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The Total Edge Monophonic Number of a Graph

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Abstract

A set S of vertices of a connected graph G is a monophonic set if every vertex of G lies on an x-y monophonic path for some elements x and y in S. The minimum cardinality of a monophonic set of G is the monophonic number of G, denoted by $m(G)$. A set S of vertices of a graph G is an edge monophonic set if every edge of G lies on an x-y monophonic for some elements x and y in S. The minimum cardinality of an edge monophonic set of G is the edge monophonic number of G, denoted by $em(G)$. A total edge monophonic set of a graph G is an edge monophonic set S such that the subgraph induced by S has no isolated vertices. The minimum cardinality of a total edge monophonic set of G is the total edge monophonic number of G, and is denoted by $em_t(G)$. It is proved that, for the integers a, b and c with $2 \leq a \leq b < c$, there exists a connected graph G having the monophonic number a, the edge monophonic number b, and the total edge monophonic number c.

Keywords : Monophonic set, monophonic number, edge monophonic set, edge monophonic number, total edge monophonic set, total edge monophonic number.

AMS Subject classification : 05C12.

1. INTRODUCTION

By a graph $G = (V, E)$ we mean a simple graph of order at least two. The order and size of G are denoted by p and q, respectively. For basic graph theoretic terminology, we refer to Harary [5]. The neighborhood of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v. The closed neighborhood of a vertex v is the set $N[v] = N(v) \cup \{v\}$. A vertex v is an extreme vertex if the sub graph induced by its neighbors is complete. A vertex v is a semi-extreme vertex of G if the sub graph induced by its neighbors has a full degree vertex in $N(v)$. In particular, every extreme vertex is a semi - extreme vertex and a semi - extreme vertex need not be an extreme vertex.

For any two vertices x and y in a connected graph G, the distance $d(x, y)$ is the length of a shortest x-y path in G. An x-y path of length $d(x, y)$ is called an x-y geodesic. A vertex v is said to lie on an x-y geodesic P if v is a vertex of P including the vertices x and y.

The closed interval $I[x, y]$ consists of all vertices lying on some x-y geodesic of G, while for $S \subseteq V$, $I[S] = \bigcup_{x,y \in S} I[x, y]$. A set S of vertices is a geodetic set if $I[S] = V$, and the minimum cardinality of a geodetic set is the geodetic number $g(G)$. A geodetic set of cardinality $g(G)$ is called *a g*-set. The geodetic number of a graph was introduced in [1, 6] and further studied in [2, 3, 4, 5]. A set S of vertices of a graph G is an edge geodetic set if every edge of G lies on an x-y geodesic for some elements x and y in S. The minimum cardinality of an edge geodetic set of G is the edge geodetic number of G denoted by eg (G). The edge geodetic number was introduced and studied in [8]. The total edge of geodetic set of a graph G is an edge geodetic set S such that the subgraph induced by S has no isolated vertices. The minimum cardinality of a total edge geodetic set of G is the total edge geodetic number of G and is denoted by $eg_t(G)$.

A chord of a path $u_1, u_2, \ldots, u_k$ in G is an edge $u_i u_j$ with $j \geq i + 2$. A u-v path P is called a monophonic path of it is a chordless path. A set S of vertices is a monophonic set if every vertex of G lies on a monophonic path joining some pair of vertices in S, and the minimum cardinality of a monophonic set is the monophonic number $m(G)$. A monophonic set of cardinality $m(G)$ is called an m - set of G. The monophonic number of a graph G was studied in [9]. A set S of vertices of a graph G is an edge monophonic set if every edge of G lies on an $x – y$ monophonic path for some elements x and y in S. The minimum cardinality of an edge monophonic set of G is the edge monophonic number of G, denoted by $em(G)$. The connected edge monophonic number of a graph was introduced and studied in [7]. A total edge monophonic set of a graph G is a edge monophonic set S such that the subgraph induced by S has no isolated vertices. The minimum cardinality of a total edge monophonic set of G is the total edge monophonic number denoted by $em_t(G)$.

