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Radio Number of Transformation Graphs of a Cycle

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Abstract

The concept of radio labeling motivated by the channel assignment problem. A radio labeling of a connected graph G is an injection $f : V(G) \to Z^+$ such that $|f(u) - f(v)| + d(u, v) \geq 1 + diam(G)$ for every pair of vertices $u, v \in V(G)$. The radio number $rn(f)$ of a radio labeling f of G is the maximum label assigned to a vertex of G. The *radio number* of G denoted by $rn(G)$ is $min\{rn(f)\}$ taken over all radio labeling f of G. In this paper we completely determine radio number of transformation graphs of a cycle.

MSC: 05C12, 05C15, 05C78.

Key words: transformation graphs, radio labeling, radio number, radio graceful

1. Introduction

All the graphs considered here are finite, simple, nontrivial, connected, and undirected. Let $G(V, E)$ be a graph on n vertices. The distance between any two vertices u, $v \in V$, denoted by $d_G(u, v)$ or simply $d(u, v)$, is the length of a shortest path between u and v. If $d_G(x, y) = p$, then the pair x, y is called a k_p-pair of G. The maximum distance between any two vertices in G is the *diameter* of G and is denoted by $diam(G)$. For the terms not defined here we refer to [1, 7].

Radio labeling introduced by W. K. Hale et.al in [8], and studied by others in [3, 4, 5, 9, 10, 11, 12, 14, 15, 16, 20]. For the similar work we refer to [13] and for entire survey work on radio labeling we refer [6]. A labeling of a connected graph G is an injection $f : V \to Z^+$, while a radio labeling is a labeling with an additional condition that $|f(u) - f(v)| + d(u, v) \geq diam(G) + 1$, for all $u, v \in V$. The radio number $rn(f)$ of a radio labeling f of G is the maximum label assigned to a vertex of G. The *radio number* of G denoted by $rn(G)$ is $min\{rn(f)\}$ taken over all radio labeling f of G. A radio labeling f of G is a *minimal radio labeling* of G if $rn(f) = rn(G)$. It is easy to see that the span of an radio-labeling f of a graph G is minimum if the label starts with the integer 1. Further for any radio labeling f of G if two vertices u and v are adjacent in a graph G, then $|f(u) - f(v)| \geq diam(G)$, also $|f(u) - f(v)| = 1$ only if u and v are diametrically opposite vertices.

The transformation graph is first introduced by B. Wu and Meng in [17] and a particular case is studied by Baoyindureng Wu, Li Zhang, and Zhao Zhang [18] and Chandarkala, K. Manjula K, and B. Sooryanaryana in [2]. For each triplet xyz, where $x, y, z \in \{+, -\}$ and a graph G, the transformation graph of G, denoted by G^{xyz}, is the graph on $V \cup E$ such that two vertices u, v are adjacent in G^{xyz} if and only if one of the following hold:

(1) $x = +$ and, u and v are adjacent vertices in G.
(2) $x = -$ and, u and v are non adjacent vertices in G.
(3) $y = +$ and, u and v are adjacent edges in G.
(4) $y = -$ and, u and v are non adjacent edges in G.
(5) $z = +$ and, u and v are incident pair in G.
(6) $z = -$ and, u and v are non incident pair in G.

For every non-trivial graph G, there are eight transformation graphs exist of which $G^{+++} = T(G)$, the total graph of G, and $G^{---} = \overline{T(G)}$. The radio numbers of all the transformation graphs of a path (i.e. $rn(P_n^{xyz})$) are determined by the authors in [19].

We now recall the following theorems [11] for immediate reference;

Theorem 1.1 (D.D.F. Liu and Xuding Zhu[11]). *Let C_n be the n-vertex cycle, $n \geq 3$. Then*

$$rn(C_n) = \begin{cases} \frac{n-2}{2}\phi(n) + 2, & if \quad n \equiv 0, 2(mod\, 4); \\ \frac{n-1}{2}\phi(n) + 1, & if \quad n \equiv 1, 3(mod\, 4) \end{cases}$$

where,

$$\phi(n) = \begin{cases} \lfloor \frac{n}{4} \rfloor + 1, & if\, n \equiv 1(mod\, 4); \\ \lfloor \frac{n}{4} \rfloor + 2, & otherwise. \end{cases}$$

