International Journal of Computational and Applied Mathematics.
ISSN 1819-4966 Volume 11, Number 1 (2016), pp. 61-69

http://www.ripublication.com

Vertex Covering and Stability in Hypergraphs

D.K.Thakkar[1] and K.N.Kalariya[2]

[1]Department of Mathematics Saurashtra Uni., Rajkot-360 005
[2]V.V.P. Engineering college, Rajkot-360 005
dkthakkar1@yahoo.co.in and kiritkalariya@gmail.com

Abstract

In this paper we consider the concepts of vertex covering and stability in hypergraphs. In particular we study the effect of removing a vertex from the hypergraph on the parameters like vertex covering number and stability number. In particular we prove that the vertex covering number may increase or remain unchanged when this operation (of removing a vertex) is performed. However, the stability number of the given hypergraph will always decrease.

AMS Subject Classifications: 05C69, 05C99, 05C07

Keywords: Vertex covering set, Minimum vertex covering set, Vertex covering number, Stable set, Maximum stable set, Stability number.

1. Introduction
The concepts of stability and vertex covering in hypergraphs have been well explored by many researchers [1]. Vertex covering and independence have found an important place in graph theory due to their various applications in optimization theory, matching theory and other areas of graph theory. In fact they have provided new techniques which are useful in other areas of mathematics.
When we think about the concepts of stability and vertex covering in hypergraph theory, they can be defined in similar manner as in graph theory and also their variants can be defined in hypergraphs. As a result we prove similar theorems in hypergraph theory and different theorems suitable for hypergraph as well.
In this paper we studied vertex covering sets and stable sets in hypergraphs which have been already defined [2].

2. Basic definitions
Definition 2.1: (Stable set, Stability number) [2]: A set $S \subseteq V(G)$ in a hypergraph

G is a Stable set if no edge is a subset of S. A Stable set with maximum cardinality is called a maximum stable set of G and is denoted as β_0-set of G.
The cardinality of a maximum Stable set is called the Stability number of G and it is denoted as $\beta_0(G)$.

Definition 2.2: (Vertex covering set, Vertex covering number) [2]: A subset S of $V(G)$ is called a vertex covering set if every edge of G has non-empty intersection with S.
A vertex covering set with minimum cardinality is called α_0-set of G.
The cardinality of a minimum vertex covering set of G is called the vertex covering number of G and it is denoted as $\alpha_0(G)$.
It is obvious to see that

i. The complement of a vertex covering set is an stable set and
ii. A subset S of $V(G)$ is a minimum vertex covering set if and only if $V(G) \setminus S$ is a maximum stable set.

Therefore, $\alpha_0(G) + \beta_0(G) = n =$ the number of vertices of hypergraph G.

Definition 2.3: (Edge degree) [2]: If v is a vertex of hypergraph G then the edge degree $\deg_e v$ is defined to be the number of edges which contain the vertex v.
Here we consider the sub-hypergraph $G\backslash v$ whose vertex set is $V(G)\backslash\{v\}$ and the edge set is sub-edges obtained by removing the vertex v from every edge of G. In this section we will consider the sub-hypergraph $G\backslash v$.
In the following lemma we shall prove that the vertex covering number cannot decrease when a vertex v is removed from the hypergraph G.
We may again assume that in a hypergraph an edge may have only one vertex.

3. Main Results:

Lemma 3.1: If G is a hypergraph and v is a vertex of G then $\alpha_0(G) \leq \alpha_0(G\backslash v)$.

Proof: Let S be a minimum vertex covering set of $V(G)\backslash\{v\}$. Let e be any edge of G. If $v \notin e$ then e is an edge of $G \backslash v$ and since S is a vertex covering set of $G \setminus v$, $e \cap S \neq \emptyset$. If e is an edge of G and $v \in e$ then $e' = e \setminus \{v\}$ is an edge of $G \backslash v$. Since S is a vertex covering set of $G \backslash v$, $e' \cap S \neq \emptyset$.Therefore, $e \cap S \neq \emptyset$.
Hence, S is a vertex covering set of G. Therefore, $\alpha_0(G) \leq |S| \leq \alpha_0(G\backslash v)$.■
Now we shall state and prove the necessary and sufficient condition under which the vertex covering number of a hypergraph does not change when a vertex is removed from the hypergraph.

