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On Some Composition Formulae for Multidimensional Fractional Integral Operators Associated Aleph $(\aleph)$ Function and General Class of Polynomial

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Abstract

In this paper the authors acquire three new composition formulae of a class of Multidimensional fractional integral operators involving generalized multivariable Polynomial and Aleph $(\aleph)$ function. On account of the general nature of the functions occurring as kernels here , Each of the results obtained in this paper would unify and extend corresponding (new and known) results involving similar function and Polynomials (of one or more variables) as special cases of our formulae. The results obtains by Erdelyi [1] , Goyal and Jain [3], Goyal, Jain and Gaur [4] and Singh and Mandia [10].

Keywords- Fractional integral operator, Aleph function, Aleph functions in series form, General class of Multivariable Polynomials.

1. INTRODUCTION

In last few years several author Erdelyi [1], Nishimoto [6,7] ,Singh and Mandia [10] have made noteworthy contributions to the fractional calculus operators involving various functions and polynomials, Srivastava and Saxena [12] have acquainted a systematic account of fractional calculus operators and their applications investigated by various authors.

The Multivariable polynomial, $S_V^{U_1,...,U_k}\left(x_1,...,x_k\right)$ adduced by Srivastava and Garg [11, p.686, eq. (1.4)] is defined in the following manner:

$$S_V^{U_1,...,U_k}\left(x_1,...,x_k\right)=\sum_{\substack{R_1,...,R_k \\ \sum_{i=1}^{k} U_i R_i \le V}} (-V)_{\sum_{i=1}^{k} U_i R_i} A\left(V,R_1,...,R_k\right)\frac{x_1^{R_1}}{R_1!}\cdots\frac{x_k^{R_k}}{R_k!} \qquad (1)$$

Where $V=0,1,2\ldots$ and $U_1,...,U_k$ arbitrary positive integers and $A\left(V_1,R_1,...,R_k\right)$ coefficients are arbitrary constants (real or complex).

Aleph ($\aleph$) function:

The Aleph function adduced by Sudland [13], however the notation and complete definition is presented in the following way in terms and the Mellin- Barnes type integrals

$$\aleph[z]=\aleph_{p_i,q_i;\tau_i;r}^{m,n}\left[z\left|\begin{matrix}(a_j,A_j)_{1,n},\left[\tau_i(a_{ji},A_{ji})\right]_{n+1,p_i}\\(b_j,B_j)_{1,m},\left[\tau_i(b_{ji},B_{ji})\right]_{m+1,q_i}\end{matrix}\right.\right]$$

$$=\frac{1}{2\pi\omega}\int_L \Omega_{p_i,q_i;\tau_i;r}^{m,n}(s)z^{-s}ds \qquad (2)$$

For all $z\neq 0$ where $\omega=\sqrt{(-1)}$ and

$$\Omega_{p_i,q_i,\tau_i;r}^{m,n}(s)=\frac{\prod_{j=1}^{m}\Gamma\left(b_j+B_js\right)\prod_{j=1}^{n}\Gamma\left(1-a_j-A_js\right)}{\sum_{i=1}^{r}\tau_i\prod_{j=m+1}^{q_i}\Gamma\left(1-b_{ji}-B_{ji}s\right)\prod_{j=n+1}^{p_i}\Gamma\left(a_{ji}+A_{ji}s\right)} \qquad (3)$$

The integration path $L=L_{i\gamma\infty},\gamma\in R$ extends from $\gamma-i\infty$ to $\gamma+i\infty$, and is such that the poles, assumed to be simple of $\Gamma(1-a_j-A_js)$,j=1,...,n do not coincide with the pole of $\Gamma(b_j+B_js)$, $j=i,...,m$ the parameter p_i,q_i are non-negative integers satisfying: $0\le n\le p_i, 0\le m\le q_i, \tau_i>0$ for $i=1,...,r$ .The $A_j,B_j,A_{ji},B_{ji}>0$ and $a_j,b_j,a_{ji},b_{ji}\in C$ The empty product in (3) is interpreted as unity. The existence conditions for the defining integral (2) are giben below:

$$\phi_l > 0, |\arg(z)| < \frac{\pi}{2}\phi_l , (l = 1,...,r) \tag{4}$$

$$\phi_l \geq 0, |\arg(z)| < \frac{\pi}{2}\phi_l \text{ and } R(\xi_l) < 0 \tag{5}$$

