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(E.A) property and rational contractive maps and common fixed point theorems in G-metric space

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Abstract

In this paper, we prove common fixed point theorems using rational inequalities satisfying (E.A) property and weak compatibility of mappings in complete G-metric spaces.

Keywords: G-metric space, fixed point, rational inequality, compatible mappings, weakly compatible mappings, property (E.A)

AMS Mathematics subject Classification: 47H10, 54H25

1. INTRODUCTION

The concept of G-metric space has introduced by Mustafa and Sims [8] in the year 2004, as a generalization of the general metric spaces. In G-metric space a non-negative real number is assigned to every element of tupple. In [12] Banach contraction mapping principle was established and a fixed point results have been proved. After that many fixed point results have been proved in this space. Some of these works may be noted in [3–4, 11–14] and [15].

Here we present the necessary definitions and results in G-metric space, which will be useful for the rest of the paper, however, for more details, we refer to [8]

Definition 1.1.([11]). Let X be a non-empty set and G: $X^3 \rightarrow [0, \infty)$ be a function satisfying the following axioms:

(G1) $G(x, y, z) = 0$ if $x=y=z$,
(G2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$
(G3) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z\ X \in$ with $z \neq y$
(G4) $G(x, y, z)=G(x, z, y)= G(y, z, x)= \cdots$(symmetry in all three variables),
(G5) $G(x, y, z)=G(x, a, a)+G(a, y, z)$for all $x, y, z, a \in X$, (rectangle inequality)

Then the function G is called a generalized metric, or specifically a G-metric on X and the pair (X, G) is called a G-metric Space.

Definition 1.2. ([11]). *Let (X, G) be a G-metric space and let $\{x_n\}$ be a sequence of points in X, a point x in X is said to be the limit of the sequence $\{x_n\}$ if $G(x,x_n,x_m) = 0$, and one says that sequence $\{x_n\}$ is G-convergent to x. Thus, if $x_n \rightarrow x$ or $x_n = x$ as $n \rightarrow \infty$, in a G-metric space (X, G), then for each $\varepsilon > 0$, there exists a positive integer N such that $G(x,x_n,x_m) < \varepsilon$ for all $m, n \geq N$.*

Proposition 1.3.([11]). *Let (X, G) be a G-metric space. Then the following are equivalent:*

i. $\{x_n\}$ is G-convergent to x,
ii. $G(x_n, x_n, x) \rightarrow 0$ as $n \rightarrow \infty$,
iii. $G(x_n, x, x) \rightarrow 0$ as $n \rightarrow \infty$,
iv. $G(x_m, x_n, x) \rightarrow 0$ as $m, n \rightarrow \infty$.

Definition 1.4.([11]). *Let (X, G) be a G-metric space. A sequence $\{x_n\}$ is called G-Cauchy if, for each $\varepsilon > 0$, there exists a positive integer N such that $G(x_n,x_m,x_l) < \varepsilon$, for all $n, m, l \geq N$, i.e., if $G(x_n, x_m, x_l) \rightarrow 0$ as $n, m, l \rightarrow \infty$.*

Proposition 1.5. ([11]). *Let (X, G) be a G-metric space. Then, for any x, y, z, a in X, it follows that:*

(i) if $G(x, y, z) = 0$, then $x = y = z$,
(ii) $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$,
(iii) $G(x, y, y) \leq 2G(y, x, x)$,
(iv) $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$,
(v) $G(x, y, z) \leq \frac{2}{3} (G(x, y, a) + G(x, a, z) + G(a, y, z))$,
(vi) $G(x, y, z) \leq G(x, a, a) + G(y, a, a) + G(z, a, a)$.

Definition 1.6. ([7]). *Let f and g be two self mappings on a G-metric space (X,G). The mappings f and g are said to be compatible if $\lim_{n\to\infty}(fgx_n, gfx_n, gfx_n)=0$ or $\lim_{n\to\infty}(gfx_n, fgx_n, fgx_n)=0$, whenever $\{x_n\}$ is a sequence in X such that*

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$$

for some $z \in X$.