Theorem 3.2: Let G be a hypergraph and $v \in V(G)$. Then $\alpha_0(G) = \alpha_0(G\backslash v)$ if and only if there is a minimum vertex covering set S of G such that $v \notin S$.

Proof: First suppose that $\alpha_0(G) = \alpha_0(G\backslash v)$. Let S be a minimum vertex covering set of $G\backslash v$. By above lemma 3.1, S is a vertex covering set of G.

If S not a minimum vertex covering set of G then $\alpha_0(G) < |S| = \alpha_0(G\backslash v)$ which is a contradiction. Hence S is a minimum vertex covering set of G and since $S \subseteq V(G)\backslash\{v\}, v \notin S$.
Conversely, let S be a minimum vertex covering set of G such that $v \notin S$. Let e' be any edge of $G\backslash v$ then $e' = e\backslash\{v\}$ for some edge e of G. Now $e \cap S \neq \emptyset$ and therefore $e' \cap S \neq \emptyset$. Because $v \notin S$. Therefore S is a vertex covering set of $G \backslash v$. Thus, $\alpha_0(G\backslash v) \leq |S| = \alpha_0(G) \leq \alpha_0(G\backslash v)$.
Hence $\alpha_0(G) = \alpha_0(G\backslash v)$, $e \cap S \neq \emptyset$. ■

Example 3.3: Consider the hypergraph G whose vertex set $V(G) = \{1,2,3 \ldots ,10\}$ and edge set $E(G) = \{\{1,2,3\}, \{3,4,5\}, \{5,6,7\}, \{7,8,9\}, \{9,10,1\}\}$ then $\{3,5,9\}$ is a minimum vertex covering set of G and also $G\backslash 1$ then $\alpha_0(G) = \alpha_0(G\backslash 1) = 3$. Note that 1 does not belong to this minimum vertex covering set.

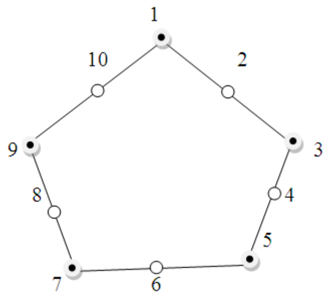

Figure: 3.1

Corollary 3.4: With notation as above $\alpha_0(G\backslash v) > \alpha_0(G)$ if and only if $v \in S$ for every minimum vertex covering set S of G. ■

Example 3.5: Consider the hypergraph G with $V(G) = \{0,1,2,3,4,5,6\}$ and $E(G) = \{\{1,0,4\}, \{2,0,5\}, \{3,0,6\}\}$. Note that, $\alpha_0(G) = 1$ However, $\alpha_0(G\backslash 0) = 3$ Thus $\alpha_0(G\backslash 0) > \alpha_0(G)$.

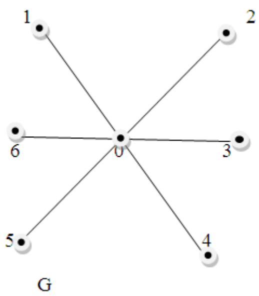

Figure: 3.2

Note that $\{0\}$ is the only minimum vertex covering set of G. Thus 0 belongs to every minimum vertex covering set of G, and thus $\alpha_0(G\backslash 0) > \alpha_0(G)$.
Here also we prove that the stability number of G reduces when a vertex is removed.

Theorem 3.6: If G is a hypergraph and $v \in V(G)$ then $\beta_0(G\backslash v) < \beta_0(G)$.

Proof: Let S be a minimum stable set of G \ {v}. We claim that $S \cup \{v\}$ is a stable set in G. For this suppose e is an edge of G such that $e \subseteq S \cup \{v\}$.
Case I: $v \notin e$.
Then $e' = e\backslash\{v\} = e$ and e' is a subset of S. Hence, S is not a stable set of $G \backslash v$, which is a contradiction.
Case II: $v \in e$.
Since, $e \subseteq S \cup \{v\}, e' = e\backslash\{v\} \subseteq S$ and thus S is not a stable set of $G\backslash v$, again a contradiction.
So, $S \cup \{v\}$ must be a stable set of G.
Therefore, $\beta_0(G) \geq |S| + 1 > |S| = \beta_0(G\backslash v)$.
Hence $\beta_0(G\backslash v) < \beta_0(G)$.■