Where

$$\phi_l = \sum_{j=1}^{n} A_j + \sum_{j=1}^{m} B_j - \tau_l \left(\sum_{j=n+1}^{p_l} A_{jl} + \sum_{j=m+1}^{q_l} B_{jl} \right) \tag{6}$$

$$\xi_l = \sum_{j=1}^{m} b_j - \sum_{j=1}^{n} a_j + \tau_l \left(\sum_{j=n+1}^{q_l} b_{jl} - \sum_{j=m+1}^{p_l} a_{jl} \right) + \frac{1}{2}(p_l - q_l), l = 1,2,...,r \tag{7}$$

For detailed introduction of Aleph (ℵ) function see [13] and [14].

II. MULTIDIMENSIONAL FRACTIONAL INTEGRAL OPERATORS

In the present Paper we study the following fractional integral operators.

$$I_x\left[f\left(t_1,...,t_s\right)\right] = I^{\rho,\sigma;e,f;\eta,\lambda}_{x:U,V;Z}\left[f\left(t_1,...,t_s\right);x_1,...,x_s\right]$$

$$= \left(\prod_{j=1}^{s} x_j^{-\rho_j - \sigma_j} \right) \int_0^{x_1} \cdots \int_0^{x_s} \left[t_j^{\rho_j} \left(x_j - t_j \right)^{\sigma_j - 1} \right]$$

$$\times S_V^{U_1,...,U_s} \left[E_1 \left(\frac{t_1}{x_1} \right)^{e_1} \left(1 - \frac{t_1}{x_1} \right)^{f_1}, ..., E_s \left(\frac{t_s}{x_s} \right)^{e_s} \left(1 - \frac{t_s}{x_s} \right)^{f_s} \right]$$

$$\times \aleph^{M,N}_{P_i,Q_i,\tau_i,r} \left[z \prod_{j=1}^{s} \left(\frac{t_j}{x_j} \right)^{\eta_j} \left(1 - \frac{t_j}{x_j} \right)^{\lambda_j} \left| \begin{matrix} \left(a_j, A_j\right)_{1,N}, \left[\tau_i\left(a_{ji}, A_{ji}\right)\right]_{N+1,P_i} \\ \left(b_j, B_j\right)_{1,M}, \left[\tau_i\left(b_{ji}, B_{ji}\right)\right]_{M+1,Q_i} \end{matrix} \right. \right] \tag{8}$$

$$\times \left(t_1,...,t_s\right) dt_1,...,dt_s$$

Where

(i) $\min \operatorname{Re}\left(e_j, f_j, \eta_j, \lambda_j\right) \geq 0$, $(j = 1,...,s)$ And all Parameters $e_j, f_j, \eta_j, \lambda_j$ are not zero simultaneously,

(ii) $\min_{1\le k\le M} \operatorname{Re}\left[1+\rho_j+U_j+\eta_j\frac{b_k}{\beta_k}\right]>0$, $\min_{1\le k\le M} \operatorname{Re}\left[\sigma_j+\lambda_j\frac{b_k}{\beta_k}\right]>0$

$$J_x\left[f\left(t_1,\dots,t_s\right)\right]=J^{\rho,\sigma;e,f;\eta,\lambda}_{x:U,V;Z}\left[f\left(t_1,\dots,t_s\right);x_1,\dots,x_s\right]$$

$$=\left(\prod_{j=1}^{s} x_j^{\rho_j}\right)\int_{x_1}^{\infty}\dots\int_{x_s}^{\infty}\left[t_j^{-\rho_j-\sigma_j}\left(t_j-x_j\right)^{\sigma_j-1}\right]$$

$$\times S_V^{U_1,\dots,U_s}\left[E_1\left(\frac{x_1}{t_1}\right)^{e_1}\left(1-\frac{x_1}{t_1}\right)^{f_1},\dots,E_s\left(\frac{x_s}{t_s}\right)^{e_s}\left(1-\frac{x_s}{t_s}\right)^{f_s}\right]$$

