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Related Fixed Point Theorems for Commuting and Weakly Compatible maps in Complete Multiplicative Metric Space

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Abstract

Retaining the concept of multiplicative metric spaces introduced by M. Özavsar, in this paper first we establish common fixed point theorem for six maps and in next theorem theorems four self maps are used to fix a common point in complete multiplicative metric space satisfying commuting and weakly compatible mappings using different type of inequality.

Keywords: Commuting mapping, weakly compatible maps, and common fixed points, multiplicative metric spaces.
MSC.46S40, 47H10, 54H25

INTRODUCTION

Abounding researchers extended the notion of a metric space such as vector valued metric space of Perov [9], a cone metric spaces of Huang and Zhang [2], a modular metric spaces of Chistyakov [3], etc. It is well know that the set of positive real numbers $\mathbb{R}_+$ is not complete according to the usual metric. To overcome this problem, In 2008, Bashirov [8] Introduced the concept of multiplicative metric spaces as follows:

Definition 1.1.[7] Let X be a nonempty set. Multiplicative metric [1] is a mapping d : X ×X → $\mathbb{R}_+$ satisfying the following conditions

(m1) d(x, y) ≥ 1 for all x, y ∈ X and d(x, y) = 1 if and only if x = y,

(m2) d(x, y) = d(y, x) for all x, y ∈ X,

(m3) d(x, z) ≤ d(x, y)·d(y, z) for all x, y, z ∈ X (multiplicative triangle inequality)

To articulate the importance of this study, we should first note that $\mathbb{R}_+$ is a complete multiplicative metric space with respect to the multiplicative metric.

Definition 1.2. [7] *Let S, T be self-maps of multiplicative metric space (X, d), then S,T are said to be compatible if* $\lim_{n\to\infty} d(STx_n, TSx_n)=0$. *Whenever* $\{x_n\}$ *is a sequence in X such that* $\lim_{n\to\infty} S\,x_n = \lim_{n\to\infty} T\,x_n = z$, *for some z ∈ X*

Definition 1.3. [9] *Two self-maps of multiplicative metric space S, T of a non-empty set X are said to be weakly compatible is STx=TSx whenever Sx=Tx.*

MAIN RESULTS

Theorem 2.1 *let (X, d) be a complete multiplicative metric space and P, Q, R, S, T and U be self-maps of X satisfying the following condition*

(1) $TU(X) \subseteq P(X)$ and $RS(X) \subseteq Q(X)$ *and*

(2) $$d(RSx, TUy) \le \begin{pmatrix} d(Px,Qy).\, d(Px,RSx).\, d(Qy,TUy). \\ d(Px,TUy).\, d(Qy,RSx).\, d(TUy,RSx) \end{pmatrix}^{\frac{\lambda}{3}}$$

for all x, y ∈ X, $\lambda \in [0,\frac{1}{4})$ *is a constant. Assume that the pairs (TU,Q), (RS,P) are weakly compatible. Pairs (T,U) , (T,Q), (U,Q), (R,S), (R,P)and (S,P) are commuting pairs of maps. Then P,Q,R,S,T and U have a unique common fixed point in X.*

Proof*: Let* $x_0 \in X$. *by (2) we can define inductively a sequence* $\{y_n\}$ *in X such that*

$y_{2n}=RSx_{2n}=Qx_{2n+1}$ and $TUx_{2n+1}=Px_{2n+2}= y_{2n+1}$ *for all n=1, 2, 3 …*

$$d(y_{2n},y_{2n+1}) = d(RSx_{2n},TUx_{2n+1})$$

$$\leq \begin{pmatrix} d(Px_{2n}, Qx_{2n+1}).\, d(Px_{2n}, RSx_{2n}).\, d(Qx_{2n+1}, TUx_{2n+1}). \\ d(Px_{2n}, TUx_{2n+1}). \\ d(Qx_{2n+1}, RSx_{2n}).\, d(TUx_{2n+1}, RSx_{2n}) \end{pmatrix}^{\frac{\lambda}{3}}$$

$$\leq \begin{pmatrix} d(y_{2n-1}, y_{2n}).\, d(y_{2n-1}, y_{2n}).\, d(y_{2n}, y_{2n+1}).\, d(y_{2n-1}, y_{2n+1}). \\ d(y_{2n}, y_{2n}).\, d(y_{2n+1}, y_{2n}) \end{pmatrix}^{\frac{\lambda}{3}}$$

