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A Fuzzy Approach To Complete Upper Semilattice And Complete Lower Semilattice

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Abstract

This study is an extension of the work "A Study of Green's Relations on Fuzzy Semigroups" [1]. Using the definition of fuzzy compatibility [1], a new concept for fuzzy partial congruences is also introduced. On the basis of the described notions and certain conditions in [1], a complete lower fuzzy semilattice and a complete upper fuzzy semilattice is found out and defined from the set of all fuzzy partial congruences on a semigroup.
The purpose of this paper is to develop the concept of a complete upper semilattice and a complete lower semilattice [5], on fuzzy views, using the percept of partially ordered and totally ordered set.

Keywords: Fuzzy compatibility, Fuzzy congruences, Fuzzy partial congruences, Dot composition of fuzzy relations, Cross composition of fuzzy relations, Complete upper fuzzy semilattices, Complete lower fuzzy semilattices.

1. INTRODUCTION:

The study introduces fuzzy partial congruences by recalling the ideas of partially ordered and totally ordered set. It is a follow up of the work [1]. Using fuzzy congruences [3, 7] and well known definitions [2], the notion of partial congruences is described.

The study concludes that the algebraic structure $<P_f(S), \circ>$ is a complete upper fuzzy semilattice, where '$\circ$'denotes the dot composition of fuzzy relations [1] *and* $< P_f(S), \diamond >$ *is a complete lower fuzzy semilattice, where* $\diamond$ denotes the cross composition of fuzzy relations [1] and $P_f(S)$ denotes the set of all fuzzy partial congruences on a semigroup.

2. BASIC NOTIONS.

Definition 2.1. Fuzzy relations: Fuzzy relation indicates the strength or association between the elements of n-tuple. When n=2 the fuzzy relation is called a fuzzy binary relation.

That is a function α defined from SXS to [0,1] is called a fuzzy binary relation on S [1],where S is a semigroup [1].

Composition of fuzzy relations

Definition 2.2 Dot composition: Let α and β be two fuzzy binary relations on a semigroup S. A dot composition of α and β, denoted by α ◦ β is defined as

α ◦ β (a,b) =$\max_{z\in S} min\{\alpha(a,z),\beta(z,b)\}$ for all a,b $\in$ s [1].

Definition.2.3.Cross composition: A cross composition of α and β, denoted by $\alpha \diamond \beta$ is defined as

$\alpha \diamond \beta(a,b) = \min_{z\in S} min\{\alpha(a,z),\beta(z,b)\}$……………………… [1].

Preposition. 2.3. Dot and Cross Composition of fuzzy relation on a semigroup is associative.

Proof: Let α, β , γ are fuzzy relations on S. By definition, ' o' is binary on S

[αo (β o γ)](a,b)=$\max_{z\in S} min\{\alpha(a,z),\beta_{o}\gamma(z,b)\}$

= $\max_{z\in S} min\{\alpha(a,z),\max_{w\in S} min[\beta(z,w)\gamma(w,z)]$

=$\max_{z,w\in S} min\{\alpha(a,z),\beta(z,w),\gamma(w,b)$

Similarly we get ,

$[(\alpha\circ\beta)\circ\gamma]$(a,b)=$\max_{z,w\in S} min\{\alpha(a,z),\beta(z,w),\gamma(w,b)\}$……………… (1)

From (1) and(2), o is associative.

By definition $\diamond$ is binary on S$[(\alpha\circ\beta)]$

$[\alpha(\diamond\beta\diamond\gamma)](a,d)$

=$\min_{b\in S} min\{\alpha(a,b),\beta\circ\gamma(b,d)\}$

=$\min_{b\in S} min\{\alpha(a,b),\min_{c\in S} min[\beta(b,c),\gamma(c,d)]\}$

=$\min_{b,c\in S} min\{\alpha(a,b),\beta(b,c),\gamma(c,d)\}$ ……………………….. (2)

Similarly, we get,

$[(\alpha\diamond\beta)\diamond\gamma](a,d)$

=$\min_{b,c\in S} min\{\alpha(a,b),\beta(b,c),\gamma(c,d)\}$………………….. (3)

From (2)and(3), $\alpha(\diamond\beta\diamond\gamma) = (\alpha\diamond\beta)\diamond\gamma$

Proposition 2.4. The set of all fuzzy binary relations $\beta_{\mu}(S)$, with respect to the operation

(1) o is a semigroup

(2) $\diamond$ is a semigroup

Proof: Result is obvious by proposition 2.3.