Example 3.7: Consider the hypergraph G with $V(G) = \{1,2,3, \ldots 11\}$ and $E(G) = \{\{1,2,3,4,5,6\}, \{3,7,8,9\}, \{5,10,11\}\}$ then $S = \{1,2,4,6,7,8,9,10,11\}$ is a stable set of G.Therefore, $\beta_0(G) = 9$. Also $S_1 = \{1,2,4,6,8,9,10,11\}$ is a stable set of $G \backslash 3$ and $\beta_0(G\backslash 3) = 8$. Hence, $\beta_0(G\backslash 3) < \beta_0(G)$.

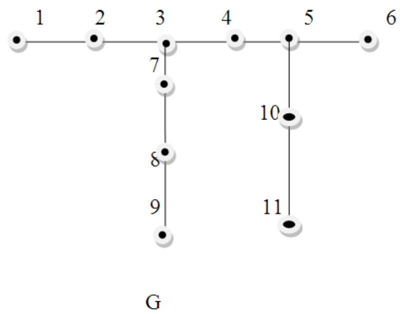

Figure: 3.3

Now we prove necessary and sufficient condition under which $\beta_0(G\backslash v) = \beta_0(G) - 1$.

Theorem 3.8: $\beta_0(G\backslash v) = \beta_0(G) - 1$ if and only if there is a maximum stable set S of G such that $v \in S$.

Proof: Suppose there is a maximum stable set S of G such that $v \in S$.
Now consider the set $S_1 = S \backslash \{v\}$. First S_1 is a Stable set in $G\backslash v$. Suppose there is an edge e' of $G\backslash v$ such that e' is a subset of S_1. Let e be any edge of G such that $e\backslash v = e'$ then it is obvious that e is a subset of S. This contradicts the fact that S is a

stable set of G.
Thus S_1 must be a stable set in $G\backslash v$. Since$\beta_0(G\backslash v) < \beta_0(G)$, S_1 must be a maximum stable set of $G\backslash v$.
Thus, $\beta_0(G\backslash v) = |S_1| = |S| - 1 = \beta_0(G) - 1$.
Conversely suppose $\beta_0(G\backslash v) = \beta_0(G) - 1$.
Let S_1 be a maximum stable set of $G\backslash v$ and $S = S_1 \cup \{v]$. First we prove that S is a stable set of G. Suppose there is an edge e of G such that e is a subset of S.
Case I:$v \notin e$.
Then, $e' = e \setminus \{v\}$. Hence e' is a subset of S_1. This contradicts the fact that S_1 is a stable set of $G\backslash v$.
Case II: $v \in e$.
Then $e' = e \setminus \{v\}$. Since e is a subset of S, e' is a subset of S_1. Again this contradicts the fact that S_1 is a stable set of $G \setminus v$.
Hence, from both the cases it follows that S is a stable set of G. Also $\beta_0(G) = \beta_0(G\backslash v) + 1$.
Therefore, S is a maximum stable of G. Note that $v \in S$. This completes the proof of the theorem. ■

Remark 3.9: From the above theorem it is clear that if S is a maximum stable set of G and $w \in S$ then $\beta_0(G) = \beta_0(G\backslash w) + 1$.
Thus, if $S_1, S_2, \dots, S_k$ are all maximum stable set of G and $S = S_1 \cup S_2 \cup \dots \cup S_k$ then $\beta_0(G) = \beta_0(G\backslash w) + 1$ if and only if $w \in S$.

Corollary 3.10: The number of vertices w in G such that $\beta_0(G\backslash w) = \beta_0(G) - 1 + |S|$ where $S = S_1 \cup S_2 \cup \dots \cup S_k$ and $\{S_1, S_2, \dots, S_k\}$ is the family of all maximum stable sets of G.