$$\leq \begin{pmatrix} d(y_{2n-1}, y_{2n}).\, d(y_{2n-1}, y_{2n}).\, d(y_{2n}, y_{2n+1}).\, d(y_{2n-1}, y_{2n}). \\ d(y_{2n}, y_{2n+1}).\, d(y_{2n+1}, y_{2n}) \end{pmatrix}^{\frac{\lambda}{3}}$$

$$d(y_{2n}, y_{2n+1}) \leq \begin{pmatrix} d^3(y_{2n-1}, y_{2n}). \\ d^3(y_{2n}, y_{2n+1}) \end{pmatrix}^{\frac{\lambda}{3}}$$

$$\leq \begin{pmatrix} d(y_{2n-1}, y_{2n}). \\ d(y_{2n}, y_{2n+1}) \end{pmatrix}^{\lambda} d(y_{2n}, y_{2n+1})$$

$$\leq d^{\frac{\lambda}{1-\lambda}}(y_{2n-1}, y_{2n})$$

$$d(y_{2n}, y_{2n+1}) \leq d^h(y_{2n-1}, y_{2n}),$$

where $h = \frac{\lambda}{1-\lambda}$. *Similarly we have,*

$$d(y_{2n+1}, y_{2n+2}) = d(TUx_{2n+1}, RSx_{2n+2})$$

$$= d(RSx_{2n+2}, TUx_{2n+1})$$

$$\leq \begin{pmatrix} d(Px_{2n+2}, Qx_{2n+1}).\, d(Px_{2n+2}, RSx_{2n+2}).\, d(Qx_{2n+1}, TUx_{2n+1}). \\ d(Px_{2n+2}, TUx_{2n+1}). \\ d(Qx_{2n+1}, RSx_{2n+2}).\, d(TUx_{2n+1}, RSx_{2n+2}) \end{pmatrix}^{\frac{\lambda}{3}}$$

$$\leq$$

$$\begin{pmatrix} d(y_{2n+1}, y_{2n}).\, d(y_{2n+1}, y_{2n+2}).\, d(y_{2n}, y_{2n+1}).\, d(y_{2n+1}, y_{2n+1}). \\ d(y_{2n}, y_{2n+2}).\, d(y_{2n+1}, y_{2n+2}) \end{pmatrix}^{\frac{\lambda}{3}}$$

$$d(y_{2n+1}, y_{2n+2}) \leq d^{\frac{\lambda}{1-\lambda}}(y_{2n}, y_{2n+1})$$

$$d(y_{2n+1}, y_{2n+2}) \leq d^h(y_{2n}, y_{2n+1})$$

where $h = \frac{\lambda}{1-\lambda}$, *Therefore,*

$$d(y_{n+1}, y_{n+2}) \leq d^{h}(y_n, y_{n+1}) \leq d^{h^2} d(y_{n-1}, y_n) \leq \cdots \leq d^{h^{n+1}}(y_0, y_1) \text{ for}$$

n=1, 2, 3, ...

now, for all m>n

$$d(y_n, y_m) \leq d(y_n, y_{n+1}).d(y_{n+1}, y_{n+2}).d(y_{n+2}, y_{n+3})\ldots d(y_{m-1}, y_m)$$

$$d(y_n, y_m) \leq (d(y_0, y_1)^{(h^{n-1}+h^{n-2}+h^{n-3}+\cdots+h^{m})} \leq (d(y_0, y_1)^{(\frac{h^m}{h-1})}$$

Which implies that, $d(x_n, x_m) \to 1$ *as* $(n, m \to \infty)$. *Hence* $\{y_n\}$ *is a Cauchy sequence, by the completeness of X, there exist z* ∈ *X such that,*

$$\lim_{n\to\infty} RSx_{2n} = \lim_{n\to\infty} Qx_{2n+1} = \lim_{n\to\infty} TUx_{2n+1} = \lim_{n\to\infty} Px_{2n+2} = z$$

Since, TU(X) ⊆ *P(X) there exist u*∈ X *such that* $z = Pu$, *then by equation* (1)

$$d(RSu,z) = d(RSu,TUx_{2n-1}).d(TUx_{2n-1}, z)$$

$$\leq \begin{pmatrix} d(Pu, Qx_{2n-1}).\, d(Pu, RSu).\, d(Qx_{2n-1}, TUx_{2n-1}) \\ .\, d(Pu, TUx_{2n-1}). \\ d(Qx_{2n-1}, RSu).\, d(TUx_{2n-1}, RSu) \end{pmatrix}^{\frac{\lambda}{3}} .d(TUx_{2n-1}, z)$$