Types of Fuzzy binary relations on a non-empty set X

Definition 2.5. A fuzzy binary relation R defined on a non-empty set X is reflexive if R(x,x)=1 for all x $\in$ **X**. If R(x,x)=1 for some and not for all x $\in$ **X ,then** R is fuzzy irreflexive, and when R(x,x)≠1 for all $x \in X$,then R is said to be anti-fuzzy reflexive. Since R is a fuzzy relation it can take any value between 0 nad 1. So we can define different grades of reflexivity, called ε-reflexivity where $0 \leq \varepsilon < 1$, then the fuzzy relation is said to be ε- reflexive. if R(x,x)=ε for all x $\in$ **X**.

Definition 2.6. Fuzzy symmetric: A fuzzy relation R on a non-empty set X is fuzzy symmetric if $R(x,y) = R(y,x)$ for all $x, y \in X$, R is fuzzy asymmetric if $R(x,y) = R(y,x)$ for some and not for all $x, y \in X$,and is fuzzy anti- symmetric if $R(x,y) \neq R(y,x)$ for all $x, y \in X$ where $x \neq y$.

Hence R is fuzzy anti- symmetric if $R(x,y) \neq R(y,x)$ implies $x \neq y$

Definition 2.7.Fuzzy transitive: A fuzzy relation defined on a non-empty set X is fuzy transitive if $R(x,z) \geq \max_{y\in X} min\{R(x,y), R(y,x)\}$ for all $x, z \in X$ or $R_{o}R \leq R$, where o'denotes the ordinary composition of fuzzy relations. A fuzzy relation is fuzzy non transitive if $R(x,z) \geq \max\min \{R(x,y), R(y,z)\}$for some and not for all $x, z \in X$.

A fuzzy relation defined on a non empty set X is fuzzy anti- transitive if $R(x,z) < \max_{y\in X} \min \{R(x,y), R(y,z)\}$ for all $x, z \in X$.

Remark 2.8: When the operation of fuzzy relation is the cross composition ,fuzzy relation α is transitive if and only if $\alpha \diamond \alpha \geq \alpha$.

Definition 2.9 Similarity relation: A fuzzy binary relation R on a non-empty set X is said to be a similarity relation if it is

1) fuzzy reflexive
2) fuzzy symmetric
3) fuzzy transitive

Definition 2.10. Fuzzy partial order relation: A fuzzy binary relation ω which is fuzzy reflexive, fuzzy anti- symmetric and fuzzy transitive is called a fuzzy partial order relation. In a non-empty set X, if x& y are ω related, we usually write (x, y)$\in \omega$. Here we write $x \preceq y$ rather than(x, y)$\in \omega$ .If in a non-empty set X, x$\preceq$y or

$y \preceq x$ *for all* $x, y \in$ X we called the partial order as, total order. Then (X,$\preceq$) denotes a totally ordered set .

When the greatest lower bond of x and y denoted by $x \wedge y$ exists for every $x, y \in X$, then (X,$\preceq$) is called a **complete lower fuzzy semilattice**.[2].

The totally ordered set (X,$\preceq$) , called **a complete upper fuzzy semilattice** if the least upperbound of x and y denoted by $x \vee y$ exists for every $x, y \in X$ [2]

3. FUZZY COMPATIBLE RELATIONS.

Ju Pil Kim , , D.R Bae., have defined fuzzy compatibility [3].Here the same definition is described more precisely in [1] .Using the newly defined fuzzy compatibility [1], we establish some important results and properties.

Definition 3.1. Fuzzy right compatible. A fuzzy binary relation α on a semigroup S is fuzzy right compatible if $\alpha(a,b) \leq \alpha(at,b)$ and $\alpha(a,b) \leq \alpha(a,bt)$ for all $a, b, t \in S$

Example 3.2. Consider the semigroup $Z^+$of all positive integers with respect to the operation ordinary multiplication. Define a fuzzy binary relation defined by $\alpha(a,b) = P[x \leq a, y \leq b], a, b \in Z^+$. That is α is a probability measure. Clearly α is always non-negative and its maximum value is 1.