Theorem 1.2 (D.D.F. Liu and M Xie[9])**.** *Let C_n^2 be an n-vertex square cycle where n is even. Then, $rn(C_n^2)$ is given by*

$$\begin{cases} \frac{2k^2+5k-1}{2}, & if \quad n = 4k \text{ and } k \text{ is odd;} \\ \frac{2k^2+3k}{2}, & if \quad n = 4k \text{ and } k \text{ is even;} \\ k^2+5k+1, & if \quad n = 4k+2 \text{ and } k \text{ is odd;} \\ k^2+4k+1, & if \quad n = 4k+2 \text{ and } k \text{ is even.} \end{cases}$$

Let $G(V,E)$ be a connected graph on n vertices with diameter d. Let $f : V \rightarrow \mathbb{Z}^+$ be an injective function. Then for any subgraph H of G, by the f-sequence of H, we mean a sequence or an arrangement $x_1, x_2, \ldots, x_{|H|}$ of the vertices of H such that $f(x_i) \leq f(x_j)$ whenever $i \leq j$. Further, if f is a radio labeling of G, then $f(x_n) - f(x_1) = \sum_{i=1}^{n-1}[f(x_{i+1}) - f(x_i)] = \sum_{i=1}^{n-1}[(d+1) - d(x_{i+1}, x_i)] = (n-1)(d+1) - \sum_{i=1}^{n-1} d(x_{i+1}, x_i)$. Thus, as $f(x_1) \geq 1$, we have;

$$f(x_n) \geq 1 + (|V(H)| - 1)(d+1) - \sum_{i=1}^{|V(H)|-1} d(x_{i+1}, x_i) \quad (1)$$

The right hand side of the inequality (1) varies only with the sum $S = \sum_{i=1}^{|V(H)|-1} d(x_{i+1}, x_i)$ for every subgraph H of G. Therefore, the minimum value of $f(x_n)$ among all the radio labeling f of G completely depends on the value of S and is minimum whenever S is as much as maximum. Therefore, problem of finding the radio number of G turns out to an integer programming problem:

$$Maximize\ S = \sum_{i=1}^{diam(G)} \alpha_i k_i \quad (2)$$

Subjected to $k_i = d(x_{i+1}, x_i)$; $\sum_{i=1}^{diam(G)} \alpha_i = |V(G)| - 1$; $\alpha_i \leq |\{\{x,y\} : d_G(x,y) = k_i\}|$; $\alpha_i \in \mathbb{Z}^+ \cup \{0\}$. Here for each i, α_i denotes the number of choices of the k_i pairs.

Remark 1.1. *From the definition of an radio labeling it is clear that every radio labeling is one-one and hence for every radio labeling f of a graph G it follows that $rn(G) \geq |V(G)|$.*

Remark 1.2. *Let $G = H = C_n^{xyz}$ and $Max\ S = \sum_{i=1}^{diam(G)} \alpha_i k_i$. Now, with $d = 3$, from Equation 1 we get,*

$$f(x_{2n}) \geq 1 + (2n-1)(3+1) - Max\ S = 8n - 3 - Max\ S \quad (3)$$

and with $d = 2$, from Equation 1 we get,

$$f(x_{2n}) \geq 1 + (2n-1)(2+1) - Max\ S = 6n - 2 - Max\ S \quad (4)$$

Remark 1.3. *For any $n \in Z^+$, $T(C_n) = C_{2n}^2$.*

Observation 1.3. *For any $n(\geq 3) \in Z^+$,*

1. $diam(C_n^{+-+}) = \begin{cases} 2, & if \quad n = 3,4,5 \\ 3, & if \quad n \geq 6 \end{cases}$

2. $diam(C_n^{-++}) = \begin{cases} 2, & if \quad n = 3,4,5 \\ 3, & if \quad n \geq 6 \end{cases}$

3. $diam(C_n^{++-}) = diam(C_n^{---}) = 2$

4. $diam(C_n^{+--}) = \begin{cases} 3, & if \quad n = 3 \\ 2, & if \quad n \geq 4 \end{cases}$

5. $diam(C_n^{-+-}) = \begin{cases} 3, & if \quad n = 3 \\ 2, & if \quad n \geq 4 \end{cases}$

6. $diam(C_n^{---}) = \begin{cases} 3, & if \quad n = 3 \\ 2, & if \quad n \geq 4 \end{cases}$

We now completely determine radio number of all transformation graphs of cycle except the total graph which is covered by Theorem 1.2 and Remark 1.3. In the next sections of this paper, we prove the following theorems which computes the actual minimum span of radio labeling for each transformation graphs of cycle.

