International Journal of Computational and Applied Mathematics.
ISSN 1819-4966 Volume 11, Number 1 (2016), pp. 1-8

http://www.ripublication.com

A Study of Green's Relations on Fuzzy Semigroup

Dr.G Hariprakash

Principal SNDP Yogam Arts &Science College
Pulpally 673579, Wayanad (Dist) Kerala State, South India.
hariprakash.g.saji@gmail.com

Abstract

In the field of research, the study of fuzzy relations assumed greater relevance, as the fuzzy relations are applicable in almost all disciplines of study and all varieties of activities. The intensive development of the subject has been continuing for the last twenty years. In any factual and vivid analytical study, it is highly necessary to represent the qualitative characteristics as quantitative. Only then we get the precise and accurate result.
Green introduced five fundamental equivalence relations on semigroup(crisp set). In this paper these relations are modified and extended to fuzzy semigroups.
In particular, the study deliberates on Green's relations on fuzzy semigroup and find the need to define fuzzy compatibility and there by fuzzy congruences.
Moreover the well defined composition of fuzzy relation is noted here as dot(.) composition of fuzzy relation, to distinguish a composition of fuzzy relation which is being newly defined and introduced in this paper noted as cross (x) composition of fuzzy relation.
The study approaches the subject by assuming a real value (between 0 and 1) to each relations defined on the fuzzy semigroup. In particular, using membership functions an ideal fuzzy relation and thereby Green's $\mathcal{L}$ and $\mathcal{R}$ relations are also being newly defined and introduced. Using the general concept 'Join' of $\mathcal{L}$ and $\mathcal{R}$; Green's $\mathcal{D}$-relation and 'meet' of $\mathcal{L}$ and $\mathcal{R}$; Green's $\mathcal{H}$-relation are find out.
This paper concludes establishing two theorems connecting
(1) Green's $\mathcal{L}$, $\mathcal{R}$ relations and Green's $\mathcal{D}$-relation,
(2) Green's $\mathcal{L}$, $\mathcal{R}$ relations and Green's $\mathcal{H}$-relation.

Key words: Fuzzy semigroup, dot composition, fuzzy relation, similarity relation, fuzzy compatible, fuzzy congruence, ideal fuzzy relation, Green's

relations, Green's $\mathcal{L}$- relations, Green's $\mathcal{R}$-relations, Green's $\mathcal{D}$- relations, Green's $\mathcal{H}$-relation, fuzzy inverse semigroup.

List of symbols:

α, β ---- fuzzy relation
S ---- Semigroup
A ---- ideal fuzzy relation
$\mathcal{D}$ ----Green's $\mathcal{D}$ relation
$\mathcal{H}$----- Green's $\mathcal{H}$ relation
$\mathcal{R}$----- Green's $\mathcal{R}$ relation
≤ ----less than or equal to
⪯----Partial order relation

Introduction:

In logical algebra fuzzy sets were introduced by Zadeh. Rosenfeld applied it to the elementary theory of groups. He used the concept of characteristics function to initiate the fuzzification of some algebraic structures. Many authors have presented various kinds of fuzzy semigroups. Green's relations are some of the most important notions in this study. This paper introduces the concept of Green's relations on fuzzy semigroup.

Definition 1.1. A semigroup S in which every element has a weight (a membership function is defined)is called a fuzzy semigroup[1].

Definition 1.2. Let S be a fuzzy semigroup. A function α from SxS to the unit interval [0, 1] α is called a fuzzy binary relation on S.

Definition 1.3. Let α and β be two fuzzy binary relations on a fuzzy semigroup S.A Composition α and β is denoted by α o β and read as α dot β is defined as
$\alpha \circ \beta(a, b) = \max_{z \in S} \min\{\alpha(a, z), \beta(z, b)\}\ for\ all\ a, b \in S.$

Definition 1.4. Let S be a fuzzy semigroup. A fuzzy binary relation α on S is called a similarity relation.
If
(i) α(x, x)=1 for all x∈S(reflexive)
(ii) α (x, y)= α (y, x) for all x, y∈S(symmetric)
(iii) $\alpha(x, y) \geq \max_{z \in S} \min\{\alpha(x, z), \alpha(z, y)\}$.
Or α ≥ αoα [2]

Definition 1.5. Let S be a fuzzy semigroup. A fuzzy binary relation α on S is called fuzzy left compatible if α (x, y)≤ α (tx, y).and α (x, y)≤ α (x, ty) for all x, y, t ∈S.