Also $\alpha(at,b) = P[x \leq at, y \leq b]$ *where* $a, b, t \in Z+$

$\geq P[x \leq a, y \leq b]$

$\geq \alpha(a,b)$

Similarly we get

$\alpha(a,bt) \geq \alpha(a,b)$

Hence α is fuzzy right compatible on Z^+.

Defiition 3.3. (Fuzzy Left Compatible). A fuzzy binary relation α on a semigroup S is fuzzy left compatible if $\alpha(a,b) \leq \alpha(a,tb)$ and $\alpha(a,b) \leq \alpha(ta,b)$ for all $a, b, t \in S$

Example 3.4 Consider $f: RxR \longrightarrow [0,\infty]$ defined by $f(x,y) = \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)}; x, y \in R$. Here R is a set of real numbers, which is a semigroup with respect to ordinary multiplication. Define a fuzzy relation $\alpha: RXR \longrightarrow [0,1)$ defined by

$$\alpha(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u,v)dudv$$

$f(u,v)$is a joint probability density function of a bivariate normal distribution.

We have

$$\alpha(a,tb)=\int_{-\infty}^{a}\int_{-\infty}^{tb} f(x,y)dx\,dy \ \ where\ a,b,t\ \in Z+$$
$$\geq \int_{-\infty}^{a}\int_{-\infty}^{b} f(x,y)dxdy$$
$$\geq \alpha(a,b)$$

Again

$$\alpha(ta,b)=\int_{-\infty}^{ta}\int_{-\infty}^{b} f(x,y)dx\,dy \ \ where\ a,b,t\ \in Z+$$
$$\geq \int_{-\infty}^{a}\int_{-\infty}^{b} f(x,y)dxdy$$
$$\geq \alpha(a,b)$$

Hence the above defined fuzzy binary relation is left compatible on Z^+.

Definition 3.5 Fuzzy Compatible:. A fuzzy binary relation α on a semigroup S is compatible if it is both fuzzy left compatible and fuzzy right compatible.

Example 3.6. Consider the semigroup of positive integers Z^+, with respect to the operation ordinarh multiplication. Define a relation α on Z^+ defined by

$$\alpha(x,y)=\int_{-\infty}^{a}\int_{-\infty}^{b} f(u.v)dvdu$$

Where $f(u,v)=\frac{1}{2\pi}e^{-\frac{1}{2}(x^2+y^2)};u,v\in Z+$

Clearly α is both fuzzy left compatible and fuzzy right compatible. Hence α is compatible.

Proposition 3.7 If α is a fuzzy right compatible relation on a semigroup S , then $\alpha(a,b)\leq \alpha(ac,bd)\, for\ all\ a,b,c,d\ \in S$. The converse is true when S is a monoid.

Proof: Given α is fuzzy right compatible. Then for any $a,b,c\ \in S$,

$\alpha(a,b)\leq \alpha(ac,b)$………………………….. (1)

Since $ac,b\ \in S$ and α is fuzzy right compatible

$\alpha(ac,b)\leq \alpha(ac,bd)\forall a,b,c,d\ \in S$ …………………………….. (2)

From (1) and (2), $\alpha(a,b)\leq \alpha(ac,bd)\, for\ all\ a,b,c,d\ \in S$

Conversely assume S is a monoid and

$\alpha(a,b)\leq \alpha(ac,bd)\, for\ all\ a,b,c,d\ \in S$ …………………… (3)

Since S is a monoid, the identity $e \in S$. Therefore from (3), $\alpha(a,b) \leq \alpha(ae,bd) \, for \, all \, a,e,b,d \in S$

That is $\alpha(a,b) \leq \alpha(a,bd) \forall a,b,d \in S$

That is $\alpha(a,b) \leq \alpha(a,bt) \forall a,b,t \in S$

Similarly we get $\alpha(a,b) \leq \alpha(at,b) \forall \, a,b,t \in S$. Hence α is fuzzy right compatible.