Theorem 1.4. *For any positive integer $n \geq 2$, and $xyz \in \{+-+, -++\}$,*

$$rn(C_n^{xyz}) = \begin{cases} 7, & if \quad n = 3 \\ 2n, & if \quad n = 4,5 \\ 3n+2, & if \quad n = 6 \\ 3(n+1), & if \quad n = 7 \\ 3n, & if \quad n \geq 8 \end{cases}$$

Theorem 1.5. *For any positive integer $n \geq 3$, $rn(C_n^{++-}) = 2n$.*

Theorem 1.6. *For any positive integer $n \geq 3$ and $xyz \in \{+--, -+-\}$,*

$$rn(C_n^{xyz}) = \begin{cases} 10, & if \quad n = 3 \\ 2n, & if \quad n \geq 4 \end{cases}$$

Theorem 1.7. *For any positive integer $n \geq 3$, $rn(C_n^{--+}) = \begin{cases} 8, & if \quad n = 3 \\ 2n, & if \quad n \geq 4 \end{cases}$*

Theorem 1.8. *For any positive integer $n \geq 4$, $rn(C_n^{---}) = 2n$.*

Throughout this paper, we label the vertices of the transformation graph C_n^{xyz} as $v_0, v_1, v_2, \ldots, v_{n-1}$ for those are the vertices of the corresponding graph C_n such that v_i is adjacent of $v_{i+1(mod n)}$ for all $i, 0 \leq i \leq n-1$, and $e_0, e_1, e_2, \ldots, e_{n-1}$ for those are the edges of the corresponding graph C_n such that $e_j = v_j v_{j+1(mod n)}$ for each $j, 0 \leq j \leq n-1$.

2. Lower Bound

In this section, we find a lower bound for the span of its radio labeling. Throughout this section, let $G = C_n^{xyz}$, f be an radio labeling of the graph G and $x_1, x_2, \ldots, x_{2n}$ be the rearrangement of the vertices of G such that $f(x_1) = 1$, $f(x_i) < f(x_{i+1})$ for all $i, 1 \leq i \leq 2n-1$.

2.1 For $xyz = +-+$ or $-++$

Now let $xyz = +-+$. The graph $C_n^{+-+} \cong C_n^{-++}$ and hence the proof follows immediately for the case $xyz = -++$.

Lemma 2.1. *For any positive integer* $n \geq 3$*,*

$$rn(C_n^{xyz}) \geq \begin{cases} 7, & if \quad n = 3 \\ 2n, & if \quad n = 4,5 \\ 3n+2, & if \quad n = 6 \\ 3(n+1), & if \quad n = 7 \\ 3n, & if \quad n \geq 8 \end{cases}$$

Proof. We prove the result in different cases as follows,

Case 1: $n = 3$

In this case for at least one i with $0 \leq i \leq 5$, $d_G(x_i, x_{i+1}) < diam(G) = 2$ (because, any vertex with degree four is adjacent all other vertices except one and it is the only option to be diametrically opposite, so that to label with consecutive integers we have to keep them as starting or ending vertex. But in this case we have three vertices with degree four so at least one of them should be labeled in middle)and hence $\alpha_1 \geq 1$ then, $Max\ S = \alpha_1 k_1 + \alpha_2 k_2 \leq 1(1) + 4(2) = 9$. By using Equation 4, we get $rn(G) = f(x_6) \geq 6n - 2 - Max\ S = 6(3) - 2 - 9 = 7$.

Case 2: $n = 4, 5$

Follows immediately by Remark 1.1

Case 3: $n = 6, 7$

If possible, let f assigns three consecutive integers for the vertices v_i, v_j and v_k of G that corresponds to any three vertices of C_n. But then $d_G(v_i, v_k) \geq 2$ (since $|f(v_i) - f(v_k)| = 2$). But, in this case for any such f we get $d_G(v_i, v_k) \leq 1$ (since $diam(G) = diam(C_n)$), a contradiction. Hence there are exactly three k_3 pairs possible if $n = 6$ or $n = 7$. Therefore, $\max S \leq \alpha_1 k_1 + \alpha_2 k_2 + \alpha_3 k_3 = 0 + 8(2) + 3(3) = 25$, if $n = 6$, and $\max S \leq \alpha_1 k_1 + \alpha_2 k_2 + \alpha_3 k_3 = 0 + 10(2) + 3(3) = 29$, if $n = 7$. Thus, from Equation 3, we get, $rn(C_n^{xyz}) = \min\{f(x_{2n})\} \geq 8n - 3 - \max S = 8(6) - 3 - 25 = 20$, if $n = 6$, and $rn(C_n^{xyz}) \geq 8n - 3 - \max S = 8(7) - 3 - 29 = 24$, if $n = 7$. Hence;

$$rn(C_n^{xyz}) = \begin{cases} 3n+2, & if \quad n = 6 \\ 3(n+1), & if \quad n = 7 \end{cases}$$