Definition 1.7. ([1]). *Two maps are said to be weakly compatible if they commute at coincidence points.*

2. MAIN RESULT

Now we come to our main result for a pair of compatible maps

Theorem 2.1. *Let (X, G) be a complete G-metric space and $f, g : X \to X$ be the self mapping on (X,G) satisfying the following conditions:*

(2.1) $f(X) \subseteq g(X)$

(2.2) *f or g is continuous,*

(2.3)

$$G(fx,fy,fz) \le \begin{pmatrix} a_1 \frac{G(fy,gy,gy)+G(fx,gx,gy)}{G(fy,gz,gz)+G(fx,gx,gz)} \, .G(gy,gy,gz), \\ +a_4 \frac{G(gz,fx,gy)+G(gy,gy,gz)}{G(fy,gy,fx)+G(fy,fy,fz)} \, .G(gx,gy,gy) \\ +a_3 \frac{G(fx,gz,gz)+G(gz,gy,fz)}{G(fx,gy,gy)+G(gz,gz,fy)} \, .G(gx,gy,fy) \\ +a_4 \frac{G(gz,fx,gy)+G(gy,gy,gz)}{G(fy,gy,fx)+G(fy,fy,fz)} \, .G(gx,gy,gy) \end{pmatrix}$$

for all x, y, z ϵ X, where $a_1,\ a_2, a_3, a_4 \ge 0, with\ \sum_{i=1}^{4} a_i < \frac{1}{3}$. Then f and g have a unique common fixed point in X provided f and g are compatible maps.

Proof. *Let $x_0 \in X$ be an arbitrary point, then by(2.1), one can choose a point $x_1 \in X$ such that $fx_0 = g\,x_1$. In general one can choose $x_{n+1} \in X$ such that $y_n = fx_n = g\,x_{n+1}$, n=0,1,2,3…*

From (2.3), we have

$$G(y_n, y_{n+1}, y_{n+1}) = G(fx_n, fx_{n+1}, fx_{n+1})$$

$$\le \begin{pmatrix} a_1 \frac{G(fx_{n+1},gx_{n+1},gx_{n+1})+G(fx_n,gx_n,gx_{n+1})}{G(fx_{n+1},gx_{n+1},gx_{n+1})+G(fx_n,gx_n,gx_{n+1})} \, .G(gx_{n+1}, gx_{n+1}, gx_{n+1}) \\ +a_2 \frac{G(fx_n,gx_{n+1},gx_{n+1})+G(fx_n,gx_{n+1},fx_{n+1})}{G(gx_{n+1},fx_n,gx_{n+1})+G(fx_n,gx_{n+1},fx_{n+1})} \, .G(gx_n, gx_{n+1}, fx_{n+1}) \\ +a_3 \frac{G(fx_n,gx_{n+1},gx_{n+1})+G(gx_{n+1},gx_{n+1},fx_{n+1})}{G(fx_n,gx_{n+1},gx_{n+1})+G(gx_{n+1},gx_{n+1},fx_{n+1})} \, .G(gx_n, gx_{n+1}, fx_{n+1}) \\ +a_4 \frac{G(gx_{n+1},fx_n,gx_{n+1})+G(gx_{n+1},gx_{n+1},gx_{n+1})}{G(fx_{n+1},gx_{n+1},fx_n)+G(fx_{n+1},fx_{n+1},fx_{n+1})} \, .G(gx_n, gx_{n+1}, gx_{n+1}) \end{pmatrix}$$

