International Journal of Computational and Applied Mathematics.
ISSN 1819-4966 Volume 12, Number 2 (2017), pp. 233-241

http://www.ripublication.com

An Accurate Evaluation of Integrals in Convex and Non convex Polygonal Domain by Twelve Node Quadrilateral Finite Element Method

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Abstract

In this paper, we present Gauss Legendre quadrature method to find numerically approximated by integral of arbitrary function over convex and non convex polygonal domain. These domain are discretised into twelve noded quadrilateral element, Deriving a new Gaussian quadrature formula for generating sampling points and its weight coefficients. Numerical results are provided to justify the usefulness of the proposed method.

Keywords: Finite Element Method, Gauss Legendre quadrature, Convex and Non convex polygonal domain.

1. INTRODUCTION

The Finite element method is a numerical method used to obtain approximate solution of integral equations are typically occurs in Boundary element method (BEM) and Surface finite element method (SFM) for solving partial differential equation on surface which requires surface integrals, the integral arising in practical problems are not always simple and some quadrature scheme cannot evaluate exactly. Numerical evaluation of integrals over triangle and square region are simple but convex and non convex polygonal region are challenging task to integration of arbitrary function, the method proposed here is termed as Gauss Legendre quadrature rule.

From the literature of review we pointed out that numerical integration of arbitrary function over triangle and quadrilateral region have been carried out [1-7], numerical integration of arbitrary function over polygonal domain are discussed in [8-10] and numerical integration of arbitrary function over polygonal domain by spline finite element method are discussed in [10,11], numerical integration of arbitrary function over convex and non convex polygonal domain by 4 node and 8- noded quadrilateral elements by using generalized Gaussian quadrature rule [12,13]. In this paper we divided the polygonal domain into 12 node quadrilateral element and then we apply Gauss Legendre quadrature rule to evaluate the numerical integration of arbitrary function over convex and non convex polygonal region.

The paper is organized as follows. In section 2. we present the mathematical formulation required for understanding the derivation and discretised the convex and Non convex polygonal domain into sub 12- node quadrilateral elements section 3 and 4. derive a new Gaussian quadrature formula over a convex and non convex region to calculate sampling points and weight coefficients of various order and also plotted the extracted sampling points in both convex and non convex region. In section 5. we compare the numerical results with some illustrative examples.

2. FORMULATION OF INTEGRALS OVER A LINEAR CONVEX QUADRILATERAL ELEMENT

The integral of an arbitrary function $f(x, y)$ over an arbitrary convex quadrilateral region Ω is given by $\mathrm{I} = \iint_{\Omega} \mathrm{f(x,y)dxdy}$ (1)

$$I = \iint_{\Omega} f(x, y)dxdy \quad (2)$$

Let us consider an arbitrary 12- node quadrilateral element in the global coordinate is mapped into 2- square in the local coordinate (ξ, η). The isoperimetric coordinate transformation from (ξ, y) plane to (ξ, η) is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \sum_{k=1}^{12} \begin{pmatrix} x_k \\ y_k \end{pmatrix} N_k(\xi, \eta) \quad (3)$$

Where (x_k, y_k), (k=1,2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12) are the vertices of the quadrilateral element in (ξ, y) plane and $N_k(\xi, \eta)$ denotes the shape function of node k such that