Proposition 3.8. If α is fuzzy left compatible relation on a semigroup S, then $\alpha(a,b) \leq \alpha(ca,db) \forall a,b,c,d \in S$. T he converse is true when S is a monoid.

Proof: Result follows from proposition 3.7 by applying the definition of fuzzy left compatibility.

Proposition 3.9. $max\{\alpha(a,b), \alpha(c,d)\} = \alpha(ac,bd)$. Converse is true when S is a monoid.

Proof: Given α is fuzzy compatible. That is α is fuzzy left compatible and fuzzy right compatible. By proposition 3.7 and 3.8 we have

$\alpha(a,b) \leq \alpha(ac,bd) \; \forall \, a,b,c,d \in S$ …………………… (4)

$\alpha(c,d) \leq \alpha(ac,bd) \; \forall \, a,b,c,d \in S$ ………………….. (5)

From (1) and (2) ,$max\{\alpha(a,b), \alpha(c,d)\} \leq \alpha(ac,bd) \forall a,b,c,d \in S$

Conversely assume, $max\{\alpha(a,b), \alpha(c,d)\} \leq \alpha(ac,bd)\} \forall a,b,c,d \in S$. That implies $\alpha(a,b) \leq \alpha(c,d) \, \forall a,b,c,d \in S \; and \; \alpha(c,d) \leq \alpha(ac,bd) \forall a,b,c,d \in S$.

Then by proposition 3.7 and 3.8 (converse part) α is fuzzy right compatible and α is fuzzy left compatible, when S is a monoid. That is α is fuzzy compatible.

Proposition 3.10. If α is fuzzy reflexive compatible fuzzy relation on a semi group S, then $\alpha(ac,bc) = \alpha(ca,cb) \forall a,b,c, \in S$.

Proof:

Given α is reflexive and compatible fuzzy relation on S. That is

$\alpha(x,x) = 1 \; \forall x \in S$ ……………………….. (6)

And by proposition 3.9

$max\{\alpha(a,b), \alpha(c,d)\} \leq \alpha(ac,bd) \forall a,b,c,d \in S$ ………………. (7)

That is $max\{\alpha(a,b), 1\} \leq \alpha(ac,bc)$. That is $1 \leq \alpha(ac,bc)$.

Since maximum value of a fuzzy relation is one, $\alpha(ac,bc) = 1$.

Similarly we get $\alpha(ca,cb)$=1. Hence the result.

Proposition 3.11. A fuzzy left compatible relation on a semigroup S is fuzzy right compatible (hence compatible), if S is commutative.

Proof: Let α be fuzzy left compatible relation on a commutative semigroup S. then
$\alpha(a,b) \leq \alpha(a,tb)\ \forall a,b,t\ \in S$,………….. (8)
And
$\alpha(a,b) \leq \alpha(ta,b)\ \forall a,b,t\ \in S$………… (9)
Since S is commutative for $t,a,b \in S; ta = tb\ and\ tb = bt.$
So (8) and (9) can be written as
$\alpha(a,b) \leq \alpha(a,bt) and\ \alpha(a,b) \leq \alpha(at,b)\ \forall a,b,t\ \in S.$
Hence α is right compatible. That is α is fuzzy left compatible and fuzzy right compatible. So α is compatible.

Proposition 3.12. A fuzzy right compatible relation on a semigroup S is fuzzy left compatible (hence compatible) when S is commutative.

Proof: Result follows from proposition 3.11 by applying the definition of right and left compatibility.

4. Fuzzy Partial Congruences

We are familiar with the definition of fuzzy congruence in groups [2] and inverse semigroup [3]. In this section we introduce a new notion, “fuzzy partial congruence” in a semigroup and arrive the conclusion to describe the set of all fuzzy partial congruences $P_f(S)$ as a complete upper fuzzy semilattice and a complete lower fuzzy semilattice .

Definition 4.1(Fuzzy Partial Congruences). A fuzzy partial ordering compatible relation on a semigroup is called a fuzzy partial congruence. That is a fuzzy binary relation α on a semigroup S is said to be a fuzzy partial congruence if

i. α is fuzzy left compatible.
ii. α is fuzzy right compatible.
iii. α is fuzzy reflexive.
iv. α is fuzzy antisymmetric.
v. α is fuzzy transitive.