The following theorems will be used in the sequel.

Theorem 1.1 [7] : Let G be a connected graph with cut vertices and let S be a edge monophonic set of G. If v is a cut vertex of G, then every component $G - v$ contain an element of S.

Theorem 1.2 [9] : Each extreme vertex of a connected graph G belongs to every monophonic set of G.
Throughout this paper G denotes a connected graph with atleast two vertices.

2. TOTAL EDGE MONOPHONIC NUMBER

Definition 2.1 : Let G be a connected graph with atleast two vertices. A total edge monophonic set of a graph G is a monophonic set S such that the sub graph induced by S has no isolated vertices. The minimum cardinality of a total edge monophonic set of G is the total edge monophonic number of G, and is denoted by $em_t(G)$.

Example 2.2 : For the graph G given in Figure 2.1, it is clear that $S_1 = \{v_1, v_5, v_6\}$ is a minimum monophonic set of G and so $m(G) = 3$, $S_2 = \{v_1, v_5, v_6\}$ is a minimum edge monophonic set of G and so $em(G) = 3$, and $S_3 = \{v_1, v_5, v_6, v_7\}$ is a minimum total edge monophonic set of G and so $em_t(G) = 4$.

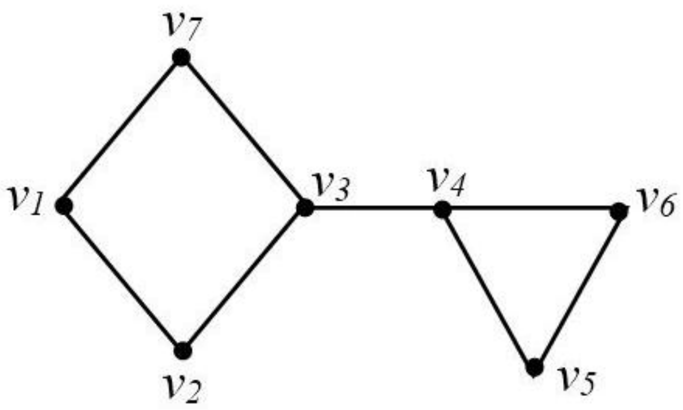

Figure 2.1 : G

Theorem 2.3 : Each extreme vertex and each support vertex of a connected graph G belong to every total edge monophonic set of G. If the set of all extreme vertices and support vertices form a total edge monophonic set, then it is the unique minimum total edge monophonic set.

Proof : Since every total edge monophonic set is a monophonic set, each extreme vertex belongs to every total edge monophonic set. Since a total edge monophonic set contains no isolated vertices, it follows that each support vertex of G also belongs to every total edge monophonic set.

Theorem 2.4: Let G be a connected graph with cut vertices and let S be a total edge monophonic set of G. If v is a cut vertex of G, then every component of G-v contains an element of S.

Proof : Since every total edge monophonic set of G is a edge monophonic set of G, the result follows from theorem 1.1.

Theorem 2.5 : For a connected graph G of order p, $2 \leq m(G) \leq em(G) \leq em_t(G) \leq p$
Proof : Any monophonic set needs atleast 2 vertices and so $m(G) \geq 2$. Since every total edge monophonic set is also a monophonic set, it follows that $m(G) \leq em_t(G)$.
Hence $2 \leq m(G) \leq em(G) \leq em_t(G) \leq p$.

Corollary 2.6 : Let G be a connected graph. If $em_t(G) = 2$, then $m(G) = 2$

Corollary 2.7 : For the complete graph, K_p $(p \geq 2)$, $em_t(G) = p$

Remark 2.8 : For the complete graph $G = K_2$, $em_t(G) = 2$ and for the complete graph K_p, $em_t(k_p) = p$, so that the total monophonic number of a graph attains its least value 2 and largest value p.

For the graph G given in figure 2.2, $m(G) = 3$, $em_t(G) = 5$ and $p = 7$, so that
$2 < m(G) < em_t(G) < p$.