$$\times\aleph^{M,N}_{P_i,Q_i,\tau_i,r}\left[z\prod_{j=1}^{s}\left(\frac{x_j}{t_j}\right)^{\eta_j}\left(1-\frac{x_j}{t_j}\right)^{\lambda_j}\left|\begin{matrix}\left(a_j,A_j\right)_{1,N},\left[\tau_i\left(a_{ji},A_{ji}\right)\right]_{N+1,P_i}\\ \left(b_j,B_j\right)_{1,M},\left[\tau_i\left(b_{ji},B_{ji}\right)\right]_{M+1,Q_i}\end{matrix}\right.\right] \tag{9}$$

$$\times\left(t_1,\dots,t_s\right)dt_1,\dots,dt_s$$

Where

(i) $\min \operatorname{Re}\left(e_j,f_j,\eta_j,\lambda_j\right)\ge 0$, $(j=1,\dots,s)$ And all Parameters e_j,f_j,η_j,λ_j are not zero simultaneously,

(ii) $\operatorname{Re}\left(W_j\right)=0$, $\min_{1\le k\le M} \operatorname{Re}\left[1+\rho_j+V_j+\eta_j\frac{b_k}{\beta_k}\right]>0$, $\min_{1\le k\le M} \operatorname{Re}\left[\sigma_j+\lambda_j\frac{b_k}{\beta_k}\right]>0$

Or $\operatorname{Re}\left(W_j\right)>0$, $\min_{1\le k\le M} \operatorname{Re}\left[\sigma_j+\lambda_j\frac{b_k}{\beta_k}\right]>0\cdot$ (10)

Throughout the paper we pretend that

$$f\left(t_1,\dots,t_s\right)=\begin{cases} O\prod_{j=1}^{s}\left(\left|t_j\right|^{U_j}\right) & \max\left\{\left|t_j\right|\right\}\to 0\\ O\prod_{j=1}^{s}\left(\left|t_j\right|^{-V_j}e^{-W_j\left|t_j\right|}\right) & \min\left\{\left|t_j\right|\right\}\to\infty\end{cases}\quad j=i,\dots,s \tag{11}$$

Such a class of function will be represented symbolically as $f(t_1,...,t_s) \in A$.

We also assume that $\int ...\int_{\Lambda_s} \left| f(t_1,...,t_s) \right| dt_1,...,dt_s < \infty$ for every bounded s- dimensional region Λ_s excluding the origin.

Chaurasia [2] gave Series representation of the Aleph function.

$$\aleph^{M,N}_{P_i,Q_i,\tau_i,r}[z] = \sum_{\nu=1}^{M} \sum_{g=0}^{\infty} \theta(S_{\nu,g})\, z^{-S_{\nu,g}} \tag{12}$$

Where

$$\theta(S_{\nu,g}) = \frac{\prod_{j=1}^{m} \Gamma(b_j + B_j S_{\nu,g}) \prod_{j=1}^{n} \Gamma(1 - a_j - A_j S_{\nu,g})(-1)^g}{\sum_{i=1}^{r} \tau_i \prod_{j=m+1}^{q_i} \Gamma(1 - b_{ji} - B_{ji} S_{\nu,g}) \prod_{j=n+1}^{p_i} \Gamma(a_{ji} + A_{ji} S_{\nu,g})\, g!\, B_\nu}, \quad S_{\nu,g} = \frac{b_\nu + g}{B_\nu} \tag{13}$$

In the Sequel, we can also use the following behavior of the Aleph function for small and large of z as recorded by [13]:

$$\aleph^{M,N}_{P_i,Q_i,\tau_i,r}[z] = O\left[|z|^a\right] \text{For small z, where } a = \min_{1\le j\le M} \left[\operatorname{Re} \frac{b_j}{B_j} \right] \tag{14}$$

$$\aleph^{M,N}_{P_i,Q_i,\tau_i,r}[z] = O\left[|z|^b\right] \text{For small z, where } b = \min_{1\le j\le N} \left[\operatorname{Re} \left(\frac{a_j - 1}{A_j} \right) \right] \tag{15}$$

And condition (4), (5), (6) and (7) are also satisfied.

