta = 0.03, \delta = 0.1, \alpha = 1] = 2657.22$

% Changes	Changes Value	Different from Original Values	Ratio=Different/Original Values	% Changes with Significant
+50	2162.44	-494.78	-0.1862	18.62% Down
+25	2079.39	-577.83	-0.2171	21.71% Down
-25	3514.69	+857.47	+0.3225	32.25% Up
-50	5111.42	+2454.20	+0.9235	92.35% Up

$TCL2\,[\lambda = 0.03, \beta = 0.03, \delta = 0.1, \alpha = 1] = 3197.35$

% Changes	Changes Value	Different from Original Values	Ratio=Different/Original Values	% Changes with Significant
+50	2346.31	-851.04	-0.2661	26.61% Down
+25	2692.54	-504.81	-0.1585	15.85% Down
-25	4008.58	+811.23	+0.2536	25.36% Up
-50	5582.23	+2384.88	+0.7456	74.56% Up

$TCF1\,[D = 100, W = 200, A = 100] = 3383.07$

% Changes	Changes Value	Different from Original Values	Ratio=Different/Original Values	% Changes with Significant
+50	5066.02	+1682.95	+0.4974	49.74% Up
+25	4224.55	+840.95	+0.2483	24.83% Up
-25	2541.59	-841.48	-0.2484	24.84% Down
-50	1700.12	-1682.95	-0.4974	49.74% Down

$TCF2\,[D = 100, W = 200, A = 100] = 3476.73$

% Changes	Changes Value	Different from Original Values	Ratio=Different/Original Values	% Changes with Significant
+50	5234.68	+1757.95	+0.5053	50.53% Up
+25	4355.70	+878.97	+0.2525	25.25% Up
-25	2579.75	-896.98	-0.2610	26.10% Down
-50	1718.78	-1757.95	-0.5053	50.53% Down

$TCL1\,[D = 100, W = 200, A = 100] = 2657.22$

% Changes	Changes Value	Different from Original Values	Ratio=Different/Original Values	% Changes with Significant
+50	-9329.24	-11986.46	-4051	451% Down
+25	-7758.22	-10414.44	-3.92	392% Down
-25	-4616.17	-7273.39	-2.73	273% Down
-50	-3045.15	-5702.37	-2.14	214% Down

$TCL2\,[D = 100, W = 200, A = 100] = 3197.35$

% Changes	Changes Value	Different from Original Values	Ratio=Different/Original Values	% Changes with Significant
+50	5038.63	+1841.28	+0.5758	57.58% Up
+25	4189.54	+992.19	+0.3102	31.02% Up
-25	2491.34	-706.01	-0.2208	32.08% Down
-50	1642.24	-1555.11	-0.4863	48.63% Down

In TCF1, By changing the value of the parameters there is a significant changes in optimum value happening with no changes the TCF1 takes amount 3383.07 % changes in different parameters makes effect less or more amount on TCF1given in tables.

As changing of parameter H in quarterly percentile the changes in TCF1 occurs 2.430 % in right manner.

As parameter δ changes quarterly the effect on TCF1 13% in opposite to right manner from its original value.

As parameter α changes quarterly the effect on TCF1 changes at very less 0.0042% from its original value.

As parameter s changes quarterly percentile the changes in TCF1 very high at 21.03 % from its original value.

As parameter c_1 changes quarterly percentile the changes in TCF1 changes at 0.59 % from its original values.

As parameter c changes quarterly percentile the changes in TCF1 changes at 0.063% from its original values.

As parameter t_d changes quarterly percentile in changes in TCF1 changes highly in right way at 27.54 % from its original values.

As parameter λ changes quarterly percentile the changes in TCF1 changes 0.1 % from its original value.

As parameter β changes quarterly percentile the changes in TCF1 changes very less at 0.07 % reverse order from its original values.

As parameter D changes quarterly percentile the changes in TCF1 changes 20.4 % from its original values.

As parameter W changes quarterly percentile the changes in TCF1 changes at 3.7 % from its original value in right order.

As parameter A changes quarterly percentile the changes in TCF1 changes at 0.8 % from its original values in right order.

As parameter F changes quarterly percentile the changes in TCF1 changes at 24.3 % in right order from its original value.

As parameter λ , β δ and α changes quarterly percentile simultaneously the changes in TCF1 is at 14.33 % from its original values in reverse order from its original value.

As parameter D , W , A changes quarterly percentile simultaneously the changes in TCF1 changes at 24.83 % from its original values in right order.

In TCF2, By changing the value of the parameters there is a significant changes in optimum value happening with no changes the TCF2 takes amount 3476.73 % changes in different parameters makes effect less or more amount on TCF2 given in tables.

As changing of parameter H in quarterly percentile the changes in TCF2 occurs 2.430 % in right manner.

As parameter δ changes quarterly the effect on TCF2 changes at 3.22% in opposite to right manner from its original value.

As parameter α changes quarterly the effect on TCF2 no changes from its original value.

As parameter s changes quarterly percentile the changes in TCF2 changes at very high 22.25% from its original value.

As parameter c_1 changes quarterly percentile the changes in TCF2 changes at 0.58 % from its original values.

As parameter c changes quarterly percentile the changes in TCF2 changes 0.03 % in reverse order from its original value.

As parameter t_d changes quarterly percentile in changes in TCF2 changes highly in right way at 1.5 % from its original values.

As parameter λ changes quarterly percentile the changes in TCF2 changes 0.09 % from its original value.