Note 4.2. $P_f(S)$ denote the set of all fuzzy partial congruence on a semi group S.

Theorem 4.3. If $\alpha,\beta \in\ P_f(S)\ and\ \alpha \circ \beta = \beta \circ \alpha\ then\ \alpha \circ \beta\ \in\ P_f(S)$.

Proof: Given $\alpha,\beta \in\ P_f(S)$. That is $\alpha\ and\ \beta$ are fuzzy left and right compatible partial ordering on S.

Now to prove that $\alpha \circ \beta \in P_f(S)$ where $\alpha, \beta \in P_f(S)$. We have to prove that α *and* β are fuzzy reflexive relation.

Then

$\alpha \circ \beta(x,x) = \max_{y \in S} min\{\alpha(x,y), \beta(y,x)\}$

$\geq \min \{\alpha(x,x), \beta(x,x)\}$

$\geq \min\{1,1\}$

≥ 1 .... (10)

Since $\alpha \circ \beta$ is a fuzzy relation, $\alpha \circ \beta \leq 1$........................... (11)

From (10) and (11), $\alpha \circ \beta(x,x) = 1$. Hence $\alpha \circ \beta$ is reflexive.

Since $\alpha, \beta \in P_f(S)$, both of them are partial ordering and hence anti- fuzzy symmetric. So

For $x \neq y, \alpha(x,y) \neq \alpha(y,x) and\ \beta(x,y) \neq \beta(y,x)$.

Suppose $x \neq y$. We have $\alpha \circ \beta(x,x) = \max_{y \in S} min\{\alpha(x,y), \beta(y,x)\}, y \in X$. Here right hand side becomes maximum for some $z \in S$, which can be in any of the three cases $x \neq z \neq y, x = z \neq y, x \neq z = y$.

Case(1) $x \neq z \neq y$;

$\alpha \circ \beta(x,y) = \max_{z \in S, x \neq z \neq y} min\{\alpha(x,z), \beta(z,y)\}, y \in X$

$\neq \max_{z \in S, x \neq z \neq y} min\{\alpha(z,x), \beta(y,z)\}$

$\neq \max_{z \in S, x \neq z \neq y} min\{\beta(y,z), \alpha(z,x)\}$

$\neq \beta \circ \alpha(y,x)$

$\neq \alpha \circ \beta(y,x)$.. (12)

Case (2) $x = z \neq y$;

$\alpha \circ \beta(x,y) = \max_{z \in S, x = z \neq y} min\{\alpha(x,z), \beta(z,y)\}$

$= \max_{z \in S, x \neq y} min\{\alpha(x,x), \beta(x,y)\}$

$= \max_{z \in S, x \neq y} min\{1, \beta(x,y)\}$

$= \beta(y,x)$ .... (13)

Again

$\alpha \circ \beta(x,y) = \beta \circ \alpha(y,x) = \max_{z \in S, x = z \neq y} min\{\beta(y,z), \alpha(z,x)\}$

$= \max_{z \in S, x = z \neq y} min\{\beta(y,x), \alpha(x,x)\}$

$= \min\{\beta(y,x), \alpha(x,x)\}$

$= \min \{\beta(y,x), 1)\}$

$= \beta(y,x)$ ……………….............................. (14)

