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Double Monophonic Number of a Graph

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Abstract

For a connected graph G of order n, a set S of vertices of G is called a *double monophonic set* of G if for each pair of vertices x, y in G there exist vertices u, v in S such that x, y lie on a u-v monophonic path. The *double monophonic number* $dm(G)$ of G is the minimum cardinality of a double monophonic set. Some general properties satisfied by double monophonic sets are discussed. The double monophonic numbers of some standard graphs are obtained. It is shown that for every pair k, n of integers with $2 \leq k \leq n$, there exists a connected graph G of order n such that $dm(G) = k$.

Keywords: monophonic set, monophonic number, double monophonic set, double monophonic number.

2010 Mathematics Subject Classification: 05C12

1. INTRODUCTION

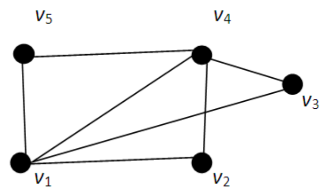
By a graph $G = (V, E)$ we mean a finite, undirected connected graph without loops or multiple edges. The *order* and *size* of G are denoted by n and m, respectively. The distance $d(x, y)$ between two vertices x and y in G is the length of the shortest x-y path in G. Any x-y path of length $d(x, y)$ is called an *x-y geodesic*. A vertex v in G is said to lie on an $x – y$ path P if v is a vertex of P including x and y. A subset S of V is called a *geodetic set* of the graph G if every vertex x of G lies on a u-v geodesic for some vertices u, v in S. A geodetic set of minimum cardinality is a *minimum geodetic set*. The cardinality of a minimum geodetic set is the *geodetic number* of G and is represented by $g(G)$. A geodetic set of cardinality $g(G)$ is also called a g – set of G. The geodetic number of a graph was introduced and studied in [1, 2, 4]. It was shown in [4] that determining the geodetic number of a graph is a NP-hard problem. For vertices x, y in G, the closed interval $I[x, y]$ consists of all vertices lying on some $x - y$ geodesic of G. Also for $S \subseteq V$, $I[S] = \bigcup_{x,y \in S} I[x, y]$. Let 2^V denote the set of all subsets of V. The mapping $I: V \text{x} V \rightarrow 2^V$ defined by $I[u, v] = \{w \in V$: w lies on a $u - v$ geodesic in $G\}$ is the *interval* function of G. One of the basic properties of I is that u, $v \in I[u, v]$ for any pair u, v. Thus the interval function captures every pair of vertices and so the problem of double geodetic set is trivially well-defined while it is clear that this fails in many graphs already for triplets(for example, complete graphs). This is the underlying idea for introducing and studying double geodetic sets in [5]. For a connected graph with at least two vertices, a set S of G is called a *double geodetic set* of G if for each pair of vertices x, y in G, there exist vertices u, v in S such that x, y lie on a $u - v$ geodesic. The *double geodetic number* $dg(G)$ of G is the minimum cardinality of a double geodetic set. A double geodetic set of cardinality $dg(G)$ is called a *dg – set* of G.

A *chord* of a path P is an edge joining two non-adjacent vertices of P. A path P is called *monophonic* if it is a chord less path. A subset S of V is called a *monophonic set* of G if every vertex v of G lies on a x-y monophonic path for some elements x and y in S. The minimum cardinality of a monophonic set of G is called the *monophonic number* of G and is represented by $m(G)$.

2. DOUBLE MONOPHONIC NUMBER

Definition 2.1 Let G be a connected graph with at least two vertices. A set S of vertices of G is called a *double monophonic set* of G if for each pair of vertices x, y in G there exist vertices u, v in S such that x, y lie on a u-v monophonic path. The *double monophonic number* $dm(G)$ of G is the minimum cardinality of a double monophonic set. Any double monophonic set of cardinality $dm(G)$ is called a *dm-set*.

Example 2. 2 For the graph G given in Figure 2. 1, it is easily seen that no 2-element subset of vertices is a double monophonic set of G. Since $S = \{v_2, v_3, v_5\}$ is a double monophonic set, it follows that $dm(G) = 3$.



G

Figure 2. 1

Example 2. 3 For the graph G given in Figure 2. 2, it is easily seen that no 3-element subset of vertices is a monophonic set of G. Let $S = \{x_1, x_2, y_1, y_2\}$. Then it is clear that S is the unique minimum monophonic set of G so that $m(G) = 4$. Since the pair of vertices u_3, v_3 does not lie on any monophonic path joining a pair of vertices from S, S is not a double monophonic set. Also, since $S_1 = \{x_1, x_2, y_1, y_2, v_3\}$ is a double monophonic set, it follows that $dm(G) = 5$. It is to be noted that $S_2 = \{x_1, x_2, y_1, y_2, u_3\}$ is also a minimum double monophonic set of G.