α is fuzzy right compatible on S if α (x, y)≤ α (xt, y) and α (x, y)≤ α (x, yt) for all x, y, t ∈S.

Definition 1.6 A fuzzy binary relation α on a fuzzy semigroup is compatible if it is both fuzzy left compatible and fuzzy right compatible.

Remark 1.7. If α is fuzzy compatible relation on a semigroup S then
Max{ α (a, b), α (c, d)}= α (ac, bd).Converse is true when S is a monoid.

Definition 1.8 A fuzzy compatible similarity relation on a fuzzy semigroup is called a fuzzy congruence relation on S. [2]

Definition 1.9 A fuzzy binary relation α on a fuzzy semigroup S is called an ideal fuzzy relation, if
α ={(x, y) ∈ $SxS: A(x) = A(y)$}
And α (x, y)= $\max_{t\in S}\{min\ [A(x), A(t)], min[A(t), A(y)]\}$
Where A is a membership function on S.

Proposition 1.10.An ideal fuzzy relation α on a fuzzy semigroup is reflexive, if and only if A(x)=1 for all x∈ S.

Proof We have α as reflexive
iff α (x, x)=1 $\forall x \in S$, that is
iff $\max_{t\in S}\{min\ [A(x), A(t)], min[A(t), A(x)]\}$=1, that is
iff A(x)=1 $\forall x \in S$.
Hence the result.

Proposition 1.11. If an ideal fuzzy relation α on a fuzzy semigroup S is reflexive, then it is a similarity relation.

2 Green's Relations on a Fuzzy Semigroup

Green introduced five fundamental equivalence relations on semigroup.With the help of Green's relations a semigroup structure can be described ([3], [4], [5])
By introducing fuzziness property on semigroup, Green's relations can be redefined as follow:

Definition 2.1. Let S be a fuzzy semigroup with membership function A defined on it.Green's £-relation on S is defined by
£={(a, b)∈ $S\ x\ S: S^1a = S^1b$ and A(xa)=A(xb) ∀ x∈ S}
£(a, b)= $\max_{z,w\in S} min\ \{min[A(za), A(wb)\}$
Green's $\mathcal{R}$-relation on S is defined by
$\mathcal{R}$={(a, b)∈SxS: $aS^1 = bS^1$ and A(ax)=A(bx)∀x∈S}

And $\mathcal{R}$(a, b)= $\max_{z,w\in S} min\ \{A(az) = A(bw)\}$
Or
£={(a, b)∈ $S\ x\ S$: $aS^1 = bS^1$and (ax, bx)∈ α}
Where α is an ideal fuzzy relation on S.

Proposition 2.2. Green's £ and $\mathcal{R}$ relations defined on a commutative fuzzy semigroups are equal.

Proposition 2.3. If α is a reflexive ideal fuzzy relation on a fuzzy semigroup S, then Green's relations £(a, b)=$\mathcal{R}$(a, b)=1 for all a, b∈S.

Proposition 2.4. If α is a reflexive ideal fuzzy relation on a commutative fuzzy semigroup, then Green's relations £ and $\mathcal{R}$ are fuzzy congruences.

Proposition 2.5. If Green's relations £ and $\mathcal{R}$ are fuzzy congruences on a fuzzy semigroup S and £ o $\mathcal{R}$ = $\mathcal{R}$ o £, then £ o$\mathcal{R}$ is a fuzzy congruence.

Proposition 2.6.If Green's relations £ and $\mathcal{R}$ are fuzzy congruence on a fuzzy semigroup S such that £ o $\mathcal{R}$=$\mathcal{R}$ o £, then £ o $\mathcal{R}$ is the smallest fuzzy congruence containing £ and $\mathcal{R}$. That is £ o$\mathcal{R}$=£ v $\mathcal{R}$.

