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The Forcing Hull Domination Number of a Graph

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Abstract

For a connected graph $G = (V, E)$, a hull set Min a connected graph $G$is called a *hull dominating set* of G if $M$is both hull set and a dominating set of G. *The hull domination number*$rh(G)$ of G is the minimum cardinality of a hull dominating set of G .Let M be a minimum hull dominating set of G. A subset $T \subseteq M$ is called a *forcing subset* for Mif M is the unique minimum hull dominating set containing T. A forcing subset for M of minimum cardinality is a *minimum forcing subset of* M. The *forcing hull domination number*of M, denoted by $f_{rh}(M)$, is the cardinality of a minimum forcing subset of M. The *forcing hull domination number* of G, denoted by $f_{rh}(G)$, is $f_{rh}(G) = min\{f_{rh}(M)\}$, where the minimum is taken over all minimum hull dominating sets M in G. Some general properties satisfied by this concept is studied. The hull domination numberof certain standard graphs are determined. The forcing hull domination number of a connected graph to be 0 and 1 are characterized.It is shown that for every positive integers a and b with $0 \leq a < b$ and $b \geq 2$, there exists a connected graph G with $r_h(G) = a$ and$f_{rh}(G) = b$.

Keywords: domination number, hull number, c hull domination number, forcing hull domination number.

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1. Introduction

By a graph $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For basic graph theoretic terminology, we refer to Harary [1]. A convexity on a finite set V is a family C of subsets of V, convex sets which are closed under intersection and which contains both V and the empty set. The pair (V, E) is called a convexity space. A finite graph convexity space is a pair (V, E), formed by a finite connected graph$G = (V, E)$ and a convexity C on V such that (V, E) is a convexity space satisfying that every member of C induces a connected subgraph of G. Thus, classical convexity can be extended to graphs in a natural way. We know that a set X of R^n is convex if every segment joining two points of X is entirely contained in it. Similarly a vertex set W of a finite connected graph is said to be convex set of G if it contains all the vertices lying in a certain kind of path connecting vertices of W[8]. The distance $d(u, v)$ between two vertices u and v in a connected graph $G$is the length of a shortest $u - v$ path in G. An $u - v$ path of length $d(u, v)$ is called an $u - v$ geodesic For two vertices u and v, let $I[u, v]$denotes the set of all vertices which lie on $u - v$ geodesic. For a set S of vertices, let $I[S] = \cup_{u,v \in S} I[u, v]$. The set S is *convex*if $I[S] = S$. Clearly if $S = \{v\}$or $S = V$, then S isconvex. The *convexity* number, denoted by $C(G)$, is the cardinality of a maximum proper convex subset of V. The smallest convex setcontaining S is denoted by $I_h(S)$ and called the *convex hull* of S. Since the intersection of two convex sets is convex, the convex hull is well defined. Note that $S \subseteq I[S] \subseteq I_h(S) \subseteq V$. A The *hull number*$h(G)$of G is the minimum order of its hullsets and any hullset of order $h(G)$ is a *minimum hull set* or simply a *h-set*of G. A set of vertices D in a graph G is a dominating set if each vertex of G is dominated by some vertex of D. The domination number of G is the minimum cardinality of a dominating set of G and is denoted by $\gamma(G)$. A dominating set of size $\gamma(G)$ is said to be a γ-set. A hull set Min a connected graph $G$is called a *hull dominating set* of G if $M$is both hull set and a dominating set of G. *The hull domination number*$rh(G)$ of G is the minimum cardinality of a hull dominating set of G. Any hull dominating set of G with cardinality $rh(G)$ is called a $rh-$set of G.These concepts are studied in [2 to 10].Throughout the following G denotes a connected graph with at least two vertices.

A recent application of the convex hull is the Onion Technique for linear optimization queries. This method is based on a theorem that a point which maximizes an arbitrary multidimensional weightening function can be found on the convex hull of the data set. The skyline of a dataset can be used to determine various point of a data sets which could optimize an unknown objective in the user's intentions. E.g. users of a booking system may search for hotels which are cheap and close to the beach. The skyline of such a query contains all possible results regardless how the user weights his criteria *beach* and *cost*. The skyline can be determined in a very similar way as the convex hull.

The following theorem is used in the sequel.