$$\leq \begin{pmatrix} a_1 \dfrac{G(fx_{n+1}, fx_n, fx_n) + G(fx_n, fx_{n-1}, fx_n)}{G(fx_{n+1}, fx_n, fx_n) + G(fx_n, fx_{n-1}, fx_n)} . G(fx_n, fx_n, fx_n) \\ +a_2 \dfrac{G(fx_n, fx_n, fx_n) + G(fx_n, fx_n, fx_{n+1})}{G(fx_n, fx_n, fx_n) + G(fx_n, fx_n, fx_{n+1})} . G(fx_{n-1}, fx_n, fx_{n+1}) \\ +a_3 \dfrac{G(fx_n, fx_n, fx_n) + G(fx_n, fx_n, fx_{n+1})}{G(fx_n, fx_n, fx_n) + G(fx_n, fx_n, fx_{n+1})} . G(fx_{n-1}, fx_n, fx_{n+1}) \\ +a_4 \dfrac{G(fx_n, fx_n, fx_n) + G(fx_n, fx_n, fx_n)}{G(fx_{n+1}, fx_n, fx_n) + G(fx_{n+1}, fx_{n+1}, fx_{n+1})} . G(fx_{n-1}, fx_n, fx_n) \end{pmatrix}$$

$$\leq \begin{pmatrix} a_1 . G(fx_n, fx_n, fx_n) \\ +a_2 . G(fx_{n-1}, fx_n, fx_{n+1}) \\ +a_3 . G(fx_{n-1}, fx_n, fx_{n+1}) \\ +a_4 \dfrac{G(fx_n, fx_n, fx_n) + G(fx_n, fx_n, fx_n)}{G(fx_{n+1}, fx_n, fx_n) + G(fx_{n+1}, fx_{n+1}, fx_{n+1})} . G(fx_{n-1}, fx_n, fx_n) \end{pmatrix}$$

$$\leq (a_2 + a_3)\big(G(fx_{n-1}, fx_{n+1}, fx_n)\big)$$

Using the rectangular inequality of g-metric space, we have

$$\begin{aligned} G(fx_{n-1}, fx_n, fx_{n+1}) &\leq G(fx_{n-1}, fx_n, fx_n) + G(fx_n, fx_n, fx_{n+1}) \\ &\leq G(fx_{n-1}, fx_n, fx_n) + 2G(fx_n, fx_{n+1}, fx_{n+1}) \end{aligned}$$

[using preposition (1.5)]

Hence, we have

$$\begin{aligned} G(fx_n, fx_{n+1}, fx_{n+1}) \leq\ & (a_2 + a_3)\big(G(fx_{n-1}, fx_n, fx_n)\big) \\ & +2(a_2 + a_3)G(fx_n, fx_{n+1}, fx_{n+1}) \end{aligned}$$

That is,

$$G(fx_n, fx_{n+1}, fx_{n+1}) \leq \frac{(a_2+a_3)}{(1-2a_2-2a_3)}\big(G(fx_{n-1}, fx_n, fx_n)\big)$$

That is,

$$G(fx_n, fx_{n+1}, fx_{n+1}) \leq \mu\big(G(fx_{n-1}, fx_n, fx_n)\big),$$

where $\mu = \frac{(a_2+a_3)}{(1-2a_2-2a_3)} < 1$

Continuing in a similar way, we have

$$G(fx_n, fx_{n+1}, fx_{n+1}) \leq \mu^n\big(G(fx_0, fx_1, fx_1)\big),$$

i.e.,

$$G(y_n, y_{n+1}, y_{n+1}) \leq \mu^n\big(G(y_0, y_1, y_1)\big)$$

Therefore, for all n,m $\in \mathbb{N}$*, n < m, we have using rectangular inequality that,*

$$G(y_n, y_m, y_m) \le G(y_n, y_{n+1}, y_{n+1}) + G(y_{n+1}, y_{n+2}, y_{n+2}) + \ ... \ + G(y_{m-1}, y_m, y_m)$$
$$\le (\mu^n + \mu^{n+1} + \mu^{n+2} + \cdots + \mu^{m-1})\big(G(y_0, y_1, y_1)\big)$$
$$\le \frac{\mu^n}{(1-\mu)}\, G(y_0, y_1, y_1)$$