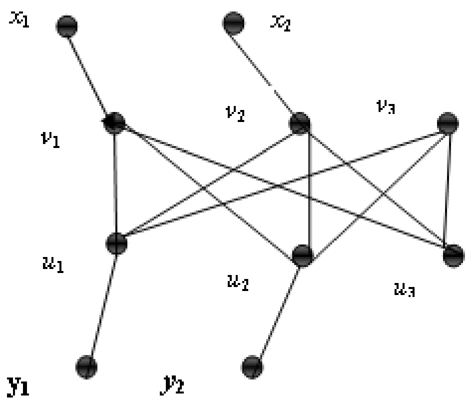


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Figure 2. 2

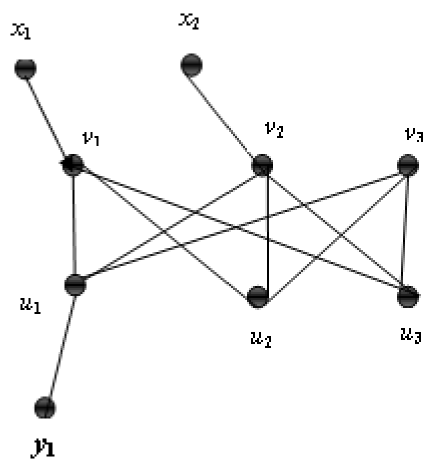

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Figure 2. 3

Also, for the graph G in Figure 2.3, the set $S = \{x_1, x_2, y_1, u_2\}$ is a minimum monophonic set so that $m(G) = 4$. Also, $S_1 = \{x_1, x_2, y_1, u_3\}$ is another minimum monophonic set. Both S_1 and S_2 are not double monophonic sets. It can be easily verified that the set $S_2 = \{x_1, x_2, y_1, u_2, v_3\}$ is a minimum double monophonic set so that $dm(G) = 5$. Also, $S_3 = \{x_1, x_2, y_1, u_2, u_3\}$ is a minimum double monophonic set of G. Thus, the double monophonic number and monophonic number of a graph can be different.

Theorem 2.4 For any graph G of order n, $2 \leq m(G) \leq dm(G) \leq n$.

Proof. A monophonic set needs at least two vertices and therefore $m(G)\geq 2$. It is clear that every double monophonic set is also a monophonic set and so $m(G)\leq dm(G)$. Also, since the set of all vertices of G is a double monophonic set of G, $dm(G)\leq n$. Hence$2\leq m(G)\leq dm(G)\leq n$.

■

Remark 2.5 The bounds in the above theorem are sharp. For the complete graph $K_n(n\geq 2)$, we have $dm(K_n) = n$. The set of the two end vertices of a non-trivial path P_n on n vertices is its unique double monophonic set so that $dm(P_n) =2$. Thus the complete graph K_n has the largest double monophonic number n and that the non-trivial paths have the smallest double monophonic number.

Theorem 2.6 Each extreme vertex of a connected graph G belongs to every double monophonic set of G. In particular, if the set of all end vertices of G is a double monophonic set, then it is the unique minimum double monophonic set of G.

Proof. Let v be an extreme vertex and let S be a double monophonic set of G. If v does not belongs to S, then v is an internal vertex of a x-y monophonic path P for some vertices x and y in S. Let u and w be the vertices adjacent to v on P. Then u and w are not adjacent so that v is not an extreme vertex, which is a contraction. Hence v belongs to every double monophonic set.

■

Corollary 2.7 For a graph G of order n with k extreme vertices, $\max\{2, k\}\leq dm(G)\leq n$.

Proof. This follows from the Theorems 2.4 and 2.6.

■

Theorem 2.8 Let G be a connected graph with a cut vertex v. Then each double monophonic set of G contains at least one vertex from each component of G-v.

Proof. First we prove that if S is a monophonic set of G, then S contains at least one vertex from each component of G-v. Let $G_1, G_2,..., G_k$ ($k \geq 2$) be the components of G-v. Now, assume that S contains no vertex from a component, say G_i ($1 \leq i \leq k$). Let u be any vertex of G_i. By Theorem 2.6, u is not an extreme vertex of G. Since S is a monophonic set of G, u lies on a $x - y$ monophonic path for some vertices x, y in S. Let this $x - y$ monophonic path be $P :x = u_0, u_1, u_2,..., u,..., u_i = y$ with $u \neq x, y$. Then both the $x - u$ sub path of P and the $u - y$ sub path of P contain v so that P is not a path, which is a contradiction. Hence every monophonic set of G contains at least one vertex from each component of G-v. Now, since every double monophonic set is a monophonic set, the theorem follows.

■

Theorem 2.9 No cut-vertex of a connected graph G belongs to any minimum double monophonic set of G.