Case 4: $n \geq 8$

To label these n vertices of G that corresponds to n vertices of C_n, f requires at least n integers. To label an edge of C_n in G after or before label a vertex or an edge of C_n in G, f should leave at least one integer (since $d_G(e_i, e_j) \leq 2$ and $d_G(e_i, v_j) \leq 2$, for all i, j and $diam(G) = 3 > 2$). Thus, all together f requires n integers for n vertices of C_n in G and $2n$ integers for n edges of C_n in G. Hence, span $f \geq n + 2n = 3n \Rightarrow rn(G) = min\{rn(f)\} \geq 3n$. Hence the lemma. □

2.2 For $xyz = ++-$

Lemma 2.2. *For any positive integer* $n \geq 3$*,* $rn(C_n^{xyz}) \geq 2n$*.*

Proof. Follows immediately by Remark 1.1. □

2.3 For $xyz = +--$ or $xyz = -+-$

Now let $xyz = +--$. Since $C_n^{+--} \cong C_n^{-+-}$ the proof follows immediately for $xyz = -+-$.

Lemma 2.3. *For any positive integer* $n \geq 3$*,*

$$rn(C_n^{xyz}) \geq \begin{cases} 10, & if \quad n = 3 \\ 2n, & if \quad n \geq 4 \end{cases}$$

Proof. We first take the case when $n = 3$. Here can have at most two k_3 pairs(since only three vertices e_0, e_1, e_2 are with eccentricity equal to diameter), and must have atleast one k_1 pair (since here three vertices of C_n are mutually adjacent and to avoid successive labeling of these an edge must be labeled in between, But to choose two k_3 pairs, edges must be labeled continuously so one pair vertices of C_n labeled successively). Or to choose one k_3 pair and other all k_2 pairs and we get $\max S = \max\{2k_3 + 2k_2 + k_1, k_3 + 4k_2\} = 11$. By using Equation 3, this yields that; $rn(C_n^{xyz}) = f(x_6) \geq 8n - 3 - \max S = 8(3) - 3 - 11 = 10$. In the second case, when $n \geq 4$, the result follows immediately by Remark 1.1. □

2.4 For $xyz = - - +$

Lemma 2.4. *For any positive integer $n \geq 3$,*

$$rn(C_n^{xyz}) \geq \begin{cases} 8, & if \quad n = 3 \\ 2n, & if \quad n \geq 4 \end{cases}$$

Proof. When $n = 3$, $G \cong C_6$ and $rn(C_n^{xyz}) = rn(C_6) = 8$ (by Theorem 1.1). The case $n \geq 4$ follows immediately by Remark 1.1. □

2.5 For $xyz = - - -$

Lemma 2.5. *For any positive integer $n \geq 4$, $rn(C_n^{xyz}) \geq 2n$.*

Proof. Result follows immediately by Remark 1.1. □

3. Upper Bound and Optimal radio labeling

In this section we actually show the lower limit, established in the previous section, for each of the transformation graphs $G = C_n^{xyz}$, is tight by executing a minimal radio labeling.

3.1 For $xyz = + - +$

Lemma 3.1. *For any positive integer $n \geq 2$,*

$$rn(C_n^{xyz}) \leq \begin{cases} 7, & if \quad n = 3 \\ 2n, & if \quad n = 4, 5 \\ 3n + 2, & if \quad n = 6 \\ 3(n+1), & if \quad n = 7 \\ 3n, & if \quad n \geq 8 \end{cases}$$

Proof. For $n \leq 7$, result follows by the radio labeling shown in Figure 1.