Proof By proposition 2.5 £ o $\mathcal{R}$ is a fuzzy congruence. Moreover,
£ o$\mathcal{R}$(a, b)= $\max_{t\in S} min\ \{£(a,t), R(t,b)\}$
≥ min{£(a, b), R(b, b)}
≥ min{£(a, b), 1}
≥ £(a, b). (1)
Also we get £ o R (a, b)≥R(a, b). (2)
Hence £o R is an upper bound of £ and $\mathcal{R}$
Consider any fuzzy congruence λ which is an upper bound of £ and $\mathcal{R}$.
That is, £ ≤λ and $\mathcal{R}$ ≤ λ.Then
£o $\mathcal{R}$ (a, b)= $\max_{t\in S} min\ \{£(a,t), R(t,b)\}$
≤ $\max_{t\in S} min\ \{\lambda(a,t), \lambda(t,b)\}$
≤ λoλ(a, b)
≤λ(a, b)
Hence £o$\mathcal{R}$ is the least fuzzy congruence containing £ and $\mathcal{R}$.That is,
£o$\mathcal{R}$=£v$\mathcal{R}$.

Definition 2.7.
The join of the Green's relations £ and $\mathcal{R}$ on a fuzzy semi group is denoted by $\mathcal{D}$, called the Green's $\mathcal{D}$-relation on a fuzzy semi group.

That is $\mathcal{D}$=£ v$\mathcal{R}$

By proposition 2.6 if £ and $\mathcal{R}$ are fuzzy congruence on a fuzzy semi group S such that £o$\mathcal{R}$=$\mathcal{R}$o£, then £o$\mathcal{R}$ is a fuzzy congruence $\mathcal{D}$-relation.

3.Cross composition of fuzzy binary relations

Definition 3.1 In the endeavor, to introduce $\mathcal{H}$ relations on a fuzzy semigroup, a new operation, say cross composition denoted by 'x' between fuzzy relations is successfully attempted on fuzzy semigroups.

The composition of fuzzy binary relations α and β on a fuzzy semi group S is defined as α x β (a, b)= $\min_{t\in S} min\,\{\alpha(a,z),\beta(z,b)\}$

Remark 3.2 Cross operation of fuzzy binary relation is always associative.

Proposition 3.3 If £ and $\mathcal{R}$ are fuzzy congruence Green's relations on a fuzzy semi group S, such that £ x$\mathcal{R}$ is reflexive and £x $\mathcal{R}$ =$\mathcal{R}$ x£, then

£ x $\mathcal{R}$ is a fuzzy congruence and is the greateset fuzzy congruence contained in £ and $\mathcal{R}$

That is, £x$\mathcal{R}$=£∧$\mathcal{R}$.

Given £x$\mathcal{R}$ is reflexive,

£x$\mathcal{R}$ (a, b)= $\min_{z\in S} min\,\{£(a,z),\mathrm{R}(z,b)\}$

= $\min_{z\in S} min\,\{R(z,b),£(a,z)\}$

= $\min_{z\in S} min\,\{R(b,z),£(z,a)\}$

Hence £x$\mathcal{R}$ is symmetric.

We have £X£≥£ and $\mathcal{R}$x$\mathcal{R}$≥$\mathcal{R}$

Then(£x$\mathcal{R}$)x(£x$\mathcal{R}$)=£x($\mathcal{R}$x£)x$\mathcal{R}$

=£x(£x$\mathcal{R}$)x$\mathcal{R}$

=(£x£)x($\mathcal{R}$x$\mathcal{R}$)

≥£x$\mathcal{R}$

So £x$\mathcal{R}$ is transitive

Again £ and $\mathcal{R}$ are compatible.So

£x$\mathcal{R}$ (a, b)= $\min_{z\in S} min\,\{£(a,z),\mathrm{R}(z,b)\}$

≤ $\min_{z\in S} min\,\{£(at,z),\mathrm{R}(z,b)\}$

≤£x$\mathcal{R}$ (at, b)……………………. (1)

Similarly we get £x$\mathcal{R}$ (a, b)≤ £x$\mathcal{R}$ (a, bt)………………… (2)

From(1)And (2)

£x$\mathcal{R}$ is compatible.That is £x$\mathcal{R}$ is compatible.That is, £x$\mathcal{R}$ isible similarity relation.