Taking n, m approaches to infinity, we have limiting value of $G(y_n, y_m, y_m)$ is zero. Thus $\{y_n\}$Is a G-Cauchy sequence in X. also it is given that G-metric space is a complete space, therefore, there exist a point $z \in X$ such that $\lim_{n\to\infty} y_n = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_{n+1} = z$. *also, it is given that the mappings, f or g is continuous, for definiteness one can assume that g is continuous, therefore*

$$\lim_{n\to\infty} gfx_n = \lim_{n\to\infty} ggx_n = g\,z.$$

Further, f and g are compatible, therefore,

$$G(fgx_n, gfx_n, gfx_n) = 0, \text{ that is, } \lim_{n\to\infty} fgx_n = g\,z$$

Using (2.3), we have

$$G(fgx_n, fx_n, fx_n) \le \begin{pmatrix} a_1 \frac{G(fx_n, gx_n, gx_n) + G(fgx_n, ggx_n, gx_n)}{G(fx_n, gx_n, gx_n) + G(fgx_n, ggx_n, gx_n)} \,.\, G(gx_n, gx_n, gx_n) \\ +a_2 \frac{G(fgx_n, gx_n, gx_n) + G(fgx_n, gx_n, fx_n)}{G(gx_n, fgx_n, gx_n) + G(fgx_n, gx_n, fx_n)} \,.\, G(ggx_n, gx_n, fx_n) \\ +a_3 \frac{G(fgx_n, gx_n, gx_n) + G(gx_n, gx_n, fx_n)}{G(fgx_n, gx_n, gx_n) + G(gx_n, gx_n, fx_n)} \,.\, G(ggx_n, gx_n, fx_n) \\ +a_4 \frac{G(gx_n, fgx_n, gx_n) + G(gx_n, gx_n, gx_n)}{G(fx_n, gx_n, fgx_n) + G(fx_n, fx_n, fx_n)} \,.\, G(ggx_n, gx_n, gx_n) \end{pmatrix}$$

Taking n approaches to infinity, we have

$$G(gz, z, z) \le \begin{pmatrix} a_1 \frac{G(z,z,z) + G(gz,gz,z)}{G(z,z,z) + G(gz,gz,z)} \,.\, G(z,z,z) \\ +a_2 \frac{G(gz,z,z) + G(gz,z,z)}{G(z,gz,z) + G(gz,z,z)} \,.\, G(gz,z,z) \\ +a_3 \frac{G(gz,z,z) + G(z,z,z)}{G(gz,z,z) + G(z,z,z)} \,.\, G(gz,z,z) \\ +a_4 \frac{G(z,gz,z) + G(z,z,z)}{G(z,z,gz) + G(z,z,z)} \,.\, G(gz,z,z) \end{pmatrix}$$

$$G(gz, z, z) \le (a_2 + a_3 + a_4) G(gz, z, z).$$

A contradiction, we have z=gz, since $a_2 + a_3 + a_4 < \frac{1}{3}$.

Now taking $x=x_n$, $y=z=z$ *Using (2.3), we have*

$$G(fx_n,fz,fz) \leq \begin{pmatrix} a_1 \frac{G(fz,gz,gz)+G(fx_n,gx_n,gz)}{G(fz,gz,gz)+G(fx_n,gx_n,gz)} \cdot G(gz,gz,gz) \\ +a_2 \frac{G(fgx_n,gz,gz)+G(fx_n,gz,fz)}{G(gz,fx_n,gz)+G(fx_n,gz,fz)} \cdot G(gx_n,gz,fz) \\ +a_3 \frac{G(fx_n,gz,gz)+G(gz,gz,fz)}{G(fx_n,gz,gz)+G(gz,gz,fz)} \cdot G(gx_n,gz,fz) \\ +a_4 \frac{G(gz,fx_n,gz)+G(gz,gz,gz)}{G(fz,gz,fx_n)+G(fz,fz,fz)} \cdot G(gx_n,gz,gz) \end{pmatrix}$$

Taking n approaches to infinity, we have

$$G(z,fz,fz) \leq (a_2+a_3).\ G(z,z,fz)$$
$$\leq 2(a_2+a_3).\ G(z,fz,fz)$$

[using preposition 1.5(iii)]

Which is a contradiction, since $(a_2+a_3) \leq \frac{1}{3}$

Hence, we have $z = fz$.