$$N1 = \frac{1}{32}(1 - x)(1 - y)(-10 + 9(x^2 + y^2)$$

$$N2 = \frac{9}{32}(1 - y)(1 - 3x)(1 - x^2)$$

$$N3 = \frac{9}{32}(1 - y)(1 + 3x)(1 - x^2)$$

$$N4 = \frac{1}{32}(1 + x)(1 - y)(-10 + 9(x^2 + y^2)$$

$N5 = \frac{9}{32}(1+x)(1-3y)(1-y^2)$

$N6 = \frac{9}{32}(1+x)(1+3y)(1-y^2)$

$N7 = \frac{1}{32}(1+x)(1+y)(-10+9(x^2+y^2)$

$N8 = \frac{9}{32}(1+y)(1+3x)(1-x^2)$

$N9 = \frac{9}{32}(1+y)(1-3x)(1-x^2)$

$N10 = \frac{1}{32}(1-x)(1+y)(-10+9(x^2+y^2)$

$N11 = \frac{9}{32}(1-x)(1+3y)(1-y^2)$

$N12 = \frac{9}{32}(1-x)(1-3y)(1-y^2)$

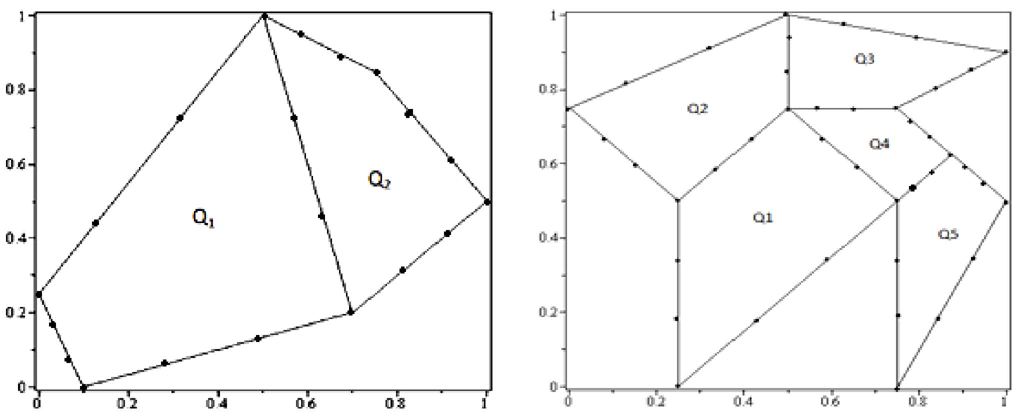
$J = \frac{\partial(x,y)}{\partial(\xi,\eta)} = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi}$

Where

$\frac{\partial x}{\partial \xi} = \sum_{k=1}^{12} x_k \frac{\partial S_k}{\partial \xi}$, $\frac{\partial x}{\partial \eta} = \sum_{k=1}^{12} x_k \frac{\partial S_k}{\partial \eta}$

and

$\frac{\partial y}{\partial \xi} = \sum_{k=1}^{12} y_k \frac{\partial S_k}{\partial \xi}$, $\frac{\partial y}{\partial \eta} = \sum_{k=1}^{12} y_k \frac{\partial S_k}{\partial \eta}$



(a) **(b)**

Fig. 1 (a) Convex polygonal Domain **(b)** Non convex polygonal Domain we test the integral domain is shown in Fig. 1(a) with two quadrilateral elements in convex polygonal domain with vertices are (0, 0.25), (0.1, 0), (0.7,0.2), (1, 0.5),(0.75, 0.85) and (0.5, 1) and Fig. 1(b) with five quadrilateral elements in non convex polygonal domain with vertices are (0, 0.75), (0.25, 0.5), (0.25, 0), (0.75, 0.5), (0.75, 0), (1, 0.5), (0.75, 0.75), (1.0, 0.9), (0.5, 1), (0.875, 0.625) and (0.5, 0.75)

3. GAUSS LEGENDRE QUADRATURE OVER A 12 NODE CONVEX REGION

Integral form of Eq. (2) rewritten as

$$I = \iint_{Q1} f(x,y)dxdy + \iint_{Q2} f(x,y)dxdy$$

$$I = \int_{-1}^{1}\int_{-1}^{1} f(x(\xi,\eta), y(\xi,\eta))\, J\, d\xi\, d\eta$$

$$= \int_{-1}^{1}\int_{-1}^{1} f(m1(\xi,\eta), n1(\xi,\eta))J1 d\xi d\eta + \int_{-1}^{1}\int_{-1}^{1} f(m2(\xi,\eta), n2(\xi,\eta))J2 d\xi d\eta$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} w_i w_j [\ f(m1(\xi_i,\eta_j), n1(\xi_i,\eta_j))J1 + \ f(m2(\xi_i,\eta_j), n2(\xi_i,\eta_j))J2]$$

