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Different Inequality and Common Fixed Points in Complex Valued Metric Space

[1]Nisha Sharma, [2]Isha and [3]Arti Mishra,

[1,3]Department of Mathematics, Manav Rachna International University, Faridabad, Haryana, India.
[2]Department of Mathematics, Pt. J.L.N Govt. College Faridabad, Haryana, India.

Abstract

This is the paper in which some common fixed point theorem for four maps in the setting of complex valued metric spaces satisfying commuting and weakly compatible conditions using different type of inequality. This theorem concludes and broadens the result of Rahul Tiwari et. al. [7] and Sandeep Bhatt et.al [8].

Keyword: fixed points, common fixed points, complex valued metric space, weakly compatible maps

INTRODUCTION AND PRELIMINARIES

The very useful concept of multiplicative calculus has studied and proved by Bashirov et al. [11]. Bashirov et al. [13] exploit the efficiency of multiplicative calculus over the Newtonian calculus. They determine that the multiplicative differential equations are more suitable than the ordinary differential equations in investigating some problems in various fields. Furthermore, Bashirov et al. [11] illustrated the effectiveness of multiplicative calculus with some alluring applications. Multiplicative absolute value functions useful to define the multiplicative distance between two nonnegative real numbers as well as between two positive square matrices. This provides the basis for multiplicative metric spaces.

Many common fixed point theorems have established and continued the approach of a metric space such as vector valued metric space of Perov [2], G-metric spaces of

Mustafa and Sims [10], a cone metric space of Huang and Zhang [6], a modular metric space of Chistyakov [9] and etc. A. Azam. B. fisher and M. khan [1] first introduced the brain wave of complex valued metric space which is more general than well-known metric spaces. Many fixed point theorems in different metric spaces have proven in several papers e.g. [3,4,5]

Let us recall a natural relation on $\mathbb{C}$,for $z_1,z_2 \in \mathbb{C}$, define a partial order $\precsim$ on $\mathbb{C}$ as follows;

$$z_1 \precsim z_2 \text{ iff } Re(z_1) \precsim Re(z_2),\ Im(z_1) \precsim Im(z_2)$$

it follows that

$$z_1 \precsim z_2$$

if one of the following conditions is satisfied:

i. $Re(z_1) = Re(z_2),\ Im(z_1) < Im(z_2)$

ii. $Re(z_1) < Re(z_2),\ Im(z_1) = Im(z_2)$

iii. $Re(z_1) < Re(z_2),\ Im(z_1) < Im(z_2)$

iv. $Re(z_1) = Re(z_2),\ Im(z_1) = Im(z_2)$

In particular, we will write $z_1 \precnsim z_2$ *if* $z_1 \neq z_2$ *and one the above conditions is not satisfied and we will write* $z_1 \prec z_2$ *if only (iii) is satisfied. Note that*

$0 \precsim z_1 \precnsim z_2 \Rightarrow |z_1| < |z_2|$,

$z_1 \precsim z_2\ ,\ z_2 \prec z_3 \Rightarrow z_1 \prec z_3$

Definition 1.1 *Let X be a nonempty set. A mapping* $d : X \times X \to \mathbb{C}$ *is called a complex valued metric on X if the following conditions are satisfied:*

(CM1) $0 \precsim d(x,y)$ *for all* $x,y \in X$ *and* $d(x,y)=0 \Leftrightarrow x = y$.

(CM2) $d(x,y) = d(y,x)$ *for all* $x, y \in X$

(CM3) $d(x,y) \precsim d(x,z)+d(z,y)$ *for all* $x, y, z \in X$.

In this case, we say that (X, d) is a complex valued metric space.

It is obvious that this concept is generalization of the classic metric. In fact, if $d : X \times X \to \mathbb{R}_+$ *satisfies((CM1)-(CM3)), then this d is a metric in the classical sense, that is, the following conditions are satisfies:*

(M1) $0 \precsim d(x,y)$ *for all* $x,y \in X$ *and* $d(x,y) = 0 \Leftrightarrow x = y$.

(M2) $d(x,y) = d(y,x)$ for all $x,y \in X$

(M3) $d(x,y) \precsim d(x,z)+d(z,y)$ for all $x,y,z \in X$.

There are so many more different and interesting type of metric spaces and classical theories of metric space for example see [7, 10].