Since for $x \neq y, \beta(x,y) \neq \beta(y,x)$

From (13) and (14),
$\alpha \circ \beta(x, y) \neq \alpha \circ \beta(y, x)$
Case (3) $x \neq z = y;$
$\alpha \circ \beta(x, y) = \max_{z \in S, x \neq z = y} min\{\alpha(x, z), \beta(z, y)\}$
$= \max_{z \in S, x \neq z = y} min\{\beta\alpha(x, y), \beta(y, y)\}$
$= \min\{\alpha(x, y), 1\}$
$= \alpha(x, y)$... (15)
Again
$\alpha \circ \beta(y, x) = \beta \circ \alpha(y, x) = \max_{z \in S, x \neq z = y} min\{\beta(y, z), \alpha(z, x)\}$
$= \min\{\beta(y, y), \alpha(y, x)\}$
$= \min\{1, \alpha(y, x)\}$
$= \alpha(y, x)$... (16)
For $x \neq y$ we have $\alpha(x, y) \neq \alpha(y, x)$ so
From (15) and (16) $\alpha \circ \beta(x, y) \neq \alpha \circ \beta(y, x)$. Hence in the three cases for $x \neq y$
$\alpha \circ \beta(x, y) \neq \alpha \circ \beta(y, x)$
Hence $\alpha \circ \beta(x, y)$ is fuzzy anti symmetric.
Now we can show that $\alpha \circ \beta$ is transitive. We have
$(\alpha \circ \beta) \circ (\alpha \circ \beta) = \alpha \circ (\beta \circ \alpha) \circ \beta$
$= \alpha \circ (\alpha \circ \beta) \circ \beta$
$= (\alpha \circ \alpha) \circ (\beta \circ \beta)$
$\leq (\alpha \circ \beta)$
Hence $(\alpha \circ \beta)$ is transitive.
Moreover $\alpha, \beta \in P_f(S)$ implies $\alpha \; and \; \beta$ are fuzzy left compatible, and fuzzy right compatible.
$\alpha \circ \beta(x, y) = \max_{z \in S} \min \{\alpha(x, z), \beta(z, y)\}$
$\leq \max_{z \in S} \min \{\alpha(tx, z), \beta(z, y\} for \; t \in S$
$\leq \alpha \circ \beta(tx, y) \; for \; t \in S$
Again
$\alpha \circ \beta(x, y) = \max_{z \in S} \min \{\alpha(x, z), \beta(z, y)\}$
$\leq \max_{z \in S} \min \{\alpha(x, z), \beta(z, ty\} \; for \; t \in S$
$\leq \alpha \circ \beta(x, ty) \; for \; t \in S$
So $\alpha \circ \beta$ is fuzzy left compatible.
Similarly we can show that $\alpha \circ \beta$ is right compatible.
Hence $\alpha \circ \beta$ is fuzzy reflexive, anti symmetric, transitive and compatible relation on S. that is $\alpha \circ \beta$ is a partial congruence on S. that is $\alpha \circ \beta \in P_f(S)$.

Theorem 4.4. If α and β are fuzzy partial congruence on a semigroup, such that

$\alpha \circ \beta = \beta \circ \alpha$ then $\alpha \circ \beta$ is the least upper bound of α and β. That is $\alpha \circ \beta = \alpha \vee \beta$.

Proof: By theorem 4.3 $\alpha \circ \beta$ is a fuzzy partial congruence.
We have
$\alpha \circ \beta = \max_{z \in S} min\{\alpha(a,z), \beta(z,b)\}$
$\geq \max_{z \in S} \min \{\alpha(a,b), \beta(b,b)\}$
$\geq \min\{\alpha(a,b), 1\}$,since β *is reflexive*
$\geq \alpha(a,b)$ .... (17)
Again
$\alpha \circ \beta(a,b) = \max_{z \in S} \min\{\alpha(a,z), \beta(z,b)\}$
$\geq \min \{\alpha(a,a), \beta(a,b)$
$\geq \min\{1, \beta(a,b)\}$
$\beta(a,b)$ .... (18)
From (17) and (18), $\alpha \circ \beta$ is an upper bound of α *and* β.
Assume any fuzzy partial congruence η which is an upper bound of *and* β . That is $\alpha \leq \eta$ *and* $\beta \leq \eta$.
$\alpha \circ \beta(a,b) = \max_{z \in S} \min\{\alpha(a,z), \beta(z,b)\}$
$\leq \max_{z \in S} \min \{\eta(a,z), \eta(z,b)$
$\leq \eta \circ \eta(a,b)$
$\leq \eta(a,b)$. Since η is a fuzzy partial congruence.
That is $\alpha \circ \beta$ is the greatest fuzzy partial congruence containing α *and* β. Hence $\alpha \circ \beta$ is the least upper bond of α *and* β .
That is $\alpha \circ \beta = \alpha \vee \beta$.