Where

$$m1 = 0.11718750\,\xi + 0.42968750\,\eta + 0.05312500\,\eta\,\xi - 0.042187500\xi^3 - 0.15468750\,\eta^3 - 0.014062500\,\eta\,\xi^3 - 0.01406250\eta^3\,\xi + 0.32500000$$

$$n1 = -0.410156250\,\xi + 0.371093750\,\eta - 0.29218750\,\eta\,\xi + 0.14765625\,\xi^3 - 0.133593750\,\eta^3 + 0.077343750\,\eta\,\xi^3 + 0.07734375\,\eta^3\xi + 0.3625000$$

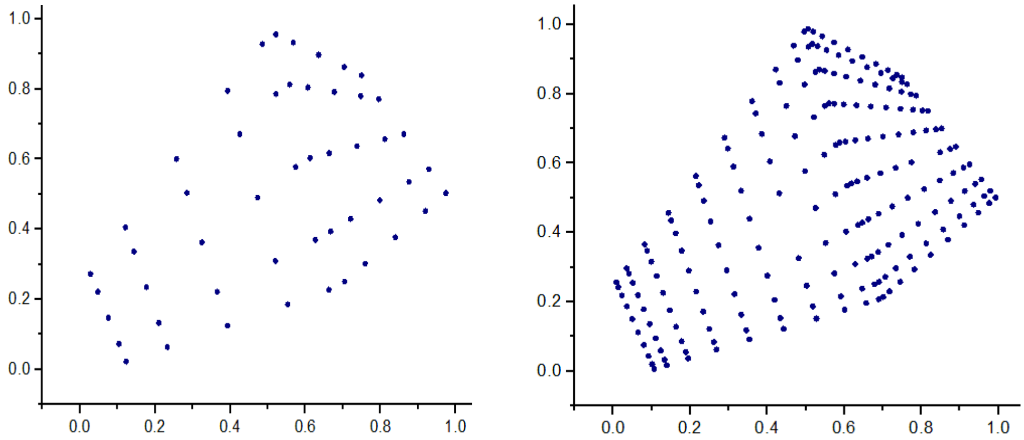
$$J1 = 0.11934326\,\xi^2 + 0.32889892\,\eta^2 + 0.105834960\,\xi - 0.0560302\,\eta + 0.10500732\,\eta\,\xi^2 - 0.0480541992\,\xi^4\,\eta - 0.0160180664\,\xi^2\,\eta^3 - 0.003262939\,\eta^2\,\xi^4 + 0.0032629394\,\eta^4\,\xi^2 - 0.1983471680\,\xi\,\eta^2 - 0.064281005\,\xi^4 - 0.17776977\eta^4 - 0.00108764\,\eta^6 - 0.028015136\,\xi^3 + 0.00108764\,\xi^6 - 0.20751953 + 0.014831542\,\eta^3 + 0.09076904\,\eta^4\xi + 0.03025634\,\eta^2\xi^3$$

$$m2 = -0.1757812500\,\eta - 0.026562500\,\eta\,\xi + 0.063281250\,\eta^3 - 0.773437500\,\xi^3 + 0.007031250\,\eta\,\xi^3 + 0.007031250\,\eta^3\xi + 0.73750000 + 0.214843750\,\xi$$

$$n2 = 0.44921875\,\eta - 0.23906250\,\eta\,\xi + 0.6375000 - 0.1617187\,\eta^3 - 0.021093750\,\xi^3 + 0.063281250\,\eta\,\xi^3 + 0.06328125\eta^3\,\xi + 0.058593750\,\xi$$

$$J2 = -0.052917480\,\eta - 0.033541259\,\xi^2 - 0.16421264\,\eta^2 + 0.099173583\,\eta\,\xi^2 - 0.04538452\,\xi^4\eta - 0.015128173\,\xi^2\eta^3 + 0.00133483\,\eta^2\xi^4 - 0.00133483\,\eta^4\xi^2 + 0.05639282\,\xi\,\eta^2 - 0.025806884\,\eta^4\xi - 0.008602294\,\eta^2\xi^3 + 0.091552734 + 0.0140075683\,\eta^3 + 0.0887420654\,\eta^4 + 0.018045043\,\xi^4 + 0.000444946\,\eta^6 + 0.00796508789\,\xi^3 - 0.00044494\,\xi^6 - 0.030090332\,\xi$$