Taking the limit as n→ ∞

$$d(RSu,z) \leq \begin{pmatrix} d(z,z).\, d(z, RSu).\, d(z,z).\, d(z,z). \\ d(z, RSu).\, d(z, RSu) \end{pmatrix}^{\frac{\lambda}{3}} .d(z,z)\,)$$

$$\leq (d(RSu, z)\,)^{\lambda}$$

which is a contradiction. $Therfore$, *RSu = Pu = z*

Since, RS(X)⊆Q(X) there exist v∈ X *such that* $z = Qv$, *then by equation* (1)

$$d(z,TUv) = d(RSu,TUv)$$

$$\leq \begin{pmatrix} d(Pu, Qv).\, d(Pu, RSu).\, d(Qv, TUv).\, d(Pu, TUv). \\ d(Qv, RSu).\, d(TUv, RSu) \end{pmatrix}^{\frac{\lambda}{3}}$$

Taking the limit as n→ ∞

$$d(z,TUv) \leq \begin{pmatrix} d(z,z).\, d(z,z).\, d(z, TUv).\, d(z, TUv). \\ d(z,z).\, d(TUv, z) \end{pmatrix}^{\frac{\lambda}{3}})$$

$$\leq \left(d^3(z, TUv)\right)^{\frac{\lambda}{3}})$$

$$\leq (d(z, TUv)\,)^{\lambda}$$ *, which is a contradiction.*

$therfore$, *TUv = Qv = z so, Pu = TUv = Qv=z*

Similarly, Q and TU are weakly compatibles, we have TUz = Qz.

Now we claim that z is a fixed point TU. If ≠z, then by (1), we have

$$d(z,TUz) = d(RSz, TUz)$$

$$\leq \begin{pmatrix} d(Pz,Qz).\,d(Pz,RSz).\,d(Qz,TUz). \\ d(Pz,TUz).\,d(Qz,RSz).\,d(TUz,RSz) \end{pmatrix}^{\frac{\lambda}{3}}$$

$$\leq \begin{pmatrix} d(z,TUz).\,d(z,z).\,d(TUz,TUz). \\ d(z,TUz).\,d(TUz,z).\,d(TUz,z) \end{pmatrix}^{\frac{\lambda}{3}}$$

$d(z,TUz) \leq \big(d(TUz,z)\big)^{\frac{4\lambda}{3}}$, *a contradiction.*

Therefore, TUz=z, hence Qz=z. we have therefore proved that TUz=Pz=Qz=z. So z is a common fixed point of TU, Q, P and RS.

By commuting property,

Tz=T(Tz)=T(UTz)=TU(Tz)

Tz=T(Pz)=P(Tz) and Uz=U(TUz)=(UT)(Uz)=(TU)(Uz)

(Uz)=(U(Pz)=P(Uz)

which follows that, Tz and Uz are common fixed points of (TU, P)

Then Tz=z=Uz=Pz=TUz

Similarly, By commuting property,

Rz=R(Rz)=R(SRz)=RS(Rz)

Rz=R(Qz)=Q(Rz) and Sz=S(RSz)=(SR)(Sz)=(RS)(Sz)

(Sz)=(S(Qz)=Q(Sz) which follows that, Rzand Sz are common fixed points of (RS, Q)

Then,

Rz=z=Sz=Qz=RSz

Therefore z is a common fixed point of T, U, R, S, P and Q.

For Uniqueness of z, Let w be other common fixed point of T, U, R, P, S and Q. Then by (1)

$$d(z,w) = d(RSz,TUw)$$

$$\leq \begin{pmatrix} d(Pz,Qw).\,d(Pz,RSz).\,d(Qw,TUw). \\ d(Pz,TUw).\,d(Qw,RSz).\,d(TUw,RSz) \end{pmatrix}^{\frac{\lambda}{3}}$$

$$= \begin{pmatrix} d(z,w).\,d(z,z).\,d(w,w). \\ d(z,w).\,d(w,z).\,d(w,z) \end{pmatrix}^{\frac{\lambda}{3}}$$

d(z,w) $\leq \left(d(z,w)\right)^{\frac{4\lambda}{3}}$, *a contradiction. So z = w*

Corollary 2.2 *let (X, d) be a complete multiplicative metric space and P, Q, S and T be self-maps of X satisfying the following condition*

(1) T(X)⊆P(X) and S(X)⊆Q(X) and

$$d(Sx,Ty) \leq \begin{pmatrix} d(Px,Qy).\,d(Px,Sx).\,d(Qy,Ty). \\ d(Px,Ty).\,d(Qy,Sx).\,d(Ty,Sx) \end{pmatrix}^{\frac{\lambda}{3}}$$