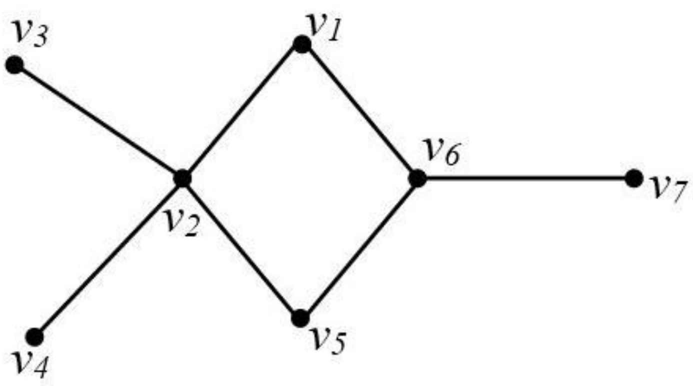

Figure : 2.2

Theorem 2.9 : For any non-trivial tree T, the set of all end vertices and support vertices of T is the unique minimum total edge monophonic set of G.
Proof : Since the set of all end vertices and support vertices of T forms a total edge monophonic set of G, the result follows from theorem 2.3.

Theorem 2.10 : For any connected graph G, $em_t(G) = 2$ if and only if $G = K_2$.
Proof : If $G = K_2$, them $em_t(G) = 2$. Conversely, let $em_t(G) = 2$. Let $S = \{v_1, v_2\}$ be a minimum total edge monophonic set of G. Then v_1v_2 is an edge. It is clear that a vertex different from u and v cannot lie on a u - v monophonic path and so $G = K_2$.

Theorem 2.11 : For the cycle $G = C_n$ ($n \geq 4$), $em_t(G) = 3$.

Proof : First, suppose that $G = C_3$, it is a complete graph, by corollary 2.6, we have $em_t(G) = 3$. For any cycle C_n ($n \geq 5$), it is easily verified that any three consecutive vertices of C_n is a minimum total edge monophonic set of C_n and so $em_t(G) = 3$.

Theorem 2.12 : For the complete bipartite graph $G = K_{r,s}$ ($2 \leq r \leq s$),

$$em_t(G) = \begin{cases} 3 & if\ 2 \leq r \leq s \\ 4 & if\ 3 \leq r \leq s \end{cases}$$

Proof : We prove this theorem by considering three cases. Let X and Y be the partite sets with $|X| = r$, $|Y| = s$.

Case 1 : $2 = r = s$, then G is cycle C_4 and by theorem 2.11, $em_t(G) = 3$.

Case 2 : $2 = r < s$, then the minimum total edge monophonic set of G is got by choosing the two elements from X and any one element from Y and so $em_t(G) = 3$.

Case 3 : $3 \leq r \leq s$, then any minimum total edge monophonic set of G is got by choosing two elements from each of X and Y so that $em_t(G) = 4$.

3. REALIZATION RESULTS

Theorem 3.1 : For every pair k, p of integers with $3 \leq k \leq p$, there exists a connected graph G of order p such that $em_t(G) = k$.

Proof : Let $P : u_1, u_2, u_3$ be a path of length 2. Let G be the graph given in figure 2.3 obtained from P by adding the new vertices $w_1, w_2, \ldots w_{k-3}$ and joining each v_i ($1 \leq i \leq k-3$) with u_1, and also, adding the new vertices $v_1, v_2, v_3, \ldots, v_{p-k}$ and joining each v_i ($1 \leq i \leq p-k$) with u_1 and u_3.

Let $S = \{w_1, w_2, \ldots w_{k-3}\}$ be the set of all extreme vertices of G. By theorem 1.2, every monophonic set, contains S. Clearly, S is not a monophonic set of G. It is clear that $S_1 = S \cup \{u_3\}$ is both the unique minimum monophonic set and unique minimum edge monophonic set of G. Also $S_2 = S_1 \cup \{u_1, u_2\}$ is a minimum total edge monophonic set of G and so $em_t(G) = k\text{-}3 + 1 + 2 = k$.