When $n \geq 8$, for each integer l, k; $0 \leq l \leq 2, 0 \leq k \leq \lceil \frac{n}{3} \rceil - 1$, define a function $f : V \to Z^+$ by $f(v_{3k+l})$

$$= \begin{cases} k + \lceil \frac{n}{3} \rceil (l+1) & \text{if } l = 1 \text{ and } n \equiv 1 \text{ (mod 3)} \\ k + \lceil \frac{n}{3} \rceil (l-1) + 1 & \text{if } l = 2 \text{ and } n \equiv 1 \text{ (mod 3)} \\ k + \lceil \frac{n}{3} \rceil l + 1 & \text{otherwise} \end{cases}$$

and

$$f(e_i) = (n+2) + 2i, \ with \ 0 \leq i \leq n-1.$$

Since $diam(G) = 3$, now to show f is a radio labeling, it is sufficient to consider the vertices that are atmost distance two apart. Let u and v be any two vertices of G with $d_G(u, v) \leq 2$.

Case 1: $u, v \in V(C_n)$.

Let $u = v_i$ and $v = v_j$ with $0 \leq i < j \leq n-1$.

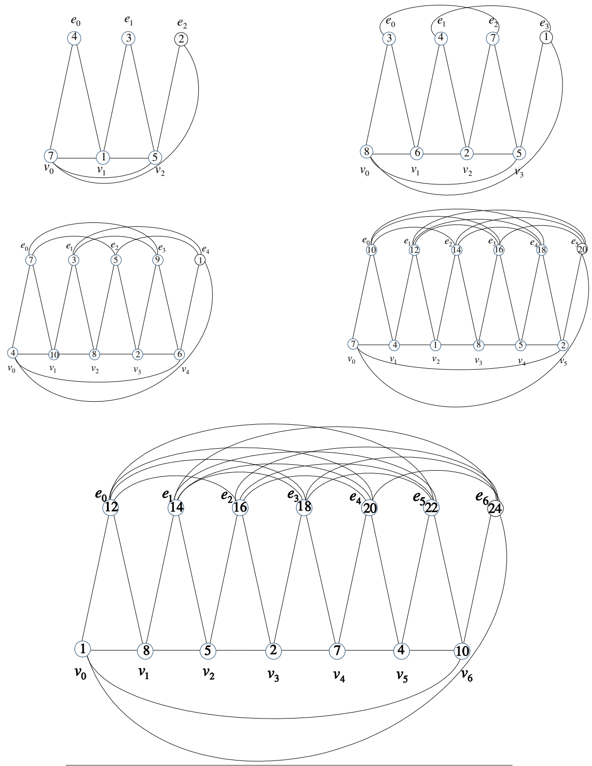

Figure 1: A radio labeling of C_n^{+-+} for $n \leq 7$.

Subcase 1: $i = 3k, j = 3k+1$

In this case, $f(u) = k + 1$ and

$$f(v) = \begin{cases} k + 2\lceil \frac{n}{3} \rceil & \text{if } n \equiv 1 \text{ (mod 3)} \\ k + \lceil \frac{n}{3} \rceil + 1 & \text{otherwise} \end{cases}$$

Therefore,

$|f(u) - f(v)| + d_G(u, v) - 1 \geq 3.$

Subcase 2: $i = 3k+1, j = 3k+2$.

In this case,

$$f(u) = \begin{cases} k + 2\lceil \frac{n}{3} \rceil & \text{if } n \equiv 1 \text{ (mod 3)} \\ k + \lceil \frac{n}{3} \rceil + 1 & \text{otherwise} \end{cases}$$

and

$$f(v) = \begin{cases} k + \lceil \frac{n}{3} \rceil + 1 & \text{if } n \equiv 1 \text{ (mod 3)} \\ k + 2\lceil \frac{n}{3} \rceil + 1 & \text{otherwise} \end{cases}$$

Therefore, $|f(u) - f(v)| + d_G(u, v) - 1 \geq 3$.

Subcase 3: $i = 3k+2, j = 3k+3 = 3(k+1)$.

In this case,

$$f(u) = \begin{cases} k + \lceil \frac{n}{3} \rceil + 1 & \text{if } n \equiv 1 \text{ (mod 3)} \\ k + 2\lceil \frac{n}{3} \rceil + 1 & \text{otherwise} \end{cases}$$

and $f(v) = (k+1)+1$.

Therefore, $|f(u)-f(v)| + d_G(u,v) - 1 \geq 3$.

Subcase 4: $i = 3k, j = 3k+2$.

In this case, $f(u) = k+1$ and

$$f(v) = \begin{cases} k+\lceil\frac{n}{3}\rceil+1 & \text{if } n \equiv 1 \pmod 3 \\ k+2\lceil\frac{n}{3}\rceil+1 & \text{otherwise} \end{cases}$$

Therefore, $|f(u)-f(v)| + d_G(u,v) - 1 \geq 3$.