For all x, y $\in X$, $\lambda \in (0,\frac{1}{4})$ *is a constant. Then P, Q,S and T have a unique common fixed point in X.*

Proof*: Let* $x_0 \in X$. *by (1) we can define inductively a sequence* $\{y_n\}$ *in X such that* $y_{2n}=Sx_{2n}=Qx_{2n+1}$ *and* $Tx_{2n+1}=Px_{2n+2}= y_{2n+1}$ *for all n=1, 2, 3 ..., using (1)*

$$d(y_{2n},y_{2n+1}) = d(Sx_{2n},Tx_{2n+1})$$

$$\leq \begin{pmatrix} d(Px_{2n},Qx_{2n+1}).\,d(Px_{2n},Sx_{2n}).\,d(Qx_{2n+1},Tx_{2n+1}). \\ d(Px_{2n},Tx_{2n+1}). \\ d(Qx_{2n+1},Sx_{2n}).\,d(TUx_{2n+1},Sx_{2n}) \end{pmatrix}^{\frac{\lambda}{3}}$$

$$\leq \begin{pmatrix} d(y_{2n-1},y_{2n}).\,d(y_{2n-1},y_{2n}).\,d(y_{2n},y_{2n+1}).\,d(y_{2n-1},y_{2n+1}). \\ d(y_{2n},y_{2n}).\,d(y_{2n+1},y_{2n}) \end{pmatrix}^{\frac{\lambda}{3}}$$

$$\leq \begin{pmatrix} d(y_{2n-1},y_{2n}).\,d(y_{2n-1},y_{2n}).\,d(y_{2n},y_{2n+1}).\,d(y_{2n-1},y_{2n}). \\ d(y_{2n},y_{2n+1}).\,d(y_{2n+1},y_{2n}) \end{pmatrix}^{\frac{\lambda}{3}}$$

$$\leq d^{\frac{\lambda}{1-\lambda}}(y_{2n-1},y_{2n}) \leq d^{h}(y_{2n-1},y_{2n}),$$

where $h=\frac{\lambda}{1-\lambda}$. *Similarly we have,*

$$d(y_{2n+1},y_{2n+2}) = d(Sx_{2n+1},Tx_{2n+2}) \leq$$

$$\begin{pmatrix} d(Px_{2n+2},Qx_{2n+1}).\,d(Px_{2n+2},Sx_{2n+2}).\,d(Qx_{2n+1},Tx_{2n+1}). \\ d(Px_{2n+2},Tx_{2n+1}). \\ d(Qx_{2n+1},Sx_{2n+2}).\,d(Tx_{2n+1},Sx_{2n+2}) \end{pmatrix}^{\frac{\lambda}{3}}$$

$$\leq \begin{pmatrix} d(y_{2n+1},y_{2n}).\,d(y_{2n+1},y_{2n+2}).\,d(y_{2n},y_{2n+1}).\,d(y_{2n+1},y_{2n+1}). \\ d(y_{2n},y_{2n+2}).\,d(y_{2n+1},y_{2n+2}) \end{pmatrix}^{\frac{\lambda}{3}}$$

$$\leq \begin{pmatrix} d(y_{2n}, y_{2n+1}).\, d(y_{2n+1}, y_{2n+2}).\, d(y_{2n}, y_{2n+1}).\, 1. \\ d(y_{2n}, y_{2n+1}).\, d(y_{2n+1}, y_{2n+2}).\, d(y_{2n+1}, y_{2n+2}) \end{pmatrix}^{\frac{\lambda}{3}}$$

$$\leq \begin{pmatrix} d(y_{2n}, y_{2n+1}). \\ d(y_{2n+1}, y_{2n+2}) \end{pmatrix}^{\lambda}$$

$$\leq d^{\frac{\lambda}{1-\lambda}}(y_{2n}, y_{2n+1})$$

$$d(y_{2n+1}, y_{2n+2}) \leq d^{h}(y_{2n}, y_{2n+1}) \leq d^{h^2}(y_{2n}, y_{2n+1}),$$