Theorem 4.5. The set $P_f(S)$ of all fuzzy partial congruence is a complete upper fuzzy semilattice. If $\alpha \circ \beta = \beta \circ \alpha$ *for* $\alpha, \beta \in P_f(S)$.

Proof: Let $\alpha, \beta \in P_f(S)$ and $\alpha \circ \beta = \beta \circ \alpha$. Then by theorem 4.4 $\alpha \circ \beta = \alpha \vee \beta$ and by theorem4.5, $\alpha \circ \beta \in P_f(S)$. That is for any $\alpha, \beta \in P_f(S)$, the least upper bond $\alpha \circ \beta = \alpha \vee \beta \in P_f(S)$. Hence $(P_f(S), \preccurlyeq)$ is a partial ordering. More over for any $\alpha, \beta \in P_f(S)$ implies $\alpha \vee \beta \in P_f(S)$. So $P_f(S)$ is a complete upper fuzzy semilattice.

Lemma 5.4 If $\alpha, \beta \in P_f(S)$ and $\alpha \diamond \beta = \beta \diamond \alpha$ then $\alpha \diamond \beta \in P_f(S)$, when it is reflexive.

Proof: Given $\alpha, \beta \in P_f(S)$. that is α *and* β are partial congruence on S. that is

$\alpha \; and \; \beta$ are fuzzy left and fuzzy right compatible partial ordering on S.

We have $\alpha \circ \beta$ is reflexive. Since $\alpha, \beta \in P_f(S)$, both $\alpha \; and \; \beta$ anti-fuzzy symmetric. That is

For any $a \neq b \Rightarrow \alpha(a) \neq \alpha(b) \; and \; \beta(a) \neq \beta(b)$………………… (20)

$\alpha \circ \beta(x,y) = \min_{z \in S} \min\{\alpha(x,z), \beta(z,y)\}$.

Here right hand side becomes minimum for some $\in S$, which can be in any of the following three cases.

1) $x \neq z \neq y$

2) $x = z \neq y$

3) $x \neq z = y$

Case (1) $x \neq z \neq y$

$\alpha \circ \beta(x,y) = \min_{z \in S} \min\{\alpha(x,z), \beta(z,y)\}$.

$\neq \min_{z \in S} \min \{\alpha(z,z), \beta(y,z)\}$ from (20)

$\neq \min_{z \in S} \min \{\beta(y,z), \alpha(z,x)\}$

$\neq \beta \circ \alpha(y,x)$

$\neq \alpha \circ \beta(y,x$

Case (2) $x = z \neq y;$

$\alpha \circ \beta(x,y) = \min_{z \in S} \min\{\alpha(x,z), \beta(z,y)\}$.

$= \min_{z \in S} \min \{\alpha(x,x), \beta(x,y) \; since \; x = z$

$= \min_{z \in S} \min \{1, \beta(x,y)$

$= \beta(x,y)$ ……………………………………………… (21)

$\alpha \circ \beta(x,y) = \beta \circ \alpha(y,x)$

$= \min_{z \in S} \min \{\beta(y,z), \alpha(z,x)\}$

$= \min_{z \in S} \min \{\beta(y,x), \alpha(x,x) \; since \; z = x$

$= \min_{z \in S} min\{\beta(y,x), 1\}$

$= \beta(y,x)$ … … … … … …. (22)

Since for $\beta(x,y) \neq \beta(y,x)$

From (21) and (22),

$\alpha \circ \beta(x,y) \neq \alpha \circ \beta(y,x)$

Case (3) $x \neq z = y;$

$\alpha \circ \beta(x,y) = \min_{z \in S} \min \{\alpha(x,z), \beta(z,y)$

$= \min_{z \in S} \min \{\alpha(x,y), \beta(y,y) \; sine \; z = y$

$= \min_{z \in S} min \{\alpha(x,y), 1\}$

$= \alpha(x,y)$ … … … … … … … … … (23)

Again $\alpha \circ \beta(y,x) = \beta \circ \alpha(y,x)$

$= \min_{z \in S} \min\{\beta(x,y), \alpha(z,x)\}$

$= \min_{z\in S} \min\{\beta(y,y), \alpha(y,x)\}$ *since* $z = y$
$= \min_{z\in S} \min\{1, \alpha(y,x)\}$
$= \alpha(y,x)$ ... ... (24)