Theorem 1.1.[10] *Each extreme vertex of a connected graph G belongs to every hull dominating set of G.*

2. Theforcing hull domination number of a graph

Definition 2.1. Let G be a connected graph and M a minimum hull dominating set of G. A subset $T \subseteq M$ is called a *forcing subset* for M if M is the unique minimum hull dominating set containing T. A forcing subset for M of minimum cardinality is a *minimum forcing subset of M*. The *forcing hull domination number* of M, denoted by $f_{rh}(M)$, is the cardinality of a minimum forcing subset of M. The *forcing hull domination number* of G, denoted by $f_{rh}(G)$, is $f_{rh}(G) = min\{f_{rh}(M)\}$, where the minimum is taken over all minimum hull dominating sets M in G.

Example 2.2. Consider the graph G given in Figure 2.1. The sets $M_1 = \{v_1, v_3, v_4\}, M_2 = \{v_2, v_4, v_5\}, M_3 = \{v_1, v_2, v_4\}$and $M_4 = \{v_1, v_4, v_5\}$ are the only three rh-sets of G such that $f_{rh}(M_1) = 1, f_{rh}(M_2) = 2, f_{rh}(M_3) = 2$and $f_{rh}(M_4) = 2$ so that$f_{rh}(G) = 1$.

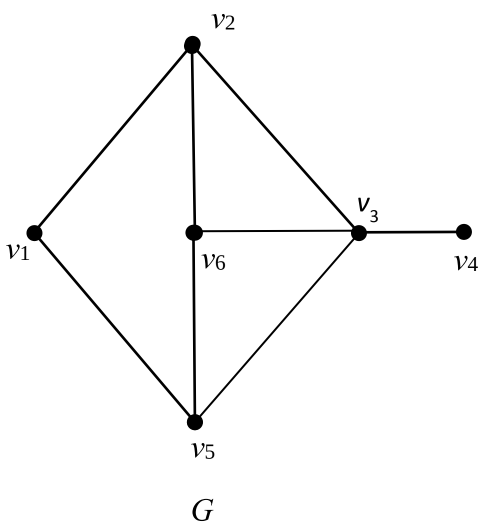

Figure 2.1

The next theorem follows immediately from the definitions of the hull domination number of a connected graph G.

Theorem 2.3. For every connected graph$G$, $0 \le f_{rh}(G) \le rh(G)$.

The following theorems characterizes graphs for which the bounds in Theorem 2.3 are attained and also graphs for which $f_{rh}(G) = 1$.

Definition 2.4.A vertex v of a graph G is said to be a *hull dominating vertex* if v belongs to every rh-set of G.

Theorem 2.5. Let G be a connected graph. Then

a) $f_{rh}(G) = 0$ if and only if G has a unique rh −set.

b) $f_{rh}(G) = 1$ if and only if G has at least two rh-sets, one of which is a unique rh-set containing one of its elements, and

c) $f_{rh}(G) = rh(G)$if and only if no rh-set of G is the unique rh-set containing any of its proper subsets.

Proof. (a)Let $f_{rh}(G) = 0$. Then by definition, $f_{rh}(S) = 0$ for some rh-set$S$ of G so that the empty set ϕ is the minimum forcing subset for S. Since the empty set ϕ is a subset of every set, it follows that S is the unique rh-setof G. The converse is clear.

(b)Let $f_{rh}(G) = 1$. Then by Theorem 2.50(a), G has at least two rh-sets. Also, since $f_{rh}(G) = 1$, there is a singleton subset T of a rh-set$S$ of G such that T is not a subset of any other rh-setof G. Thus S is the unique rh-set containing one of its elements. The converse is clear.

(c) Let $f_{rh}(G) = rh(G)$.Then $f_{rh}(S) = rh(G)$for every rh-set$S$in G. Also, by Theorem 2.34,$rh(G) \ge 2$ and hence $f_{rh}(G) \ge 2$. Then by Theorem 2.50(a), G has at least two rh-setsand so the empty set ϕ is not a forcing subset for any rh-setof G. Since $f_{rh}(S) = rh(G)$,no proper subset of S is a forcing subset of S. Thus no rh-set of G is the unique rh-setcontaining any of its proper subsets. Conversely, the data implies that G contains more than one rh-setand no subset of any rh-set$S$ other than S is a forcing subset for S. Hence it follows that $f_{rh}(G) = rh(G)$. ■

Theorem 2.6. Let$G$ be a connected graph and let $\mathfrak{F}$ be the set of relative complementsof the minimum forcing subsets in their respective connected rh-sets in$G$. Then$\bigcap_{F \in \mathfrak{F}} F$is the set of all connected hull vertices of G.