Subcase 5: $i = 3k+1, j = 3k+3 = 3(k+1)$.

In this case,

$$f(u) = \begin{cases} k+2\lceil\frac{n}{3}\rceil & \text{if } n \equiv 1 \pmod 3 \\ k+\lceil\frac{n}{3}\rceil+1 & \text{otherwise} \end{cases}$$

and $f(v) = (k+1)+1$.

Therefore, $|f(u)-f(v)| + d_G(u,v) - 1 \geq 3$.

Subcase 6: $i = 3k+2, j = 3k+4 = 3(k+1)+1$

In this case,

$$f(u) = \begin{cases} k+\lceil\frac{n}{3}\rceil+1 & \text{if } n \equiv 1 \pmod 3 \\ k+2\lceil\frac{n}{3}\rceil+1 & \text{otherwise} \end{cases}$$

and

$$f(v) = \begin{cases} (k+1)+2\lceil\frac{n}{3}\rceil & \text{if } n \equiv 1 \pmod 3 \\ (k+1)+\lceil\frac{n}{3}\rceil+1 & \text{otherwise} \end{cases}$$

Therefore, $|f(u)-f(v)| + d_G(u,v) - 1 \geq 3$.

Subcase 7: $i = 0, j = n-1 = 3k+l$.

In this case, $f(u) = 1$ and

$$f(v) = \begin{cases} k+1 & \text{if } l = 0 \\ k+\lceil\frac{n}{3}\rceil+1 & \text{if } l = 1 \\ k+\lceil\frac{n}{3}\rceil+1 & \text{if } l = 2 \end{cases}$$

Therefore, $|f(u) - f(v)| + d_G(u,v) - 1 = \begin{cases} k & \text{if } l = 0\ (n \geq 10) \\ k+\lceil\frac{n}{3}\rceil & \text{if } l = 1\ (n \geq 8) \\ k+\lceil\frac{n}{3}\rceil & \text{if } l = 2\ (n \geq 9) \end{cases} \geq 3.$

Subcase 8: $i = 0, j = n-2 = 3k+l$

In this case, $f(u) = 1$ and

$$f(v) = \begin{cases} k+1 & \text{if } l = 0 \\ k+\lceil\frac{n}{3}\rceil+1 & \text{if } l = 1 \\ k+\lceil\frac{n}{3}\rceil+1 & \text{if } l = 2 \end{cases}$$

Therefore, $|f(u) - f(v)| + d_G(u,v) - 1 = \begin{cases} (k)+1 & \text{if } l = 0\ (n \geq 8) \\ (k+\lceil\frac{n}{3}\rceil)+1 & \text{if } l = 1\ (n \geq 9) \\ (k+\lceil\frac{n}{3}\rceil)+1 & \text{if } l = 2\ (n \geq 10) \end{cases} \geq 3.$

Subcase 9: $i = 1, j = n-1 = 3k+l$.

In this case,

$$f(u) = \begin{cases} 2\lceil\frac{n}{3}\rceil & \text{if } n \equiv 1 \pmod 3 \\ \lceil\frac{n}{3}\rceil+1 & \text{otherwise} \end{cases}$$

and

$$f(v) = \begin{cases} k+1 & \text{if } l = 0 \\ k+\lceil\frac{n}{3}\rceil+1 & \text{if } l = 1 \\ k+2\lceil\frac{n}{3}\rceil+1 & \text{if } l = 2 \end{cases}$$

Therefore, $|f(u) - f(v)| + d_G(u,v) - 1 = \begin{cases} |k+1-2\lceil\frac{n}{3}\rceil|+1 & \text{if } l = 0\ (n \geq 10) \\ (k)+1 & \text{if } l = 1\ (n \geq 8) \\ (k+\lceil\frac{n}{3}\rceil+1)+1 & \text{if } l = 2\ (n \geq 9) \end{cases} \geq 3.$

Case 2: $u, v \in E(C_n)$.

Let $u = e_i$ and $v = e_j$ with $0 \leq i < j \leq n-2$. Then $|f(u) - f(v)| = |f(e_i) - f(e_j)| = 2(j-i)$ is equal to 2 if $d_G(u,v) = 2$ (i.e $j = i+1$) and, at least 4 if $d_G(u,v) = 1$ (i.e $j > i+1$). In each of these cases $|f(u)-f(v)|+d_G(u,v)-1 \geq 4 > diam(G)$.

Case 3: $u \in V(C_n)$ and $v \in E(C_n)$.