That is, £x$\mathcal{R}$ is a fuzzy congruence on S. (3)

Now £x$\mathcal{R}$ (a, b)= $\min_{z\in S} min\,\{£(a,z), \mathrm{R}(z,b)\}$
≤min{£(a, b), R(b, b)}
≤min{£(a, b), 1}
≤£(a, b) (4)
Also £x$\mathcal{R}$ (a, b)= $\min_{z\in S} min\,\{£(a,z), \mathrm{R}(z,b)\}$
≤ min{£(a, a), R(a, b)}
≤min{1, R(a, b)}
≤$\mathcal{R}$ (a, b) (5)
(3), (4) and (5) show :£ x$\mathcal{R}$ is a fuzzy congruence contained in £ and R.
Consider any fuzzy congruence δ contained in £ and $\mathcal{R}$.That is δ≤£ and $\delta \leq$ R
£x$\mathcal{R}$ (a, b)= $\min_{z\in S} min\,\{£(a,z), \mathrm{R}(z,b)\}$
≥ $\min_{z\in S} min\,\{£(a,z), \delta(z,b)\}$
≥ δx $\delta(a,b)$.
≥ δ(a, b)
Hence £ xR is the greatest fuzzy congruence contained in £ and R.That is £ x$\mathcal{R}$=£∧$\mathcal{R}$

Definition 3.4 The meet of Green's relations £ and $\mathcal{R}$ on a fuzzy semigroup S is denoted by $\mathcal{H}$ and called Green's $\mathcal{H}$relation on a fuzzy semigroup.

Then by proposition 3.3, if £ and $\mathcal{R}$ are fuzzy congruence, Green's relations on a fuzzy semigroup S, such that $\mathcal{R}$ o £, reflexive and £o$\mathcal{R}$=$\mathcal{R}$o£.Then £o$\mathcal{R}$ is a fuzzy congruence, Green's $\mathcal{H}$ relation.

Definition 3.5 A fuzzy semigroup S is called an inverse semigroup if every element a∈ S posses a inverse. ie for any a∈ S aa'a=a and a'aa'=a'.
An inverse semigroup in which every element has a weight is called a fuzzy inverse semigroup.

Definition3.6 (Ordered relation on fuzzy inverse semigroup).If a and b are any two elements in a fuzzy inverse semigroup, we write a ⪯ b, if there exists an idempotent e in S such that a=eb.The relation defined above is clearly a partial order relation on fuzzy inverse semigroup S.

Lemma 3.7 Let S be a fuzzy inverse semigroup in which partial order relation ⪯ is defined. If Green's relation £ and $\mathcal{R}$ defined on S are compatible, then £o$\mathcal{R}$=$\mathcal{R}$o£.

Proof £o$\mathcal{R}$ (a, b)= $\max_{t\in S} min\,\{£(a,t), R(t,b)\}$
= $\max_{t\in S} min\,\{R(t,b), £(a,t)\}$
Since t is any element in fuzzy inverse semigroup S, it can be equal to $bc^{-1}a$ in some cases.Therefore,
£o$\mathcal{R}$ (a, b)≥ $\max_{c\in S} min\,\{R(\mathrm{bc}^{-1}\mathrm{a},b)\}, £(a,bc^{-1}a),$

Since partial order relation '⪯' is defined on S, for b, c ∈S, there exists e such that b=ec and for a, c ∈S there exists f such that a=fc where e and f is idempotents in S.

£o$\mathcal{R}$ (a, b)≥ $\max_{c\in S}$ min $\{R(bc^{-1}a, ec)\}, £(fc, bc^{-1}a),$

≥ $\max_{c\in S} min\,\{R(a,c)\}, £(c,b),$ since R and are £ compatible

≥$\mathcal{R}$o £ (a, b)

Similarly, $\mathcal{R}$o £≥£o$\mathcal{R}$. Hence £o$\mathcal{R}$= $\mathcal{R}$o £

Theorem 3.8:If Green's relations £ and $\mathcal{R}$ are fuzzy congruences on a fuzzy inverse semigroup S, in which a partial order relation '⪯' is defined, then £o$\mathcal{R}$ Is a fuzzy congruence $\mathcal{D}$ –relation

Proof:By Lemma 3.7 £o$\mathcal{R}$.= $\mathcal{R}$o £ and by proposition 2.6, £o$\mathcal{R}$. Is a fuzzy congruence $\mathcal{D}$ –relation.

Lemma 3.9:Let S be a Fuzzy inverse semigroup in which a partial order relation '⪯' is defined.If Green's relation £ and R are compatible on S, then £X$\mathcal{R}$= $\mathcal{R}$X£.