Proof. Let W be the set of all connected hull vertices of G. We are to show that $W = \bigcap_{F \in \mathfrak{F}} F$. Let $v \in W$. Then v is a connected hull vertex of G that belongs to every rh-set$S$of G. Let $T \subseteq S$ be any minimum forcing subset for any rh-set$S$of G. We claim that $v \notin T$. If $v \in T$, then $T' = T - \{v\}$ is a proper subset of T such that S is the unique rh-setcontaining T' so that $T'$is a forcing subset for S with $|T'| < |T|$ which is a contradiction to T is a minimum forcing subset for S. Thus $v \notin T$ and so $v \in F$, where F is the relative complement of T in S. Hence $v \in \bigcap_{F \in \mathfrak{F}} F$so that $W \subseteq \bigcap_{F \in \mathfrak{F}} F$.

Conversely, let $v \in \bigcap_{F \in \mathfrak{F}} F$. Then v belongs to the relative complement of T in S for every T and every S such that $T \subseteq S$, where T is a minimum forcing subset for S. Since F is the relative complement of Tin S, we have $F \subseteq S$ and thus $v \in S$

for every S, which implies that v is a hull dominating vertex of G. Thus $v \in W$ and so $\bigcap_{F\in\mathfrak{F}} F \subseteq W$. Hence $W = \bigcap_{F\in\mathfrak{F}} F$. ■

Corollary 2.7. Let G be a connected graph and S a rh-set of G. Then noconnected hull vertex of G belongs to any minimum forcing subset of S.

Proof. The proof is contained in the proof of the first part of Theorem 2.50(a). ■

Theorem 2.8. Let G be a connected graph and W be the set of all hull dominating vertices of G. Then$f_{rh}(G) \le rh(G) - |W|$.

Proof. Let S be any rh-set of G.Then $rh(G) = |S|$, $W \subseteq S$and $W$is the unique ■

rh-set containing $S - W$. Thus$f_{rh}(G) \le |S - W| = |S| - |W| = rh(G) - |W|$.

Corollary 2.9. If $G$is a connected graph with k extreme vertices,then $f_{rh}(G) \le rh(G) - k$.

Proof. This follows from Theorems 2.4 and 2.31. ■

Theorem 2.10. For any complete graph $G = K_p (p \ge 2)$ or any non-trivial tree $G = T, f_{rh}(G) = 0$.

Proof. For the complete graph$G = K_p$, it follows from Theorem 1.1 that the set of all vertices of G is the unique hull dominating set of G. Hence it follows from Theorem 2.5(a) that $f_{rh}(G) = 0$. For any non-trivial tree G, the hull domination number $rh(G)$ equals the number of end vertices in G. In fact, the set of all end vertices of G is the unique rh −set of G and so $f_{rh}(G) = 0$ by Theorem 2.5(a). ■

Theorem 2.11. For a complete bipartite graph $G = K_{r,s}$

$$f_{rh}(G) = \begin{cases} 0 \ ; r = 1, \quad s \ge 2 \\ 1 \ ; r = 2, s \ge 2 \\ 2 \ ; 3 \le r \le s \end{cases}$$

Proof. If $r = 1,\ s \ge 2$, the result follows from Theorem 2.10. For $r = 2, s \ge 2$, let $U = \{u_1, u_2\}$ and $V = \{v_1, v_2, \dots, v_s\}$ be a bipartition of G. Then $M = \{u_1, u_2\}$ is a mh-set of G. It is clear that M is the only rh-set containing u_1 so that $f_{rh}(G) = 1$. For $3 \le r \le s$, let $U = \{u_1, u_2, \dots, u_r\}$ and $V = \{v_1, v_2, \dots, v_s\}$ be a bipartition of G. Let $M_1 = \{u, v\}$. Suppose that $u$and$v$ are adjacent. Then $u\{u, v\}$ is not a hull dominating set of G. Therefore$u$ and v are independent. It is clear that M_1 is a hull dominating set of G. so that $rh(G) = 2$ and by Theorem 2.3, $0 \le f_{rh}(G) \le 2$. Suppose $0 \le f_{rh}(G) \le 1$. Since $rh(G) = 2$and the rh −set of G is not unique, by Theorem 2.5 (b),$f_{rh}(G) = 1$. Let $S = \{u, v\}$be a rh-set of G. Let us assume that$f_{rh}(G) = 1$. By Theorem 2.5(b), S is the only rh-set containing $u$or v. Let us assume that S is the only mh-set containing u. Then $r = 2$, which is a contradiction to $r \ge 2$. Therefore$f_{rh}(G) = 2$. ■