As parameter β changes quarterly percentile the changes in TCF2 changes very less at 0.09 % reverse order from its original values.

As parameter D changes quarterly percentile the changes in TCF2 changes 19.90 % from its original values.

As parameter W changes quarterly percentile the changes in TCF2 changes at 4.7 % from its original value in right order.

As parameter A changes quarterly percentile the changes in TCF2 changes at 0.7 % from its original values in right order.

As parameter F changes quarterly percentile the changes in TCF2 changes at 1.2 % in right order.

As parameter λ , β , δ and α changes quarterly percentile simultaneously the changes in TCF2 is at 13.60 % from its original values in reverse order.

As parameter D , W , A changes quarterly percentile simultaneously the changes in TCF2 changes at 25.25 % from its original values in right order .

In TCL1, By changing the value of the parameters there is a significant changes in optimum value happening with no changes the TCL1 takes amount 2657.22 % changes in different parameters makes effect less or more amount on TCL1given in tables.

As changing of parameter H in quarterly percentile the changes in TCL1 occurs 2.84 % in reverse manner.

As parameter δ changes quarterly the effect on TCL1 18.04% in opposite to right manner from its original value in reverse order.

As parameter α changes quarterly the effect on TCL1 changes at very less 2.96% from its original value in reverse order.

As parameter s changes quarterly percentile the changes in TCL1 very high at 21.03% from its original value.

As parameter c_1 changes quarterly percentile the changes in TCL1 changes at 0.75 % from its original values in reverse order.

As parameter c changes quarterly percentile the changes in TCL1 changes 1.23% in reverse order from its original value in right manner.

As parameter t_d changes quarterly percentile in changes in TCL1 changes highly in right way at 64.84 % from its original values in reverse order.

As parameter λ changes quarterly percentile the changes in TCL1 changes 1% from its original value in reverse order.

As parameter β changes quarterly percentile the changes in TCL1 changes very less at 0.22 % reverse order from its original values in right manner.

As parameter D changes quarterly percentile the changes in TCL1 changes 20.3 % from its original values in right manner.

As parameter W changes quarterly percentile the changes in TCL1 changes at 2.9 % from its original value in reverse order.

As parameter A changes quarterly percentile the changes in TCL1 changes at 1.0 % from its original values in right order.

As parameter F changes quarterly percentile the changes in TCL1 changes at 0.003 % in reverse order from its original value.

As parameter λ , β , δ and α changes quarterly percentile simultaneously the changes in TCL1 is at 21.71 % from its original values in reverse order from its original value.

As parameter D , W , A changes quarterly percentile simultaneously the changes in TCL1 changes at 392 % from its original values inverse order.

In TCL2, By changing the value of the parameters there is a significant changes in optimum value happening with no changes the TCL2 takes amount 3197.35 % changes in different parameters makes effect less or more amount on TCL2given in tables.

As changing of parameter H in quarterly percentile the changes in TCL2 occurs 2.14 % in right manner.

As parameter δ changes quarterly the effect on TCL2 15.95 % in opposite to right manner from its original value in reverse order.

As parameter α changes quarterly the effect on TCL2 changes at very less 0.015% from its original value in reverse order.

As parameter S changes quarterly percentile the changes in TCL2 very high at 20.46% from its original value.

As parameter c_1 changes quarterly percentile the changes in TCL2 changes at 0.0063 % from its original values in reverse order.

As parameter C changes quarterly percentile the changes in TCL2 changes 0.06% in reverse order from its original value in reverse manner.

As parameter t_d changes quarterly percentile in changes in TCL2 changes highly in right way at 2.25 % from its original values in right order.

As parameter λ changes quarterly percentile the changes in TCL2 changes 0.41 % from its original value in right order.

As parameter β changes quarterly percentile the changes in TCL2 no changes from its original values in right manner.

As parameter D changes quarterly percentile the changes in TCL2 changes 26.1% from its original values in right manner.

As parameter W changes quarterly percentile the changes in TCL2 changes at 2.4 % from its original value in right order.

As parameter A changes quarterly percentile the changes in TCL2 changes at 0.78 % from its original values in right order.

As parameter F changes quarterly percentile the changes in TCL2 changes at 0.13 % in reverse order from its original value.

As parameter λ , β , δ and α changes quarterly percentile simultaneously the changes in TCL2 is at 25.36 % from its original values in reverse order from its original value.

As parameter D , W , A changes quarterly percentile simultaneously the changes in TCL2 changes at 32.08 % from its original values in right order.

5. CONCLUSION

In this study, a two-warehouse inventory model for non-instantaneous deteriorating items with exponential demand rate under time varying holding cost has been developed. Shortages are allowed and completely backlogged. The deterioration rate in OW is assumed to be higher than in RW and the holding cost in RW is greater than that in OW because of the difference in the storage facilities of both warehouse. While formulating the inventory model for deteriorating items, usually it is assumed that the items start deteriorating as soon as they enter into the warehouse. However, there are numerous products like food items(dry fruits, food grains etc.), electronic items (refrigerator, television, etc.), that have a shelf-life and start deteriorating after a time lag that is termed as non-instantaneous deterioration. Moreover, the price discount, low cost storage, huge demand etc. and under such a situation one may decide to procure large quantity of the items which would arise the problem of storing. As the capacity of own warehouse is limited, therefore one has to hire another rented warehouse to store the excess quantity.

The proposed model can further be extended by including some more realistic features, such as inventory-level-dependent demand, price-dependent demand, inflation and permissible delay in payments etc.

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