III COMPOSITION FORMULA FOR THE MULTIDIMENSIONAL FRACTIONAL INTEGRAL OPERATORS

Result 1

$$I^{\rho,\sigma;e,f;\eta,\lambda}_{x:U,V;Z} \left\{ I^{\rho',\sigma';e',f';\eta',\lambda'}_{y:U',V';Z'} \left[f(t_1,...,t_s) \right] \right\}$$

$$= \left(\prod_{j=1}^{s} x_j^{-\rho_j'-1} \right) \int_0^{x_1} \ldots \int_0^{x_s} \left(\prod_{j=1}^{s} t_j^{\rho_j'} \right) G\left(\frac{t_1}{x_1}, \ldots, \frac{t_s}{x_s} \right) f\left(t_1, \ldots, t_s\right) dt_1 \ldots dt_s \quad (16)$$

Where $G\,(t_1, \ldots, t_s) = \sum_{\substack{R_1, \ldots, R_s \\ \sum_{j=1}^{s} U_j R_j \le V}} (-V)_{\sum_{j=1}^{s} U_j R_j} A\left(V, R_1, \ldots, R_s\right) \frac{E_1^{R_1}}{R_1!} \cdots \frac{E_s^{R_s}}{R_s!}$

$$\times \sum_{\substack{R_1', \ldots, R_s' \\ \sum_{j=1}^{s} U_j' R_j' \le V'}} (-V')_{\sum_{j=1}^{s} U_j' R_j'} A\left(V', R_1', \ldots, R_s'\right) \frac{E_1'^{R_1'}}{R_1'!} \cdots \frac{E_s'^{R_s'}}{R_s'!}$$

$$\times \sum_{\nu=1}^{M'} \sum_{g=0}^{\infty} \theta\left(S_{\nu,g}\right) z'^{-S_{\nu,g}} \Gamma\left(\sigma_j' + f_j' R_j' - \lambda_j' S_{\nu,g} \right) t_j^{e_j' R_j' - \lambda_j' S_{\nu,g}}$$

$$\left(1-t_j\right)^{\sigma_j + \sigma_j' + f_j R_j + f_j' R_j' - \lambda_j' S_{\nu,g} + n - 1} \aleph^{M, N+2s}_{P_i+2s, Q_i+2s, \tau_i, r} \left[z \prod_{j=1}^{s} \left(1-t_j\right)^{\lambda_j} \middle| \begin{matrix} A^* \\ B^* \end{matrix} \right] \quad (17)$$

Where

$$A^* = \left(a_j, A_j\right)_{1,N}, \left[\tau_i\left(a_{ji}, A_{ji}\right)\right]_{N+1, P_i}, \left(1-n-\sigma_j - f_j R_j; \lambda_j\right)_{1,s},$$

$$\left(-\rho_j + \rho_j' + \sigma_j' - e_j R_j + \left(e_j' + f_j'\right) R_j' - \left(\eta_j' + \lambda_j'\right) S_{\nu,g}; \eta_j\right)_{1,s}$$

$$B^* = \left(b_j, B_j\right)_{1,M}, \left(-\rho_j + \rho_j' + \sigma_j' + n - e_j R_j + f_j R_j + e_j' R_j' - \left(\eta_j' + \lambda_j'\right) S_{\nu,g}; \eta_j\right)_{1,s},$$

$$\left(1-n-\sigma_j - \sigma_j' - f_j R_j - f_j' R_j' - \lambda_j' S_{\nu,g}; \lambda_j\right)_{1,s}, \left[\tau_i\left(b_{ji}, B_{ji}\right)\right]_{M+1, Q_i}$$

Where $\theta\left(S_{v,g}\right)$ and $S_{v,g}$ are given by (13). And following conditions are satisfied:

$$\min_{1\le k\le M} \operatorname{Re}\left[1+\rho_j+\eta_j\frac{b_k}{\beta_k}\right]>0\ ,\ \min_{1\le k\le M} \operatorname{Re}\left[1+\rho'_j+U_j+\eta'_j\frac{b_k}{\beta_k}\right]>0$$

Or $\quad \operatorname{Re}\left(W_j\right)=0\,,\operatorname{Re}\left(W_j\right)>0\ ,\ \min_{1\le k\le M} \operatorname{Re}\left[1+\rho_j++\sigma_j+V_j+\eta'_j\frac{b_k}{\beta_k}\right]>0.$