Where ξ_{i}, η_j are sampling points and w_i, w_j weight coefficients to computed the sampling points and corresponding weights of order N = 5, 10, 15 and plotted the distribution of sampling points in convex polygonal domain of various order



N = 5 N = 10

Fig. 2 Distribution of Sampling points in convex polygonal domain

4. GAUSS LEGENDRE QUADRATURE OVER A 12 NODED NON CONVEX REGION

Integral form of Eq. (2) rewritten as

$$I = \iint_{Q_1} f(x,y)dxdy + \iint_{Q_2} f(x,y)dxdy + \iint_{Q_3} f(x,y)dxdy + \iint_{Q_4} f(x,y)dxdy + \iint_{Q_5} f(x,y)dxdy$$

$$I = \int_{-1}^{1}\int_{-1}^{1} f(x(\xi,\eta), y(\xi,\eta))\, J\, d\xi\, d\eta$$

$$= \int_{-1}^{1}\int_{-1}^{1} f(m1(\xi,\eta), n1(\xi,\eta))J1 d\xi d\eta + \int_{-1}^{1}\int_{-1}^{1} f(m2(\xi,\eta), n2(\xi,\eta))J2 d\xi d\eta + \int_{-1}^{1}\int_{-1}^{1} f(m3(\xi,\eta), n3(\xi,\eta))J3 d\xi d\eta$$

$$+ \int_{-1}^{1}\int_{-1}^{1} f(m4(\xi,\eta), n4(\xi,\eta))J4 d\xi d\eta + \int_{-1}^{1}\int_{-1}^{1} f(m5(\xi,\eta), n5(\xi,\eta))J5 d\xi d\eta$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} w_i w_j [\ f(m1(\xi_i,\eta_j), n1(\xi_i,\eta_j))J1 + f(m2(\xi_i,\eta_j), n2(\xi_i,\eta_j))J2$$

$$+ f(m3(\xi_i,\eta_j), n3(\xi_i,\eta_j))J3 + f(m4(\xi_i,\eta_j), n4(\xi_i,\eta_j))J4 + f(m5(\xi_i,\eta_j), n5(\xi_i,\eta_j))J5]$$

Where

$m1 = -0.0351562500\xi^3 + 0.035156250\,\xi^3\,\eta + 0.03515625\,\xi\eta^3 - 0.1328125000\xi\eta - 0.10546875\,\eta^3 + 0.3125000 + 0.097656250\xi + 0.2929687500\eta$

$n1 = 0.087890625\xi^3 + 0.01757812\xi^3\eta + 0.01757812500\xi\eta^3 + 0.7187500000 - 0.066406250\xi\,\eta - 0.052734375\,\eta^3 - 0.2441406250\,\xi + 0.1464843750\,\eta$

$J1 = -0.02313995\xi^4 + 0.00061798\eta^6 - 0.05472564\eta^4 - 0.0102996\xi^3 - 0.000617\xi^6 - 0.06675720 + 0.04267883\xi^2 + 0.1015167\eta^2 - 0.048614501\,\eta\xi^2 + 0.02224731\xi^4\eta + 0.00741577\xi^2\eta^3 + 0.00185394\eta^2\xi^4 - 0.001853942\eta^4\xi^2 - 0.07292175\xi\eta^2 + 0.011123657\xi^3\eta^2 + 0.03337097\eta^4\xi - 0.006866455\eta^3 + 0.03890991\xi + 0.02593994\,\eta$