Uniqueness:

Let w be the other common fixed point other than z of functions f and g . Then using (2.3), we have

$$G(z,w,w) = G(fz,fw,fw)$$

$$\leq \begin{pmatrix} a_1 \frac{G(fw,gw,gw)+G(fz,gz,gw)}{G(fw,gw,gw)+G(fz,gz,gw)} \cdot G(gw,gw,gw) \\ +a_2 \frac{G(fz,gw,gw)+G(fz,gw,fw)}{G(gw,fz,gw)+G(fz,gw,fw)} \cdot G(gz,gw,fw) \\ +a_3 \frac{G(fz,gw,gw)+G(gw,gw,fw)}{G(fz,gw,gw)+G(gw,gw,fw)} \cdot G(gz,gw,fw) \\ +a_4 \frac{G(gw,fz,gw)+G(gw,gw,gw)}{G(fw,gw,fz)+G(fw,fw,fw)} \cdot G(gz,gw,gw) \end{pmatrix}$$

$$\leq \begin{pmatrix} a_1 \frac{G(w,w,w)+G(z,z,w)}{G(w,w,w)+G(z,z,w)} \cdot G(w,w,w) \\ +a_2 \frac{G(z,w,w)+G(z,w,w)}{G(w,z,w)+G(z,w,w)} \cdot G(z,w,w) \\ +a_3 \frac{G(z,w,w)+G(w,w,w)}{G(z,w,w)+G(w,w,w)} \cdot G(z,w,w) \\ +a_4 \frac{G(w,z,w)+G(w,w,w)}{G(w,w,z)+G(w,w,w)} \cdot G(z,w,w) \end{pmatrix}$$

$$\leq (a_2 + a_3 + a_3).G(z,w,w)$$

This is possible only if G(z,w,w)=0 i.e., z=w. which completes the proof.

***Theorem 2.2**. Let f and g be weakly compatible self maps of G-metric space (X, G) satisfying conditions (2.1) and (2.3) and any one of the subspace f(X) and g(X) is complete. Then f and g have a unique common fixed point.*

***Proof.** Using theorem (2.1), it is concluded that $\{y_n\}$ is a Cauchy sequence. Since either f(X) and g(X) is complete, for definiteness assume that g(X) is subspace of X then the subsequence of $\{y_n\}$ must get a limit point in g(X). Let the limit point be z. let $u \in g^{-1}z$. then gu=z as $\{y_n\}$ is a Cauchy sequence containing a convergent subsequence, therefore the sequence $\{y_n\}$ is also convergent implying thereby the convergence of subsequence of the convergent sequence. Now we show that fu=z*

On setting x=u, y= x_n and z= x_n in (2.3), we have

$$G(fu, fx_n, fx_n) \leq \begin{pmatrix} a_1 \frac{G(fx_n,gx_n,gx_n)+G(fu,gu,gx_n)}{G(fx_n,gx_n,gx_n)+G(fu,gu,gx_n)} .G(gx_n, gx_n, gx_n) \\ +a_2 \frac{G(fu,gx_n,gx_n)+G(fu,gx_n,fx_n)}{G(gx_n,fu,gx_n)+G(fu,gx_n,fx_n)} .G(gu, gx_n, fx_n) \\ +a_3 \frac{G(fu,gx_n,gx_n)+G(gx_n,gx_n,fx_n)}{G(fu,gx_n,gx_n)+G(gx_n,gx_n,fx_n)} .G(gu, gx_n, fx_n) \\ +a_4 \frac{G(gx_n,fu,gx_n)+G(gx_n,gx_n,gx_n)}{G(fx_n,gx_n,fu)+G(fx_n,fx_n,fx_n)} .G(gu, gx_n, gx_n) \end{pmatrix}$$