Definition 1.2 *Let $\mathbb{C}$ be a complex valued metric space then,*

i. *We say that a sequence $\{x_n\}$ is said to be a Cauchy sequence be a sequence in $x \in X$ If for every $\varepsilon \in \mathbb{C}$, with $0 \prec \varepsilon$ there is $n_0 \in N$ such that $d(x_n,x_m) \prec \varepsilon$ for all $n > n_0$*
ii. *We say that a sequence $\{x_n\}$ converges to an element x If for every $x \in \mathbb{C}$, with $0 \prec \varepsilon$ there exist an integer $n_0 \in N$ such that $d(x_n,x) \prec \varepsilon$ for all $n > n_0$ and we write $x_n \xrightarrow[d]{} x$.*
iii. *We say that (X, d) is complete if every Cauchy sequence in X converges to a point in X.*

Example 1.3 *Let $X=\mathbb{C}$. Define the mapping $d : X \times X \to \mathbb{C}$ by*

$d(z_1, z_2) = e^{pi|z_1 - z_2|}$, for all $z_1, z_2 \in X$ and $p \in \mathbb{R}$

Then (X, d) is a complex valued metric space.

Lemma 1.4 *Any sequence $\{x_n\}$ in complex valued metric space (X, d), converges to x if and only if $|d(x_n,x)| \to 0$ as $n \to \infty$*

Lemma 1.5 *Any sequence $\{x_n\}$ in complex valued metric space (X, d) is a Cauchy sequence if and only if $|d(x_n,x_{n+m})| \to 0$ as $n \to \infty$, where $m \in \mathbb{N}$.*

MAIN RESULT

Theorem 2.1. *Let (X, d) be a complex valued metric space. Let $G, F : X \times X \to X$ and $g, f : X \to X$ are four mappings. Suppose that there exist non-negative constants $a_i \in [0,1)$, $i=1,2,3\ldots,8$ such that $\sum_{i=1}^{8} a_i < 1$ and for all $x, y, u, v \in X$*

(2.1) $d(FG(x,y),FG(u,v)) \precsim a_1 d(fgx,fgu)+a_2 d(fgy,fgv)$

$$+\frac{a_3 d(fgx,FG(y,x))d(fgu,FG(u,v))}{d(fgx,fgv)}$$

$$+\frac{a_4 d(fgx,FG(u,v))d(fgu,FG(x,y))}{d(fgx,fgv)}$$

$$+\frac{a_5 d(fgy,FG(x,y))d(fgv,FG(v,u))}{d(fgy,fgu)}$$

$$+\frac{a_6 d(fgy,FG(v,u))d(fgv,FG(y,x))}{d(fgy,fgu)}$$

$$+\frac{a_7 d(fgx,FG(y,x))d(fgv,FG(v,u))}{d(fgx,fgv)}$$

$$+\frac{a_8 d(fgy,FG(v,u))d(fgu,FG(x,y))}{d(fgy,fgu)}$$

Suppose $G(X \text{ x } X)\ F(X \text{ x } X) \subseteq g(X)\ f(X)$ *and* $g(X)$ *and* $f(X)$ *is a complete subspace of* X*. Then F, G and f, g have a coupled coincidence point* $(x^*, y^*) \in X \text{ x } X$.

Proof *Let* x_0, y_0 *are arbitrary elements of X. Let* $fgx_1=FG(x_0,y_0)$*,* $fgy_1=FG(y_0,x_0)$*, this can be done because* $G(X \text{ x } X)\ F(X \text{ x } X) \subseteq g(X)\ f(X)$*. Continuing this process, we obtain two sequences* $\{x_n\}$ *and* $\{y_n\}$ *such that* $fgx_{n+1} = FG(x_n,y_n)$ *for all* $n \geq 0$*. Then using (2.1) we have*

$$d(fgx_n,fgx_{n+1}) = d\big(FG(x_{n-1},y_{n-1}),FG(x_n,y_n)\big)$$

$$\precsim a_1 d(fgx_{n-1},fgx_n) + a_2 d(fgy_{n-1},fgy_n)$$

$$+\frac{a_3 d(fgx_{n-1},FG(y_{n-1},x_{n-1}))d(fgx_n,FG(x_n,y_n))}{d(fgx_{n-1},fgy_n)}$$

$$+\frac{a_4 d(fgx_{n-1},FG(x_n,y_n))d(fgx_n,FG(x_{n-1},y_{n-1}))}{d(fgx_{n-1},fgy_n)}$$

$$+\frac{a_5 d(fgy_{n-1},FG(x_{n-1},y_{n-1}))d(fgy_n,FG(y_n,x_n))}{d(fgy_{n-1},fgx_n)}$$

$$+\frac{a_6 d(fgx_{n-1},FG(y_n,x_n))d(fgy_n,FG(y_{n-1},x_{n-1}))}{d(fgy_{n-1},fgx_n)}$$

$$+\frac{a_7 d(fgx_{n-1},FG(y_{n-1},x_{n-1}))d(fgy_n,FG(y_n,x_n))}{d(fgx_{n-1},fgy_n)}$$