$m2 = -0.1328125\xi\eta - 0.10546875\xi^3 + 0.0351562\eta^3 + 0.0351562\xi^3\eta + 0.0351562\xi\eta^3 + 0.4375 + 0.29296875\xi - 0.09765625\eta$

$n2 = -0.19921875\xi\eta - 0.087890625\xi^3 - 0.087890625\eta^3 + 0.052734375\xi^3\eta + 0.052734375\xi\eta^3 + 0.244140625\xi + 0.244140625\eta + 0.40625$

$J2 = -0.1809539795\xi^2 - 0.025039675\eta^2 + 0.1701507568\,\eta\,\xi^2 - 0.077865600\xi^4\eta - 0.025955200\xi^2\eta^3 + 0.005561828\eta^2\xi^4 - 0.0055618286\eta^4\xi^2 - 0.024307250\xi\eta^2 + 0.013801574\eta^4 - 0.003433227\xi^3 - 0.001853942\xi^6 + 0.095367431 + 0.024032592\eta^3 + 0.097434997\xi^4 + 0.0018539428\eta^6 + 0.011123657\eta^4\xi + 0.0037078857\xi^3\eta^2 + 0.0129699707\xi - 0.0907897949\eta$

$m3 = -0.06640625\xi\eta - 0.052734375\xi^3 + 0.017578125\eta^3 + 0.017578125\xi^3\eta + 0.017578125\xi\eta^3 - 0.048828125\eta + 0.146484375\xi + 0.84375$

$n3 = -0.19921875\xi\eta - 0.087890625\xi^3 - 0.087890625\eta^3 + 0.052734375\xi^3\eta + 0.052734375\xi\eta^3 + 0.244140625\xi + 0.244140625\eta + 0.40625$

$J3 = 0.085075378\eta\xi^2 + 0.002780914\eta^2\xi^4 - 0.00278091430\eta^4\xi^2 - 0.03893280\xi^4\eta - 0.012977600\xi^2\eta^3 - 0.012153625\xi\eta^2 + 0.000926971\eta^6 + 0.006900787\eta^4 + 0.048717498\xi^4 - 0.00171661377\xi^3 - 0.0009269714\xi^6 + 0.047683715 + 0.012016296\eta^3 + 0.005561828\eta^4\xi + 0.0018539428\xi^3\eta^2 - 0.09047698\xi^2 - 0.0125198364\eta^2 - 0.045394897\eta + 0.0064849853\xi$

$m4 = 0.066406250\xi\eta + 0.0527343750\eta^3 - 0.0527343750\xi^3 - 0.0175781250\xi^3\eta - 0.0175781250\xi\eta^3 + 0.7187500000 - 0.1464843750\eta + 0.1464843750\xi$

$n4 = -0.03515625\eta^3 - 0.03515625\xi^3 + 0.09765625\eta + 0.09765625\xi + 0.62500$

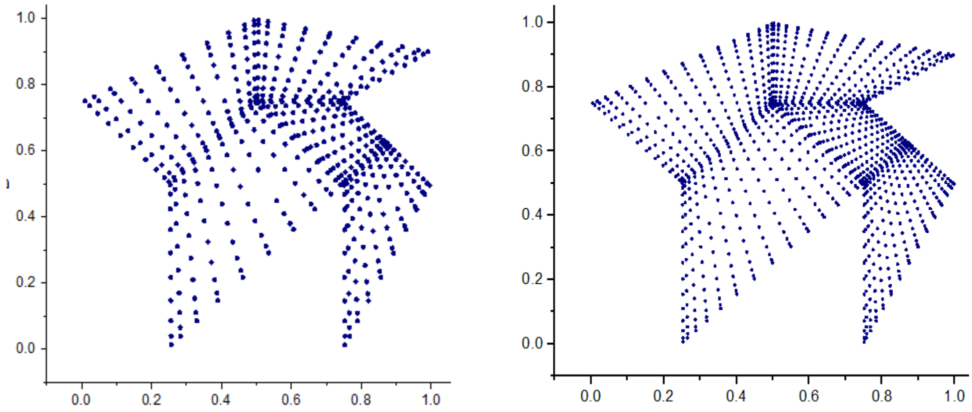
$J4 = -0.0121536254\eta\xi^2 + 0.005561828\xi^4\eta - 0.001716613\eta^3 + 0.001853942\xi^2\eta^3 + 0.02861022949 - 0.030899047\xi^2 + 0.0064849853\eta + 0.016685485\xi^4 + 0.012153625\xi\eta^2 - 0.0055618286\xi\eta^4 - 0.030899047\eta^2 + 0.001716613\xi^3 - 0.001853942\xi^3\eta^2 - 0.006484985\xi + 0.016685485\eta^4$