Remark 3.11: From the proof of the above theorem 4.2.12 it is clear that if $\beta_0(G) = \beta_0(G\backslash w) + 1$ and if S_1 is a maximum stable set of $G \setminus v$ then $S_1 \cup \{v\}$ is a maximum stable set of G containing v.
Conversely, if S is a maximum stable set of G containing v then $S_1 = S \setminus \{v\}$ is a maximum stable set of $G \setminus v$.
Also we make the following conclusions from the above theorems (under the assumption that $\beta_0(G\backslash v) = \beta_0(G) - 1$.):

1) There is a one-one correspondence between the maximum stable sets of $G\backslash v$ and the maximum stable sets of G containing the vertex v.
2) It is also obvious that the number of maximum stable sets of G is greater than or equal to the number of maximum stable sets of $G \setminus v$.
3) It is clear to that the number of maximum stable sets of G equal the number of maximum stable sets of $G \setminus v$ if and only if v belongs to the intersection of all maximum stable sets of G.
4) Also it may be noted that the number of vertices v such that $\beta_0(G\backslash v) = \beta_0(G) - 1 \geq \beta_0(G)$.

We state the following corollary.

Corollary 3.12: $\beta_0(G\backslash v) = \beta_0(G) - 1$ if and only if $v \in S$ for any maximum stable set S of G.
Now we consider a hypergraph $G\backslash v$ in which the vertex set is $V(G)\backslash\{v\}$ and the edge set is equal to the set of those edges of G which do not contain the vertex v. This is called the partial sub-hypergraph of G.
First we established that if $v \in V(G)$ then $\alpha_0(G\backslash v) \leq \alpha_0(G)$.

Proposition 3.13: Let G be a hypergraph and if $v \in V(G)$ then $\alpha_0(G\backslash v) \leq \alpha_0(G)$.

Proof: Note that every edge of $G \setminus v$ is also an edge of G.
Let S be a minimum vertex covering set of G.
Case I**:** $v \in S$.
If e' is any edge of $G \setminus v$ then e' is also an edge of G and hence $e' \cap S \neq \emptyset$. Since $v \notin e'$, $e' \cap (S \setminus \{v\}) \neq \emptyset$.Thus $S \setminus \{v\}$ is a vertex covering set of $G \setminus v$.
Case II**:** $v \notin S$.
Let e' be any edge of $G \setminus v$ then e' is also an edge of G. Hence, $e' \cap S \neq \emptyset$.
Thus, S is a vertex covering set of $G \setminus v$. Therefore, S or $S \setminus \{v\}$ is a vertex covering set of $G \setminus v$. Hence $\alpha_0(G\backslash v) \leq \alpha_0(G)$.■

Proposition 3.14: If G is a hypergraph and $v \in V(G)$. If $\alpha_0(G\backslash v) < \alpha_0(G)$ then $\alpha_0(G\backslash v) = \alpha_0(G) - 1$.

Proof: Let S_1 be a minimum vertex covering set of $G\backslash v$ then S_1 cannot be a vertex covering set of G. Therefore there is an edge e of G such that $e \cap S \neq \emptyset$. Then it implies that $v \in e$. Note that $S = S_1 \cup \{v\}$ must be a vertex covering set of G. Therefore, $\alpha_0(G) = |S_1| + 1 = \alpha_0(G\backslash v) + 1$. Thus, $\alpha_0(G\backslash v) = \alpha_0(G) - 1$.■

Theorem 3.15: Let G be a hypergraph and $v \in V(G)$. Then $\alpha_0(G\backslash v) < \alpha_0(G)$ if and only if there is a minimum vertex covering set S of G such that $v \in S$.

Proof: Suppose $\alpha_0(G\backslash v) < \alpha_0(G)$.
Let S_1 be a minimum vertex covering set of $G \setminus v$ then $S = S_1 \cup \{v\}$ is a minimum vertex covering set of G. Thus, $v \in S$ and S is a minimum vertex covering set of G.
Conversely, let S be a minimum vertex covering set of G such that $v \in S$. Consider the set $S_1 = S \setminus \{v\}$. Let e' be any edge of $G \setminus v$ then e' is also an edge of G. Since S is a vertex covering set of G, $e' \cap S \neq \emptyset$.
Since $v \in e'$, $e' \cap (S\ \{v\}) \neq \emptyset$. Thus $S \setminus \{v\}$ is a vertex covering set of $G \setminus v$.
Therefore, $\alpha_0(G\backslash v) \leq |S \setminus \{v\}| < |S| = \alpha_0(G)$. Hence, $\alpha_0(G\backslash v) < \alpha_0(G)$.■

Corollary 3.16: Let G be a hypergraph and $v \in V(G)$. Then $\alpha_0(G\backslash v) = \alpha_0(G)$ if and only if for every minimum vertex covering set S of G, $v \notin S$.