$$+\frac{a_8 d(fgy_{n-1},FG(y_n,x_n))d(fgx_n,FG(x_{n-1},y_{n-1}))}{d(fgy_{n-1},fgx_n)}$$

$$d(fgx_n, fgx_{n+1}) \precsim a_1 d(fgx_{n-1}, fgx_n) + a_2 d(fgy_{n-1}, fgy_n) + \frac{a_3 d(fgx_{n-1}, fgy_n) d(fgx_n, fgx_{n+1})}{d(fgx_{n-1}, fgy_n)} + \frac{a_4 d(fgx_{n-1}, fgx_{n+1}) d(fgx_n, fgx_n)}{d(fgx_{n-1}, fgy_n)} + \frac{a_5 d(fgy_{n-1}, fgx_n) d(fgy_n, fgy_{n+1})}{d(fgy_{n-1}, fgx_n)} + \frac{a_6 d(fgy_{n-1}, fgy_{n+1}) d(fgy_n, fgy_n)}{d(fgy_{n-1}, fgx_n)} + \frac{a_7 d(fgx_{n-1}, fgy_n) d(fgy_n, fgy_{n+1})}{d(fgx_{n-1}, fgy_n)} + \frac{a_8 d(fgy_{n-1}, fgy_{n+1}) d(fgx_n, fgx_n)}{d(fgy_{n-1}, fgx_n)}$$

$$d(fgx_n, fgx_{n+1}) \precsim a_1 d(fgx_{n-1}, fgx_n) + a_2 d(fgy_{n-1}, fgy_n) + a_3 d(fgx_n, fgx_{n+1}) + a_5 d(fgy_n, fgy_{n+1}) + a_7 d(fgy_n, fgy_{n+1})$$

Which implies that

(2.2) $$|d(fgx_n, fgx_{n+1})| \precsim a_1 |d(fgx_{n-1}, fgx_n)| + a_2 |d(fgy_{n-1}, fgy_n)| + a_3 |d(fgx_n, fgx_{n+1})| + (a_5 + a_7) |d(fgy_n, fgy_{n+1})|$$

Similarly, we can prove that

(2.3) $$|d(fgy_n, fgy_{n+1})| \precsim a_1 |d(fgy_{n-1}, fgy_n)| + a_2 |d(fgx_{n-1}, fgx_n)| + a_3 |d(fgy_n, fgy_{n+1})| + (a_5 + a_7) |d(fgx_n, fgx_{n+1})|$$

Substituting, $d_n = |d(fgx_n, fgx_{n+1})| + |d(fgy_n, fgy_{n+1})|$

adding equations (2.2) and (2.3), we have

$$d_n \precsim (a_1 + a_2) d_{n-1} + (a_3 + a_5 + a_7) d_n$$

$$d_n \precsim \frac{(a_1 + a_2)}{1 - (a_3 + a_5 + a_7)} d_{n-1}$$

$$d_n \precsim h.\, d_{n-1} \text{ where } h = \frac{(a_1 + a_2)}{1 - (a_3 + a_5 + a_7)} < 1$$

$$d_n \precsim h^1.d_{n-1} \precsim h^2.d_{n-2} \precsim h^3.d_{n-3} \precsim \dots \precsim h^n.d_0$$

hence,

(2.4) $$d_n \precsim h^n.d_0$$

to show that $\{x_n\}$ *and* $\{y_n\}$ *are Cauchy sequences, if* $m>n$, *then we have*

$$\begin{aligned}|d(fgx_n,fgx_m)|+|d(fgy_n,fgy_m)| &\precsim |d(fgx_n,fgx_{n+1})| +|d(fgy_n,fgy_{n+1})| +|d(fgx_{n+1},fgx_{n+2})| \\ &\quad + \dots + |d(fgx_{m-1},fgx_m)| +|d(fgy_{m-1},fgy_m)| \\ &\precsim d_n+d_{n+1}+d_{n+2}+d_{n+3}+\dots+d_{m-1} \\ &\precsim h^n.d_0 + h^{n+1}.d_0+h^{n+2}.d_0+h^3.d_0+\dots+h^{m-1}.d_0 \\ &\precsim (h^n+h^{n+1}+\dots+h^{m-1})\, d_0 \\ &\precsim \frac{h^n}{(1-h)} d_0 \to 0 \ as\ n \to \infty\end{aligned}$$

Hence $\{fgx_n\}$ *and* $\{fgy_n\}$ *are Cauchy sequences in* $fg(X)$. *Since* $fg(X)$ *is complete, there exists* x^* *and* $y^* \in X$ *such that* $fgx_n \to fgx^*$ *and* $fgy_n \to fgy^*$ *as* $n \to \infty$