$m5 = 0.6250000000 + 0.1953125000\xi - 0.07031250000\xi^3$

$n5 = 0.1855468750\eta + 0.80625000 - 0.0097656\xi - 0.146093750\xi\eta - 0.06679680\eta^3 + 0.00351562\xi^3 + 0.038671875\xi^3\eta + 0.038671875\xi\eta^3$

$J5 = 0.0534759521\eta\xi^2 - 0.0022247314\xi^4 - 0.024472045\xi^4\eta - 0.008157348\xi^2\eta^3 + 0.00411987304\xi^2 - 0.0285339355\eta + 0.0075531005\eta^3 - 0.0019073486$

sampling points and corresponding weights are calculated by the above equations for order N = 5, 10, 15 and plotted the distribution of sampling points in Non convex polygonal domain



(a) N = 10 (b) N = 15

Fig. 3 Distribution of Sampling points in Non convex polygonal domain

5. NUMERICAL RESULTS

We have compared the numerical results obtained using Gauss Legendre quadrature rule with that of numerical results arrived in [10] and [11] of various order N = 5, 10, 15 and are tabulated in Table 1 and 2

TABLE. 2 Convex region

Exact value	Order	Computed value
$\iint_C e^{-100((x-0.5)^2+(y-0.5)^2)}dxdy$ 0.0314145286323 [Ref. 10]	**N=5** **N=10** **N=15**	0.03141452073 0.03141452851 0.03141452863
$\iint_C \sqrt{(x-0.5)^2+(y-0.5)^2}\,dxdy$ 0.1568251255862 [Ref. 10]	**N=5** **N=10** **N=15**	0.15682512772 0.15682512545 0.15682512558
$\iint_C \lvert x^2+y^2-0.25\rvert dxdy$ 0.1990625494351 [Ref. 10]	**N=5** **N=10** **N=15**	0.19906254728 0.19906250651 0.19906254943
$\iint_C \sqrt{\lvert 3-4x-3y\rvert}\,dxdy$ 0.5453868050054 [Ref. 10]	**N=5** **N=10** **N=15**	0.54538689115 0.54538684539 0.54538680501

TABLE. 3 Non convex region

Exact value	Order	Relative error	Relative error[11]
$\iint_N e^{-100((x-0.5)^2+(y-0.5)^2)}dxdy$ = 0.031220839802646214	N=5 N=10 N=15	0.0000000762 0.0000000045 0.0000000003	0.0000027
$\iint_N \sqrt{(x-0.5)^2+(y-0.5)^2}\,dxdy$ = 0.15767705312825664	N=5 N=10 N=15	0.0000001108 0.0000000763 0.0000000005	0.0000011
$\iint_N \left\|(x-\frac{1}{2})^2+(y-\frac{1}{2})^2-\frac{1}{16}\right\|dxdy$ = 0.0354330513939807	N=5 N=10 N=15	0.0000000910 0.0000000085 0.0000000096	0.000000821
$\iint_N \sqrt{\|3-4x-3y\|}\,dxdy$ = 0.5169123732639505	N=5 N=10 N=15	0.0000000033 0.0000000046 0.0000000001	0.00000282

5. CONCLUSIONS

In this paper, Gauss Legendre quadrature rule is applied for the numerical integration of arbitrary function over convex and non convex polygonal domain is discretised into 12 – noded quadrilateral element The results obtained are in excellent agreement with exact values.

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