Where $h=\frac{\lambda}{1-\lambda}$

Therefore, $d(y_{n+1}, y_{n+2}) \leq d^{h}(y_n, y_{n+1}) \leq d^{h^2} d(y_{n-1}, y_n) \leq \cdots \leq d^{h^{n+1}}(y_0, y_1)$

for n=0,1, 2, 3, … and m>n

$$d(y_n, y_m) \leq d(y_n, y_{n+1}).d(y_{n+1}, y_{n+2}).d(y_{n+2}, y_{n+3})\dots d(y_{m-1}, y_m)$$

$$d(y_n, y_m) \leq (d(y_0, y_1)^{(h^{n-1}+h^{n-2}+h^{n-3}+\dots+h^m)} \leq (d(y_0, y_1)^{(\frac{h^m}{h-1})}$$

which implies that, $d(x_n, x_m) \to 1$ *as* $(n, m \to \infty)$*. Hence* $\{y_n\}$ *is a Cauchy sequence, by the completeness of X, there exist z∈X such that,*

$$\lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Qx_{2n+1} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} Px_{2n+2} = z$$

Since, $T(X) \subseteq P(X)$ *there exist* $u \in X$ *such that* $z = Pu$, *then by equation* (1)

$$d(Su,z) = d(Su, Tx_{2n-1}).d(Tx_{2n-1}, z)$$

$$\leq$$

$$\begin{pmatrix} d(Pu, Qx_{2n-1}).\, d(Pu, Su).\, d(Qx_{2n-1}, Tx_{2n-1}). \\ d(Pu, Tx_{2n-1}).\, d(Qx_{2n-1}, Su).\, d(Tx_{2n-1}, Su) \end{pmatrix}^{\frac{\lambda}{3}} .d(Tx_{2n-1}, z)$$

Taking the limit as $n \to \infty$

$$d(Su,z) \leq \begin{pmatrix} d(z,z).\, d(z,Su).\, d(z,z).\, d(z,z). \\ d(z,Su).\, d(z,Su) \end{pmatrix}^{\frac{\lambda}{3}} .d(z,z)$$

$$\leq \left(d^3(Su,z)\right)^{\frac{\lambda}{3}}$$

$$\leq (d(Su,z))^{\lambda}$$

which is a contradiction. Therfore,

$$Su = Pu = z$$

Since, S(X)⊆Q(X) there exist v$\in X$ *such that* $z = Qv$, *then by equation* (1)

$$d(z,Tv) = d(Su,Tv)$$

$$\leq \begin{pmatrix} d(Pu,Qv).\,d(Pu,Su).\,d(Qv,Tv).\,d(Pu,Tv). \\ d(Qv,Su).\,d(Tv,Su) \end{pmatrix}^{\frac{\lambda}{3}}$$

Taking the limit as n→ ∞

$$d(z,Tv) \leq \begin{pmatrix} d(z,z).\,d(z,z).\,d(z,Tv).\,d(z,Tv). \\ d(z,z).\,d(Tv,z) \end{pmatrix}^{\frac{\lambda}{3}}$$

$$\leq (d(z,Tv))^{\lambda}$$ *, which is a contradiction.*

$therfore$, *Tv = Qv = z so, Pu = Tv = Qv = z*

Similarly, Q and TU are weakly compatibles, we have Tz=Qz.

Now we claim that z is a fixed point TU. If ≠ z, then by (1), we have

$$d(z,Tz) = d(Sz, Tz)$$

$$\leq \begin{pmatrix} d(Pz,Qz).\,d(Pz,Sz).\,d(Qz,Tz). \\ d(Pz,Tz).\,d(Qz,Sz).\,d(Tz,Sz) \end{pmatrix}^{\frac{\lambda}{3}}$$

$$\leq \begin{pmatrix} d(z,Tz).\,d(z,z).\,d(Tz,Tz). \\ d(z,Tz).\,d(Tz,z).\,d(Tz,z) \end{pmatrix}^{\frac{\lambda}{3}}$$

$$\leq \left(d(Tz,z)\right)^{\frac{4\lambda}{3}},$$

a contradiction. Therefore, Tz = z, hence Qz = z. we have therefore proved that Tz = Pz = Qz = z. So z is a common fixed point of T, Q, P and S.

$$d(z,w) = d(Sz,Tw) \leq \begin{pmatrix} d(Pz,Qw).\,d(Pz,Sz).\,d(Qw,Tw). \\ d(Pz,Tw).\,d(Qw,Sz).\,d(Tw,Sz) \end{pmatrix}^{\frac{\lambda}{3}}$$

$$= \begin{pmatrix} d(z,w).\,d(z,z).\,d(w,w). \\ d(z,w).\,d(w,z).\,d(w,z) \end{pmatrix}^{\frac{\lambda}{3}}$$

$$d(z,w) \leq \left(d(z,w)\right)^{\frac{4\lambda}{3}}$$ *, a contradiction. So z = w*

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