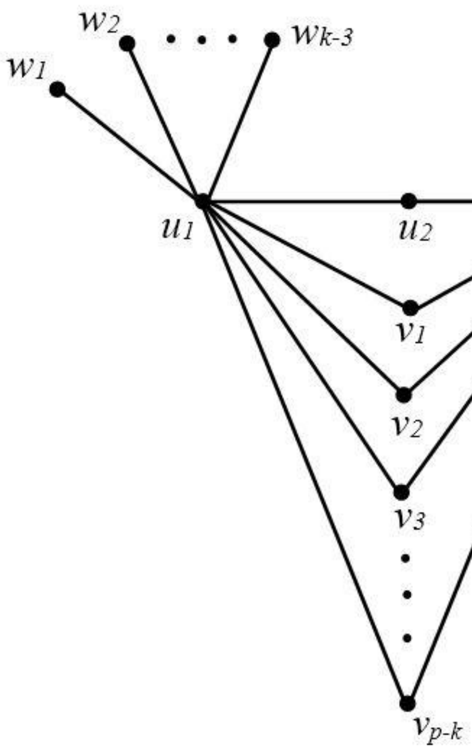

Figure 3.1 : G

Theorem 3.2 : For any positive integers $2 \leq a \leq b < c$, there exists a connected graph G such that $m(G) = a$, $em(G) = b$, $em_t(G) = c$.

Proof : Case 1. $a = b < c$.

Take a copy of star $K_{1,a}$ with leaves $x_1, x_2, \ldots.. x_a$ and the support vertex x. Subdivide the edge xx_i, where $1 \leq i \leq c - b - 1$, calling the new vertices $y_1, y_2, \ldots, y_{c-b-1}$, where x_i is adjacent to y_i and y_i is adjacent to x for all $i \in \{1, 2, \ldots, c\text{-}b\text{-} 1\}$. The graph G is shown in figure 3.2.

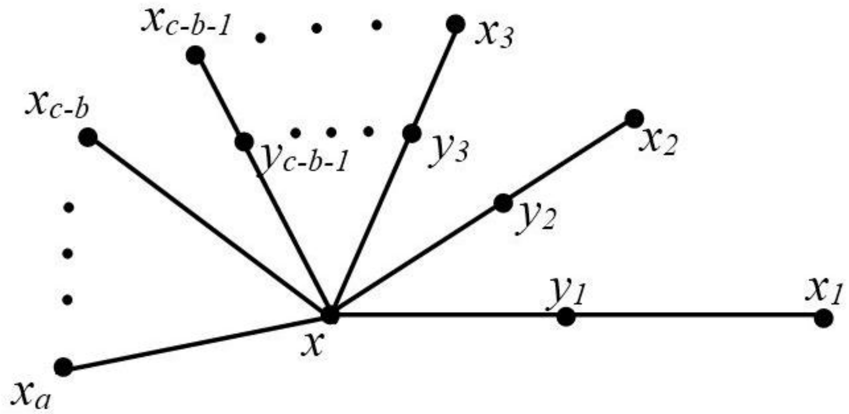

Figure 3.2 : G

Let S = {x_2, x_3, …, x_a} be the set of all extreme vertices of *G*. By Theorem 1.2, *S* is a subset of every monophonic set of *G*. It is clear that $S_1 = S \cup \{x_1\}$ is both the unique minimum monophonic set and unique minimum edge monophonic set of *G* and so $m(G) = em(G) = a$. Also $S_2 = S_1 \cup \{y_1, y_2, \ldots, y_{c-b-1}, x\}$ is a minimum total edge monophonic set of *G* and so $em_t(G) = c$.