Taking n approaches to infinity in above inequality, we have

$$G(fu, z, z) \leq \begin{pmatrix} a_1 \frac{G(z,z,z)+G(fu,z,z)}{G(z,z,z)+G(fu,z,z)} .G(z, z, z) \\ +a_2 \frac{G(fu,z,z)+G(fu,z,z)}{G(z,fu,z)+G(fu,z,z)} .G(z, z, z) \\ +a_3 \frac{G(fu,z,z)+G(z,z,z)}{G(fu,z,z)+G(z,z,z)} .G(z, z, z) \\ +a_4 \frac{G(z,fu,z)+G(z,z,z)}{G(z,z,fu)+G(z,z,z)} .G(z, z, z) \end{pmatrix} = 0$$

Which implies that fu=z

Therefore fu=gu=z, i.e., u is a coincidence point of f and g. since f and g are weakly compatible, it follows that fgu=gfu, i.e., fz=gz.

Now, it is to be shown that fz=z, suppose that fz≠z, therefore G(fz,z,z)>0. From (2.3), on substituting x=z, y=u, z=u, we have

$$G(fz,fu,fu) \leq \begin{pmatrix} a_1 \frac{G(fu,gu,gu)+G(fz,gz,gu)}{G(fu,gu,gu)+G(fz,gz,gu)} . G(gu,gu,gu) \\ +a_2 \frac{G(fz,gu,gu)+G(fz,gu,fu)}{G(gu,fz,gu)+G(fz,gu,fu)} . G(gz,gu,fu) \\ +a_3 \frac{G(fz,gu,gu)+G(gu,gu,fu)}{G(fz,gu,gu)+G(gu,gu,fu)} . G(gz,gu,fu) \\ +a_4 \frac{G(gu,fz,gu)+G(gu,gu,gu)}{G(fu,gu,fz)+G(fu,fu,fu)} . G(gz,gu,gu) \end{pmatrix}$$

$$G(fz,z,z) \leq \begin{pmatrix} a_1 \frac{G(z,z,z)+G(fz,fz,z)}{G(z,z,z)+G(fz,fz,z)} . G(z,z,z) \\ +a_2 \frac{G(fz,z,z)+G(fz,z,z)}{G(z,fz,z)+G(fz,z,z)} . G(fz,z,z) \\ +a_3 \frac{G(fz,z,z)+G(z,z,z)}{G(fz,z,z)+G(z,z,z)} . G(fz,z,z) \\ +a_4 \frac{G(z,fz,z)+G(z,z,z)}{G(z,z,fz)+G(z,z,z)} . G(fz,z,z) \end{pmatrix}$$

$$G(fz,z,z) \leq \left(a_2 . G(fz,z,z) + a_3 . G(fz,z,z) + a_4 . G(fz,z,z)\right)$$

$G(fz,z,z) \leq (a_2 + a_3 + a_4)G(fz,z,z)$, *a contradiction*

Which implies that fz=z.

Therefore, fz=gz=z i.e., z is a common fixed point of f and g. uniqueness follows easily.

3. PROPERTY (E.A.) IN G-METRIC SPACE

Generalization of non compatible maps as property (E.A.) in metric space has been introduced by Amari and Moutawakil [2] and it is as follows:

Definition 3.1. *Let A, S are two self- maps of metric space (X,d). The pair (A,S) is said to satisfy property (E.A.) is there exist a sequence { x_n } in X such that*

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ *for some z∈X.*

Now, we prove a common fixed point theorem for a pair of weakly compatible mappings along the property (E.A.).

Theorem 3.2. *Let (X, G) be a complete G-metric space and f, g: X → X be the two self mapping satisfying (2.3) and the following conditions:*
(3.1) f and g satisfy property (E.A.).
(3.2) g(X) is a closed subspace of X.

Then f and g have a unique common fixed point in X provided f and g are weakly compatible maps.

***Proof*.** *Since f and g satisfy property (E.A.), and hence there exist a sequence { x_n } in X such that* $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u \in X$. *since g(X) is closed subspace of X, therefore every Cauchy sequence is convergent in that space. i.e every convergent sequence of points in g(X) has a limit point in g(X). Therefore*

$\lim_{n\to\infty} fx_n = u = \lim_{n\to\infty} gx_n$ *for some* $a \in X$. *This implies that* $u=ga \in X$.