Proof: $\mathcal{R}$X£.(a, b)= $\min_{c\in S} min\,\{R(a,c), £(c,b)\}.$

Since partial order relation ' ⪯ ' is defined on S, for a, c ∈S there exists f ∈ E(S) such that a=fc and for b, c ∈ S, there exists e ∈ E(S) such that b=ec.

$\mathcal{R}$x£.(a, b)≤ $\min_{c\in S} min$ {R$(a, ec), £(fc, b)$}.since £ is left compatible.

≤ $\min_{c\in S} min$ {R$(bc^{-1}a, b), £(a, bc^{-1}a)$}.

≤ $\min_{c\in S} min$ {R$(t, b), £(a, t)$}. Where t= $bc^{-1}a$ ∈S

≤ $\min_{c\in S} min$ {$£(a,t)$, R(t,b)}.

≤ £x$\mathcal{R}$(a, b)

Similarly, £x$\mathcal{R}$(a, b)≤$\mathcal{R}$x£.(a, b)

Hence £x$\mathcal{R}$= $\mathcal{R}$x£.

Theorem 3.10: If £ and $\mathcal{R}$ are fuzzy congruence Green's relations on a fuzzy inverse semigroup S in which partial order relation is defined, then £x$\mathcal{R}$ is a fuzzy congruence Green's $\mathcal{H}$–relation when if £x$\mathcal{R}$ is reflexive.

Proof: By Lemma 3.9 £o$\mathcal{R}$=$\mathcal{R}$o£., and by proposition 3.3 and definition 3.4, £x$\mathcal{R}$ is a fuzzy congruence Green's $\mathcal{H}$–relation.

The objective application of this study is to find solutions to some problems to the need of researchers in different faculties where they have to analyse and compare certain characteristic, properties like attachment, association, joined capacity etc, by applying Green's relations.

In 1905 Albert Einstein revolutioned the concept of space and motion by putting forward, the theory of relativity. It provides a framework which embraces practically all the branches of physical science. The principle of relativity was first stated by Newton. Newton's laws explained the microscopic world with amazing success. A series of study cuminated in Maxwell's equations of the electromagnetic field describe electricity, magnetism and light in the uniform system. According to Newton the motion of bodies included in a given space are the same among themselves. Now the question is whether the space or system is standing still or a moving system. Again Maxwell equations does not hold good to obey the Principle of relativity. Their form does not remain the same in a moving space and in stationary stage.In short velocity of particles and light depends on the observers in different inertial frames. That is electromagnetic radiation will not be same for different inertial observers.

Galilean transformation, Maxwell's equations, introduction of ether frames try to maintain the statusquo.As a result the final result of the experiment related to velocity of particles and light becomes approximate and never gets accurate and absolute.

In the experiment connected to velocity of electromagnetic radiation say light, which varies with medium of propagation. Let η be the velocity of light (electromagnetic radiation) in vaccum η_1, η_2, η_3 etc. be its velocities in the media M_1, M_2, M_3 etc. respectively. Consider its velocity in vaccum as the basic define a membership function from the set of medias to [0, 1] defined by $\mu(M_{i)}= \frac{\eta+\eta i}{2\eta}$

Using this membership function fuzzy logic can be applied and with the help of above Green's relations many form of laws introduced by, Newton (Laws), Maxwell (equations)Galilean(transformation)can be verified and modified.

References

[1] Zadeh, L. A., 1965, "Fuzzy sets, " pp.338-353

[2] Kim J P., and Bae, D.R, 1997, "Fuzzy congruences in groups, " Fuzzy sets and systems, 85, pp.115-120.

[3] Howie, J.M., 1976, " An Introduction to Semigroup Theory, " Lond Math Soc., Academic press Inc., London.

[4] Clifford, A.H., and Preston, G.B, 1967, "The algebraic theory of semigroups, " Maths survey no, 7, Amer.Math.Soc.vol I

[5] Clifford, A.H., and Preston, G.B, 1967 "The algebraic theory of semigroups, " Maths survey no.7, Amer.Math Soc.vol II.

[6] Kuroki, N.K., 1997, "Fuzzy congruence on inverse semigroup, " Fuzzy sets and systems, 86, pp. 335-340.