Let $u = v_{3k+l}$ with $0 \leq l \leq 2$, $0 \leq k \leq \lceil\frac{n}{3}\rceil - 1$ and $v = e_j$ with $0 \leq i < j \leq n-1$.

Subcase 1: $j = (3k+l)-1$.

In this case, $d_G(u,v) = 1$, Therefore

a) when $l = 1$ and $n \equiv 1 \pmod 3$

$|f(u)-f(v)|+d_G(u,v)-1 = f(e_{(3k+l)-1})-f(v_{3k+l}) = (n+2)+2(3k+l-1)-k-\lceil\frac{n}{3}\rceil\times(l+1) = n+2-2\lceil\frac{n}{3}\rceil > 3$ for all $n \geq 10$

b) when $l = 2$ and $n \equiv 1 \pmod 3$.

$|f(u)-f(v)|+d_G(u,v)-1 = f(e_{(3k+l)-1})-f(v_{3k+l} = (n+2)+2(3k+l-1)-k-\lceil\frac{n}{3}\rceil\times(l-1)-1 = n-\lceil\frac{n}{3}\rceil+3 > 3$ for all $n \geq 10$

c) when $l \neq 1, 2$ or $n \not\equiv 1 \pmod 3$.

$|f(u)-f(v)|+d_G(u,v)-1 = f(e_{(3k+l)-1})-f(v_{3k+l}) = (n+2)+2(3k+l-1)-k-\lceil\frac{n}{3}\rceil\times l-1 = n-2\lceil\frac{n}{3}\rceil+3 > 3.$

Subcase 2: $j = (3k+l)$.

In this case, $d_G(u,v) = 1$ and $|f(u) - f(v)| + d_G(u,v) - 1 = f(e_{3k+l}) - f(v_{3k+l}) = (n+2) + 2(3k+l) - f(v_{3k+l}) = (n+2) + 2(3k+l-1) - f(v_{3k+l}) + 2 = f(e_{(3k+l)-1}) - f(v_{3k+l}) + 2 > 3 + 2 > 3$ (by subcase1 of Case3).

Subcase 3: $j \notin \{(3k+l)-1, (3k+l)\}$

In this case, $d_G(u,v) = 2$ and hence it suffices to show $|f(u) - f(v)| > 1$. In fact $|f(u) - f(v)| = f(v) - f(u) = (n+2) + 2j - f(u) \geq (n+2) + 2j - n = 2(j+1) > 1$.

Thus, from all the above subcases, f is an radio labeling and $span f = 3n$. Therefore $rn(G) \leq 3n$ for all $n \geq 8$. □

3.2 For $xyz = -++$

Remark 3.1. *As $C_n^{+-+} \cong C_n^{-++}$ result follows immediately by Lemma 3.1 for $xyz = -++$.*

3.3 For $xyz = ++-$

Lemma 3.2. *For any positive integer $n \geq 3$, $rn(C_n^{xyz}) \leq 2n$.*

Proof. Consider a function $f : V \to Z^+$ defined by $f(v_i) = 2i+1,\ with\ 0 \leq i \leq n-1$, and $f(e_i) = 2(i+1),\ with\ 0 \leq i \leq n-1$. Since $diam(G) = 2$, now to show f is a radio labeling it is sufficient to consider the vertices that are adjacent. Let u and v be any two vertices of G with $d_G(u,v) = 1$.

Case 1: $u, v \in V(C_n)$.

Let $u = v_i$ and $v = v_j$ with $0 \leq i < j \leq n-1$. Then $|f(u) - f(v)| = |f(v_i) - f(v_j)| = 2(j-i)$ is at least 2 if $d_G(u,v) = 1$. In this case $|f(u) - f(v)| + d_G(u,v) - 1 \geq 2$.

Case 2: $u, v \in E(C_n)$.

Let $u = e_i$ and $v = e_j$ with $0 \leq i < j \leq n-1$. Then $|f(u) - f(v)| = |f(e_i) - f(e_j)| = 2(j-i)$ is at least 2 if $d_G(u,v) = 1$. In this case $|f(u) - f(v)| + d_G(u,v) - 1 \geq 2$.

Case 3: $u \in V(C_n)$ and $v \in E(C_n)$.

Let $u = v_i$ with $0 \leq i \leq n-1$ and $v = e_j$ with $0 \leq j \leq n-1$. Then $|f(u) - f(v)| = |f(v_i) - f(e_j)| = |2(i-j) - 1|$ is at least 3 if $d_G(u,v) = 1$ (i.e $|i-j| \geq 2$). In this case $|f(u) - f(v)| + d_G(u,v) - 1 \geq 3$.