Example 3.17: Let G be the hypergraph whose vertex set $V(G) = \{0,1,2,3,4,5,6\}$ and $E(G) = \{\{1,0,4\},\{2,0,5\},\{3,0,6\}\}$. Let $v = 0$ then the partial sub-hypergraph $G \setminus 0$ has no edges. Also 0 belongs to the only minimum vertex covering set {0}. Therefore, $\alpha_0(G\backslash 0) < \alpha_0(G)$. In fact, we may note that $\alpha_0\ (G) = 1$ and $\alpha_0(G\backslash 0) = 0$.

Corollary 3.18: Let G be a hypergraph and $v \in V(G)$ such that v is not isolated and $\alpha_0(G\backslash v) = \alpha_0(G)$ then for every minimum vertex covering set S of G, $N(v) \cap S \neq \emptyset$.

Proof: Let S be any minimum vertex covering set of G then $v \notin S$. Let e be any edge containing v then $e \cap S \neq \emptyset$. Thus $N(v) \cap S \neq \emptyset$.■

Remark 3.19: It may be noted that the compliment of a vertex covering set is an independent set. Also it may be noted that if $v \in V(G)$ and $v \in S$ for some minimum vertex covering set S then $\alpha_0(G\backslash v) < \alpha_0(G)$. Thus we introduce the following two notations.
$V_{cr}^{-} = \{v \in V(G): \alpha_0(G\backslash v) < \alpha_0(G)\}$ and $V_{cr}^{0} = \{v \in V(G): \alpha_0(G\backslash v) = \alpha_0(G)\}$ then
$V_{cr}^{-} = \cup\, \{S: S\ is\ a\ minimum\ vertex\ covering\ set\ of\ G\}$.
Thus, $V_{cr}^{0} = V(G) \setminus \cup\, \{S: S\ is\ a\ minimum\ vertex\ covering\ set\ of\ G\}$.
$= \cap\, \{V(G) \setminus S: S\ is\ a\ minimum\ vertex\ covering\ set\ of\ G\}$.
Since the compliment of every vertex covering set is a stable set, it follows that V_{cr}^{0} is also a stable set.
Thus, we have the following corollary.

Corollary 3.20: V_{cr}^{0} is a stable set.
Since every edge containing the vertex intersect every vertex covering set of G, it follows that $V_{cr}^{-} \geq$ the minimum edge degree of a hypergraph G, provided there is vertex in the hypergraph G such that $v \in V_{cr}^{0}$. If there is no vertex in V_{cr}^{0} then every vertex $v \in V_{cr}^{-}$ and therefore $V_{cr}^{-} \geq$ the minimum edge degree of a hypergraph G.

Corollary 3.21: Suppose G is a hypergraph and for this hypergraph G, $V_{cr}^{0} \neq \emptyset$. Then $\alpha_0(G) \geq \deg_e v$.

Proof: Let S be a minimum vertex covering set of G and $v \in V_{cr}^{0}$. By corollary 4.2.23, if e is an edge of g containing then $e \cap S \neq \emptyset$. Also note that if e_1 and e_2 are distinct edges of G containing v then $e_1 \cap S \neq e_2 \cap S$ (because G is a hypergraph and no two edges can intersect in more than one vertex).
Thus, $|S| \geq$ the number of edges containing vertex v. That is $\alpha_0(G) \geq \deg_e v$.■
We may prove as in the case of a vertex covering that the stability number of a hypergraph does not decrease when a vertex is removed from the hypergraph G.

Lemma 3.22: If G is a hypergraph and $v \in V(G)$ then $\beta_0(G\backslash v) \leq \beta_0(G)$.