Now using inequality (2.3), we have

$$G(fa, fx_n, fx_n) \le \begin{pmatrix} a_1 \frac{G(fx_n,gx_n,gx_n)+G(fa,ga,gx_n)}{G(fx_n,gx_n,gx_n)+G(fa,ga,gx_n)} . G(gx_n, gx_n, gx_n) \\ +a_2 \frac{G(fa,gx_n,gx_n)+G(fa,gx_n,fx_n)}{G(gx_n,fa,gx_n)+G(fa,gx_n,fx_n)} . G(ga, gx_n, fx_n) \\ +a_3 \frac{G(fa,gx_n,gx_n)+G(gx_n,gx_n,fx_n)}{G(fa,gx_n,gx_n)+G(gx_n,gx_n,fx_n)} . G(ga, gx_n, fx_n) \\ +a_4 \frac{G(gx_n,fa,gx_n)+G(gx_n,gx_n,gx_n)}{G(gx_n,gx_n,fa)+G(fx_n,fx_n,fx_n)} . G(ga, gx_n, gx_n) \end{pmatrix}$$

Taking $n\to\infty$, *we have*

$$G(fa,u,u) \le \begin{pmatrix} a_1 \frac{G(u,u,u) + G(fa,u,u)}{G(u,u,u) + G(fa,u,u)} . G(u,u,u) \\ +a_2 \frac{G(fa,u,u) + G(fa,u,u)}{G(u,fa,u) + G(fa,u,u)} . G(u,u,u) \\ +a_3 \frac{G(fa,u,u) + G(u,u,u)}{G(fa,u,u) + G(u,u,u)} . G(u,u,u) \\ +a_4 \frac{G(u,fa,u) + G(u,u,u)}{G(u,u,fa) + G(u,u,u)} . G(u,u,u) \end{pmatrix}$$

$$G(fa,u,u) = 0$$

$wehave, fa = u. this\ implies\ that$

$$fa = ga = u.$$

$thusaisthecoincidencepointoffandg. sincefandgareweakly$
$compatible, therefore$

$$fu = fga = gfa = gu.$$

Again from (2.3), we have

$$G(fu,fa,fa) \leq \begin{pmatrix} a_1 \frac{G(fa,ga,ga)+G(fu,gu,ga)}{G(fa,ga,ga)+G(fu,gu,ga)} . G(ga,ga,ga) \\ +a_2 \frac{G(fu,ga,ga)+G(fu,ga,fa)}{G(ga,fu,ga)+G(fu,ga,fa)} . G(gu,ga,fa) \\ +a_3 \frac{G(fu,ga,ga)+G(ga,ga,fa)}{G(fu,ga,ga)+G(ga,ga,fa)} . G(gu,ga,fa) \\ +a_4 \frac{G(ga,fu,ga)+G(ga,ga,ga)}{G(ga,ga,fu)+G(fa,fa,fa)} . G(gu,ga,ga) \end{pmatrix}$$

$$G(fu,u,u) \leq \begin{pmatrix} a_1 \frac{G(u,u,u)+G(fu,fu,u)}{G(u,u,u)+G(fu,fu,u)} . G(u,u,u) \\ +a_2 \frac{G(fu,u,u)+G(fu,u,u)}{G(u,fu,u)+G(fu,u,u)} . G(fu,u,u) \\ +a_3 \frac{G(fu,u,u)+G(u,u,u)}{G(fu,u,u)+G(u,u,u)} . G(fu,u,u) \\ +a_4 \frac{G(u,fu,u)+G(u,u,u)}{G(u,u,fu)+G(u,u,u)} . G(fu,u,u) \end{pmatrix}$$

$$G(fu,u,u) \leq (a_2 + a_3 + a_4) G(fu,u,u)$$

Implies, fu=u, since $a_2 + a_3 + a_4 < \frac{1}{3}$

And Uniqueness follows easily.

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