In view of Theorem 2.3, we have the following realization result.

Theorem 2.12. For every pair a, b of integers with $0 \le a < b$, $b \ge 2$, there exists a connected graph Gsuch that $f_{rh}(G) = a$ and $rh(G) = b$.

Proof. If $a = 0$, let $G = K_b$. Then by Theorem 1.1, $rh(G) = b$and by Theorem 2.10, $f_{rh}(G) = 0$. For$a \ge 1$, Let $C_i: u_i, v_i, w_i, x_i, u_i; (1 \le i \le a)$ be a copy of the cycle C_4. Let Q_i be the graph obtained from C_i by joining the vertex $v_i$with $x_i (1 \le i \le a)$. Let G be the graph obtained from $Q_i (1 \le i \le a)$ by adding new vertices $x, z_1, z_2, \dots, z_{b-a}$and joining x with each $u_i, w_i$and$z_i (1 \le i \le a)$The graph G is shown in Figure 2.2.Let $Z = \{z_1, z_2, \dots, z_{b-a}\}$ be the set of end-vertices of G. By Theorem 1.1,Z is a subset of every hull dominating set of G. For $1 \le i \le a$,let $F_i = \{v_i, x_i\}$. We observe that every rh −set of G must contain at least one vertex from each F_i and so $rh(G) \ge b - a + a = b$. Now $M_1 = Z \cup \{x, x_1, x_2, x_3, \dots x_a\}$ is a hull dominating set of G so that $rh(G) \le b - a + a = b$. Thus $rh(G) = b$. Next we show that $f_{rh}(G) = a$. Since every rh-set contains Z, it follows from Theorem 2.8 that $f_{rh}(G) \le h(G) - |Z| = b - (b - a) = a$. Now, since $rh(G) = b$ and every rh −set of G contains Z, it is easily seen that every rh-set M is of the form $Z \cup \{x\} \cup \{d_1, d_2, d_3, \dots d_a\}$, where $d_i \in F_i (1 \le i \le a)$. Let T be any proper subset of $M$with $|T| < a$. Then it is clear that there exists some j such that $T \cap H_j = \Phi$, which shows that $f_{rh}(G) = a$. ■

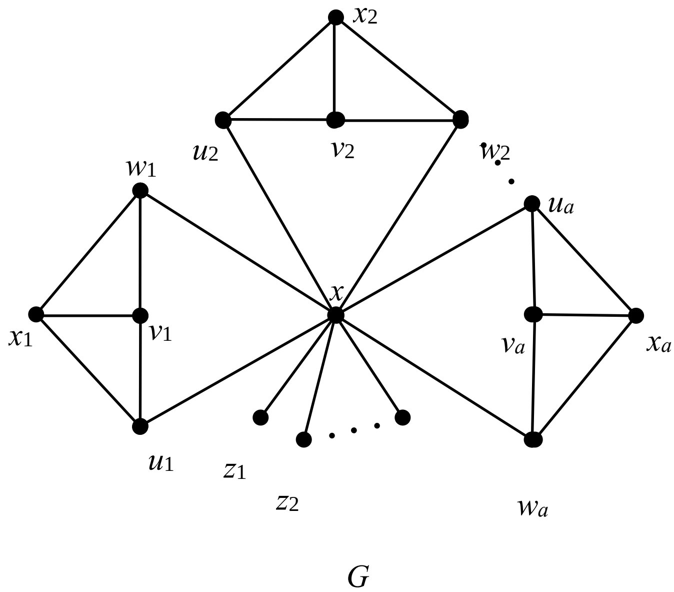

Figure 2.2

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