Proof: Let S be a maximum stable set of $G\backslash v$ then obviously S is also an stable set of G. Thus, $\beta_0(G\backslash v) \leq |S| \leq \beta_0(G)$. Hence, $\beta_0(G\backslash v) \leq \beta_0(G)$.■

Example 3.23: Consider the hypergraph G whose vertex set $V(G) = \{0,1,2,3,4,5,6\}$ and edge set $E(G) = \{ \{1,2\}, \{2,3\}, \{3,4\}, \{4,5\}, \{5,6\}, \{6,1\}, \{1,0,4\}, \{2,0,5\}, \{3,0,6\}\}$. In this hypergraph the set $\{0, 1, 3, 5\}$ and $\{0, 2, 4, 6\}$ are maximum stable sets and its stability number $\beta_0(G) = 4$.
Now let $v = 0$. The sub-hypergraph $G \setminus v$ has the vertex set is $V(G \setminus \{0\}) = \{1,2,3,4,5,6\}$ and the edge set is $E(G \setminus\{0\}) = \{ \{1,2\}, \{2,3\}, \{3,4\}, \{4,5\}, \{5.6\}, \{6,1\}\}$. The set $\{1, 3, 5\}$ and $\{2, 4, 6\}$ are maximum stable sets of $G \setminus v$ and its stability number is $\beta_0(G\setminus 0) = 3$. Note that $\beta_0(G) > \beta_0(G\setminus 0)$.
Now we prove the necessary and sufficient condition under which the stability number of hypergraph G does not change.

Theorem 3.24: $\beta_0(G\setminus v) = \beta_0(G)$ if and only if there is a maximum stable set S of G such that $v \notin S$.

Proof: Suppose $\beta_0(G\setminus v) = \beta_0(G)$. Let S be a maximum stable set of $G\setminus v$. Now S is also a stable set of G.
Also $\beta_0(G) = |S|$ and thus S is a maximum stable set of G not containing the vertex v.
Conversely, let S be a maximum stable set of G such that $v \notin S$. Since $v \notin S$, S is also a stable set of $G \setminus v$ and therefore $|S| \leq \beta_0(G\setminus v)$. Thus, $\beta_0(G) \leq \beta_0(G\setminus v) \leq \beta_0(G)$. Therefore, $\beta_0(G\setminus v) = \beta_0(G)$.■

Corollary 3.25: $\beta_0(G\setminus v) < \beta_0(G)$ if and only if v belongs to every maximum stable set of G.
Now we introduce the following notations,
$I^0 = \{v \in V(G)\colon \beta_0(G\setminus v) = \beta_0(G)\}$.
$I^- = \{v \in V(G)\colon \beta_0(G\setminus v) < \beta_0(G)\}$.
Thus we can deduce that, $I^- = \cap\ \{S\colon S\ is\ a\ maximum\ stable\ set\ of\ G\}$. Also it may be noted that
$I^0 = V(G) \setminus \cup \{S\colon S\ is\ a\ maximum\ stable\ set\ of\ G\}$.
Now we prove that when $\alpha_0(G)$ decreases $\beta_0(G)$ remains same and when $\alpha_0(G)$ remains same $\beta_0(G)$ decreases.(when vertex is removed from the hypergraphs).

Theorem 3.26:
(1) If $\alpha_0(G\setminus v) < \alpha_0(G)$ then $\beta_0(G\setminus v) = \beta_0(G)$.
(2) If $\alpha_0(G\setminus v) = \alpha_0(G)$ then $\beta_0(G\setminus v) < \beta_0(G)$.

Proof: (1) $\alpha_0(G) + \beta_0(G) = n$= the number of vertices of hypergraph G.
Now $\alpha_0(G\setminus v) + \beta_0(G\setminus v) = n - 1$.
Thus $\alpha_0(G) - 1 + \beta_0(G\setminus v) = n - 1$.
Hence $\alpha_0(G) + \beta_0(G\setminus v) = n$.
From this it follows that $\beta_0(G\setminus v) = \beta_0(G)$.
Proof of (2) is similar. ■

REFERENCES

[1] B.D. Acharya, Domination in Hypergraphs, AKCE J. Graphs.Combin., No.2 (2007), pp.117-126

[2] C.BERGE, Hypergraphs, North-Holland Mathematical Library, New York Volume-45 (1989).

[3] T.W.Haynes, S.T.Hedetniemi, P.J Slater, Fundamentals of domination in graphs, Marcel Dekker, New York, 1997.

[4] T.W.Haynes, S.T.Hedetniemi, P.J Slater, Domination in graphs Advanced Topics, Marcel Dekker, New York, 1998.