Proof. Let S be any minimum double monophonic set of G. Suppose that S contains a cut vertex z of G. Let G_1, G_2,... $G_k(k\geq 2)$ be the components of G-z. Let $S_1 = S - \{z\}$. First, we prove that S_1 is a double monophonic set of G. Let x, y be any pair of vertices in G. Since S is a double monophonic set of G, the vertices x and y lie on a u –v monophonic path for some vertices u and v in S. Now, if $z \neq u$, v, then u and v are in S_1 and so S_1 is a double monophonic set of G, which is a contradiction. So, assume that $z = u$ or v. For convenience, let $z = u$. Assume without loss of generality that v is in S_1. By Theorem 2.7, it is possible to take a vertex w from $G_k(k \neq 1)$ such that $w\in S$. Since z is a cut-vertex of G, it follows that z lies on a $w - v$ monophonic path. That is, both z and w lie on a $w - v$ monophonic path. Thus the pair of vertices x and y lie on a $w - v$ monophonic path with both w, v in S_1, which shows that S_1 is a double monophonic set of G. This is a contradiction to the fact that S is a minimum double monophonic set of G. This proves that no cut vertex belongs to a minimum double monophonic set of G.

■

The following corollary is a consequence of Theorems 2.6 and 2.9.

Corollary 2.10 For any tree T, the double monophonic number $dm(T)$ equals the number of end vertices in T. In fact, the set of all end vertices of T is the unique *dm-set* of T.

Theorem 2.11 For every pair k, n of integers with $2\leq k\leq n$, there exists a connected graph G of order n such that $dm(G) = k$.

Proof. For $k = n$, let $G = K_n$. Then by Theorem 2.5 $dm(G) = n= k$.
Now, let $2\leq k<n$. Then there exists a tree of order n with k end vertices. Therefore, by Corollary 2.10, $dm(G) = k$.

■

Theorem 2.12 For a non-trivial connected graph G, $m(G) = 2$ if and only if $dm(G) =2$.

Proof. It is proved in Theorem 2.4 that $2 \leq m(G) \leq dm(G) \leq n$. If $dm(G) = 2$, then it follows that $m(G)=2$. Suppose that $m(G) =2$. Let $S =\{u, v\}$ be a minimum monophonic set of G. Let x, y be any pair of vertices in G. Then it is clear that x, y lie on a u-v monophonic path so that S is also a double monophonic set. Thus $dm(G) =2$.

■

In the following theorem, we determine the double monophonic number of certain standard graphs.

Theorem 2. 13

(i) For the complete graph $G = K_n$, $dm(G) = n$.
(ii) For the star $G = K_{1, n-1}$, $dm(G) = n$-1.
(iii) For the complete bipartite graph $G = K_{m, n}(2 \leq m \leq n)$, $dm(G) = \min\{m, n\}$.
(iv) For the cycle $G = C_n$ $(n\geq 4)$, $dm(G) = 2$.
(v) For the wheel $G = W_{1, n-1}$, $dm(G) = 2$.
(vi) For the graph $G = K_n - e$, $dm(G) = 2$.

Proof.

(i) Since every vertex of K_n is extreme, this follows from Theorem 2. 6.

(ii) This follows from Corollary 2. 10.

(iii) Let X and Y be the partite sets of $K_{m,n}$. Let S be a double monophonic set of G. First we prove that $X \subseteq S$ or $Y \subseteq S$. Otherwise, $X \not\subseteq S$ and $Y \not\subseteq S$. This means that there exist vertices $x \in X$, $y \in Y$ such that $x, y \notin S$. Since S is a double monophonic set, the vertices x, y lie on a monophonic path joining a pair of vertices of S. Since the monophonic paths have length either 1 or 2, it follows that the pair of vertices x, y lie only on monophonic paths of the type $x– y$, $x– t$, and $s – y$ for some $t \in X$ and $s \in Y$. Hence it is clear that $x \in S$ or $y \in S$. This is a contradiction to $x, y \notin S$. Thus the claim that $X \subseteq S$ or $Y \subseteq S$ is proved. Also, it is easily seen that both X and Y are double monophonic sets and so $dm(G)$ = min$\{m, n\}$.

(iv) Any set S of vertices of G consisting of two non-adjacent vertices is clearly a double monophonic set so that $dm(G) = 2$.

(v) Let $V(G) = \{ v, v_1, v_2,..., v_{n-1}\}$ with v the central vertex and $v_1, v_2,..., v_{n-1}$ the cycle C_{n-1}. Let S be any two non-adjacent vertices on the cycle C_{n-1}. It is clear that S is a double monophonic set of G so that $dm(G) = 2$.

(vi) Let e be the edge $e = uv$. Then u and v are the only extreme vertices of G and it is clear that $S = \{u, v\}$ is a double monophonic set so that $dm(G) = 2$. ∎

In view of certain results proved above, we leave the following problem as an open question.

Problem 2.14 Characterize the class of graphs G of order n for which (i) $dm(G) = n$-1 and (ii) $dm(G) = n$.

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