Case 2. $2 < a < b < c$ **.**

Take a copy of star $K_{1,a}$ with leaves $x_1, x_2, \ldots x_a$ and the support vertex *x*. Subdivide the edge xx_i, where $1 \le i \le c - b - 1$, calling the new vertices $y_1, y_2, \ldots, y_{c-b-1}$ where x_i is adjacent to y_i and y_i is adjacent to *x* for all $i \in \{1, 2, \ldots c\text{-}b\text{-}1\}$. Let *G* be the graph obtained by adding $b - a$ new vertices $w_1, w_2, \ldots w_{b-a}$ and joining each w_i ($1 \le i \le b - a$) with x, x_1 and y_1. The graph *G* is shown in figure 3.3.

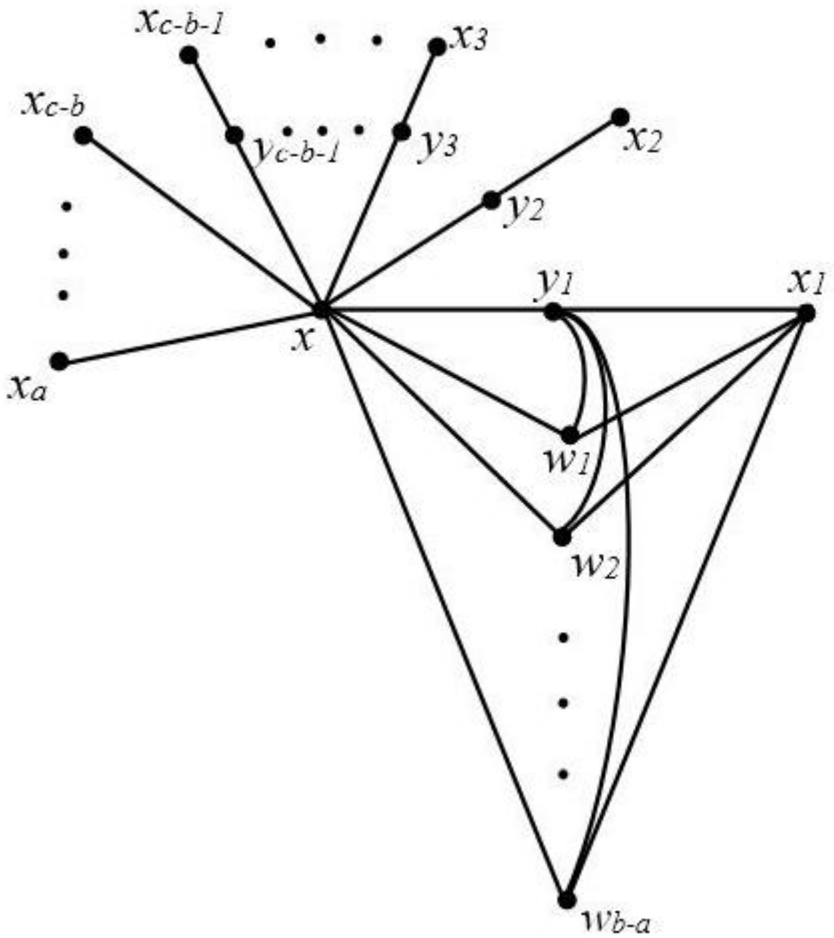

Figure 3.3 : G

Let $S = \{x_2, x_3 \dots x_a\}$ be the set of all extreme vertices of G. It is clear that S is not a monophonic set of G. By theorem 1.2, every monophonic set of G contains S. Clearly $S_1 = S \cup \{x_1\}$ is a monophonic set of G, so that $m(G) = a$.
Let $S_2 = S_1 \cup \{w_1, w_2 \dots w_{b-a}\}$. It is clear that S_2 is an edge monophonic set of G so that $em(G) = a+b-a = b$. Let $S_3 = S_2 \cup \{y_1, y_2 \dots y_{c-b-1}, x\}$. By theorem 2.3, every total edge monophonic set of G contains S. It is clear that S_3 is a minimum total edge monophonic set of G, So that $em_t(G) = b+c-b-1+1=c$.

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