On the other hand, we have

$$\begin{aligned}d(FG(x^*,y^*),fgx^*) &\precsim d(FG(x^*,y^*),fgx_{n+1})+d(fgx_{n+1},fgx^*) \\ &= d(FG(x^*,y^*),FG(x_n, y_n))+d(fgx_{n+1},fgx^*) \\ &\precsim a_1 d(fgx^*,fg x_n)+a_2 d(fg y^*,fgy_n) \\ &\quad + \frac{a_3 d(fgx^*,FG(y^*,x^*))d(fgx_n,FG(x_n,y_n))}{d(fgx^*,fgy_n)} \\ &\quad + \frac{a_4 d(fgx^*,FG(x_n,y_n))d(fgx_n,FG(x^*,y^*))}{d(fgx^*,fgy_n)} \\ &\quad + \frac{a_5 d(fgy^*,FG(x^*,y^*))d(fgx_n,FG(y_n,x_n))}{d(fgy^*,fgx_n)} \\ &\quad + \frac{a_6 d(fgy^*,FG(y_n,x_n))d(fgy_n,FG(y^*,x^*))}{d(fgy^*,fgx_n)} \\ &\quad + \frac{a_7 d(fgx^*,FG(y^*,x^*))d(fgy_n,FG(y_n,x_n))}{d(fgx^*,fgy_n)} \\ &\quad + \frac{a_8 d(fgy^*,FG(y_n,x_n))d(fgx_n,FG(x^*,y^*))}{d(fgy^*,fgx_n)} + d(fgx_{n+1},fgx^*)\end{aligned}$$

[using 2.1]

$$d(FG(x^*,y^*),fgx^*) \precsim a_1 d(fgx^*,fgx_n)+a_2 d(fgy^*,fgy_n)$$
$$+\frac{a_3 d(fgx^*,FG(y^*,x^*))[d(fgx_n,fgx^*)+d(fgx^*,fgx_{n+1})]}{d(fgx^*,fgy_n)}$$
$$+\frac{a_4[d(fgx^*,fgx_{n+1})]d(fgx_n,FG(x^*,y^*)}{d(fgx^*,fgy_n)}$$
$$+\frac{a_5 d(fgy^*,FG(y^*,x^*))[d(fgy_n,fgy^*)+d(fgy^*,fgy_{n+1})]}{d(fgy^*,fgx_n)}$$
$$+\frac{a_6[d(fgy^*,fgy_{n+1})]d(fgy_n,FG(y^*,x^*))}{d(fgy^*,fgx_n)}$$
$$+\frac{a_7 d(fgx^*,FG(y^*,x^*))[d(fgy_n,fgy^*)+d(fgy^*,fgy_{n+1})]}{d(fgx^*,fgy_n)}$$
$$+\frac{a_8[d(fgy^*,fgy_{n+1})]d(fgx_n,FG(x^*,y^*))}{d(fgy^*,fgx_n)}+d(fgx_{n+1},fgx^*)$$

which implies that

$$|d(FG(x^*,y^*),fgx^*)| \precsim a_1|d(fgx^*,fgx_n)|+a_2|d(fgy^*,fgy_n)|$$
$$+\frac{a_3|d(fgx^*,FG(y^*,x^*))|\{|d(fgx_n,fgx^*)|+|d(fgx^*,fgx_{n+1})|\}}{|d(fgx^*,fgy_n)|}$$
$$+\frac{a_4\{|d(fgx^*,fgx_{n+1})|\}|d(fgx_n,FG(x^*,y^*)|}{|d(fgx^*,fgy_n)|}$$
$$+\frac{a_5[|d(fgy_n,fgy^*)|+|d(fgy^*,fgy_{n+1})|]}{|d(fgy^*,fgx_n)|}$$
$$+\frac{a_6[|d(fgy^*,fgy_{n+1})|]|d(fgy_n,FG(y^*,x^*))|}{|d(fgy^*,fgx^*)|}$$
$$+\frac{a_7|d(fgx^*,FG(y^*,x^*))|[d(fgy_n,fgy^*)|+|d(fgy^*,fgy_{n+1})|]}{|d(fgx^*,fgy_n)|}$$
$$+\frac{a_8[|d(fgy^*,fgy_{n+1})|]|d(fgx_n,FG(x^*,y^*))|}{|d(fgy^*,fgx_n)|}+|d(fgx_{n+1},fgx^*)|$$

Since $fgx_n \to fgx^*$ *and* $fgy_n \to fgy^*$ *as* $n \to \infty$, *we have* $|d(FG(x^*,y^*),fgx^*)| \precsim 0$

that is,

$FG(x^*,y^*) = fgx^*$, *similarly one can show that* $FG(y^*, x^*) = fgy^*$

Hence (x^*, y^*) *is a coupled coincidence point of* f *and* g.

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