Thus, from all the above subcases, f is an radio labeling and $span f = 2n$. Therefore $rn(G) \leq 2n$ □

3.4 For $xyz = +--$

Lemma 3.3. *For any positive integer $n \geq 3$,*

$$rn(C_n^{xyz}) \leq \begin{cases} 10, & if \quad n = 3 \\ 2n, & if \quad n \geq 4 \end{cases}$$

Proof. For $n = 3$, result follows by the radio labeling shown in Figure 2.

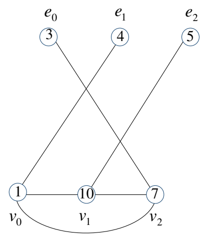

Figure 2: An radio labeling of C_3^{+--}

For $n \geq 4$, proof is similar to that of Lemma 3.2. □

3.5 For $xyz = -+-$

Remark 3.2. *As $C_n^{+--} \cong C_n^{-+-}$ proof follows immediately by Lemma3.3 for $xyz = -+-$.*

3.6 For $xyz = --+$

Lemma 3.4. *For any positive integer $n \geq 3$,*

$$rn(C_n^{xyz}) \leq \begin{cases} 8, & if \quad n = 3 \\ 2n, & if \quad n \geq 4 \end{cases}$$

Proof. For $n = 3$, result follows by the radio labeling shown in Figure 3.

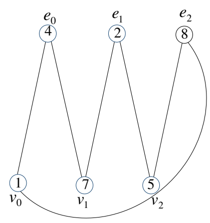

Figure 3: An radio labeling of C_3^{--+}

For $n \geq 4$, define a function $f : V \to Z^+$ by $f(v_i) = i + 1,\ with\ 0 \leq i \leq n-1$; $f(e_i) = (n+1)+i,\ with\ 0 \leq i \leq n-1$. Since $diam(G) = 2$, now to show f is a radio labeling it is sufficient to consider the vertices that are adjacent. Let u and v be any two vertices of G with $d_G(u,v) = 1$.

Case 1: $u, v \in V(C_n)$.

Let $u = v_i$ and $v = v_j$ with $0 \leq i < j \leq n-1$. Then $|f(u) - f(v)| = |f(v_i) - f(v_j)| = (j-i)$ is at least 2 if $d_G(u,v) = 1$ (i.e $j \geq i+2$). In this case $|f(u) - f(v)| + d_G(u,v) - 1 \geq 2$.

Case 2: $u, v \in E(C_n)$.

Let $u = e_i$ and $v = e_j$ with $0 \leq i < j \leq n-2$. Then $|f(u) - f(v)| = |f(e_i) - f(e_j)| = (j-i)$ is at least 2 if $d_G(u,v) = 1$ (i.e $j \geq i+2$). In this case $|f(u) - f(v)| + d_G(u,v) - 1 \geq 2$.

Case 3: $u \in V(C_n)$ and $v \in E(C_n)$.

Let $u = v_i$ with $0 \leq i \leq n-1$ and $v = e_j$ with $0 \leq j \leq n-1$.

Subcase 1: $i = j$.

In this case $|f(u) - f(v)| = |f(v_i) - f(e_j)| = |i - j - n|$ is at least 4 (since $n \geq 4$). Therefore, $|f(u) - f(v)| + d_G(u,v) - 1 \geq 3$.

Subcase 2: $i = j+1$.

In this case $|f(u) - f(v)| = |f(v_i) - f(e_j)| = |i - j - n|$ is at least 3 (since $n \geq 4$). Therefore, $|f(u) - f(v)| + d_G(u,v) - 1 \geq 3$.

Subcase 3: $i = 0$ and $j = n-1$.

In this case $|f(u) - f(v)| = |f(v_i) - f(e_j)| = |0 + 1 - 2n|$ is at least 7 (since $n \geq 4$). Therefore, $|f(u) - f(v)| + d_G(u,v) - 1 \geq 3$.

Thus, from all the above subcases, f is an radio labeling and $span f = 2n$.

Therefore $rn(G) \leq 2n$ for all $n \geq 4$. □

3.7 For $xyz = ---$

Lemma 3.5. *For any positive integer $n \geq 4$, $rn(C_n^{xyz}) \leq 2n$.*

Proof. Proof is similar to that of Lemma 3.2. □

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