From (23) and (24) $\alpha \circ \beta(x,y) \neq \alpha \circ \beta(y,x)$ since $\alpha(y,x) \neq \beta(y,x)$.
Hence in the three cases $\alpha \circ \beta$ is anti-fuzzy symmetric.
Since $\alpha, \beta \in P_f(S)$, both of them are transitive. That is $\alpha \circ \alpha \geq \alpha, \beta \circ \beta \geq \beta$.
Now $(\alpha \circ \beta) \circ (\alpha \circ \beta) = \alpha \circ (\beta \circ \alpha) \circ \beta$
$= \alpha \circ (\alpha \circ \beta) \circ \beta$
$= (\alpha \circ \alpha) \circ (\beta \circ \beta)$
$\leq \alpha \circ \beta$.

Hence $\alpha \circ \beta$ is transitive
Also
$\alpha \circ \beta(x,y) = \min_{z\in S} \min\{\alpha(x,z), \beta(z,y)\}$
$\leq \min_{z\in S} \min\{\alpha(xt,z), \beta(z,y)\}$
$\leq \alpha \circ \beta(xt,y)$.

Similarly we get $\alpha \circ \beta(x,y) \leq \alpha \circ \beta(x,yt)$
Hence $\alpha \circ \beta$ is fuzzy left compatible.
That is $\alpha \circ \beta$ is fuzzy reflexive, fuzzy anti- symmetric, fuzzy transitive and fuzzy compatible relations on S. That is $\alpha \circ \beta$ is a fuzzy partial congruence on S. so $\alpha \circ \beta \in P_f(S)$.
Hence the result.

Lemma 5.5. If α *and* β are fuzzy partial congruence on a semi group S such that $\alpha \circ \beta = \beta \circ \alpha$, then
$\alpha \circ \beta$ is the greatest lower bound of α *and* β.

Proof: By lemma 5.4 $\alpha \circ \beta \in P_f(S)$.
Again $\alpha \circ \beta \in (x,y) = \min_{z\in S} \min\{\alpha(x,z), \beta(x,y\}$
$\leq \min\{\alpha(x,x), \beta(x,y)$
$\leq \min\{1, \beta(x,y)\}$
$\leq \beta(x,y)$............................... (25)

Also $\alpha \circ \beta \in (x,y) = \min_{z\in S} \min\{\alpha(x,z), \beta(x,y\}$
$\leq \min\{\alpha(x,y), \beta(y,y)\}$
$\leq \min\{\alpha(x,y), 1)\}$
$\leq \alpha(x,y)$.......................... (26)

From (25) and (26) $\alpha \circ \beta$ is a lower bound of α *and* β. Consider any lower bound $\eta \in P_f(S)$ of α *and* β. That is ,$\eta \leq \alpha$ *and* $\eta \leq \beta$

We have $\eta(x,y) \leq \eta \circ \eta(x,y)$
$\leq \min_{z \in S} \min \{\eta(x,z), \eta(z,y\}$
$\leq \min_{z \in S} \min \{\alpha(x,z), \beta(z,y)\}$
$\leq \alpha \circ \beta(x,y)$.

Hence any lower bound η is less than or equal to $\circ \beta$. That is $\alpha \circ \beta$ is the greatest lower bound of $and\ \beta$ .that is, $\alpha \circ \beta = \alpha \wedge \beta$.

Theorem 5.6. The set $P_f(S)$ of all fuzzy partial congruence is a complete lower fuzzy semilattice if $\alpha \circ \beta$ is reflexive and $\alpha \circ \beta = \beta \circ \alpha$.

Proof: By Lemma 5.4 $\alpha \circ \beta \in P_f(S)$ and by lemma 5.5 $\alpha \circ \beta$ is the greatest lower bound of $\alpha, \beta \in P_f(S)$. That is for any $\alpha, \beta \in P_f(S)$, $\alpha \circ \beta = \alpha \wedge \beta \in P_f(S)$. So $P_f(S)$ is a complete upper semi lattice

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