Result 2

$$J^{\rho,\sigma;e,f;\eta,\lambda}_{x:U,V;Z}\left\{J^{\rho',\sigma';e',f';\eta',\lambda'}_{y:U',V';Z'}\left[f\left(t_1,\dots,t_s\right)\right]\right\}$$

$$=\left(\prod_{j=1}^{s}x_j^{-\rho'_j}\right)\int_{x_1}^{\infty}\dots\int_{x_s}^{\infty}\left(\prod_{j=1}^{s}t_j^{\rho'_j-1}\right)G\left(\frac{x_1}{t_1},\dots,\frac{x_s}{t_s}\right)f\left(t_1,\dots,t_s\right)dt_1\dots dt_s \qquad (18)$$

Where $G\,(t_1,\dots,t_s)$ is given by (17), the composition operator defined by the LHS of (18) exists and the following conditions are satisfied:

$$\min_{1\le k\le M} \operatorname{Re}\left[\sigma_j+\lambda_j\frac{b_k}{\beta_k}\right]>0\ ,\ \min_{1\le k\le M} \operatorname{Re}\left[\sigma'_j+\lambda'_j\frac{b_k}{\beta_k}\right]>0$$

Or $\quad \operatorname{Re}\left(W_j\right)=0\,,\operatorname{Re}\left(W_j\right)>0\ ,\ \min_{1\le k\le M} \operatorname{Re}\left[1+\rho_j++\sigma_j+V_j+\eta_j\frac{b_k}{\beta_k}\right]>0.$

Result 3

$$I^{\rho,\sigma;e,f;\eta,\lambda}_{x:U,V;Z}\left\{J^{\rho',\sigma';e',f';\eta',\lambda'}_{y:U',V';Z'}\left[f\left(t_1,\dots,t_s\right)\right]\right\}$$

$$= \left(\prod_{j=1}^{s} x_j^{-\rho_j - 1} \right) \int_0^{x_1} \dots \int_0^{x_s} \left(\prod_{j=1}^{s} t_j^{\rho_j} \right) G\left(\frac{t_1}{x_1}, \dots, \frac{t_s}{x_s} \right) f\left(t_1, \dots, t_s \right) dt_1 \dots dt_s$$

$$+ \left(\prod_{j=1}^{s} x_j^{\rho'_j} \right) \int_{x_1}^{\infty} \dots \int_{x_s}^{\infty} \left(\prod_{j=1}^{s} t_j^{-\rho'_j - 1} \right) G'\left(\frac{x_1}{t_1}, \dots, \frac{x_s}{t_s} \right) f\left(t_1, \dots, t_s \right) dt_1 \dots dt_s \quad (19)$$

Where

$$G(t_1, \dots, t_s) = \sum_{\substack{\sum_{j=1}^{s} U_j R_j \le V \\ R_1, \dots, R_s}} \left(-V \right)_{\sum_{j=1}^{s} U_j R_j} A\left(V, R_1, \dots, R_s \right) \frac{E_1^{R_1}}{R_1!} \dots \frac{E_s^{R_s}}{R_s!}$$

$$\times \sum_{\substack{\sum_{j=1}^{s} U'_j R'_j \le V' \\ R'_1, \dots, R'_s}} \left(-V' \right)_{\sum_{j=1}^{s} U'_j R'_j} A\left(V', R'_1, \dots, R'_s \right) \frac{E_1'^{R'_1}}{R'_1!} \dots \frac{E_s'^{R'_s}}{R'_s!}$$

$$\times \sum_{\nu=1}^{M'} \sum_{g=0}^{\infty} \theta\left(S_{\nu,g} \right) z'^{-S_{\nu,g}} \left(1 - t_j \right)^{\sigma_j + \sigma'_j + f_j R_j + f'_j R'_j - \lambda'_j S_{\nu,g} - 1} t_j^{e'_j R'_j + n}$$

$$\times \Gamma\left(\sigma'_j + f'_j R'_j - \lambda'_j S_{\nu,g} + n \right) \aleph^{M, N+2s}_{P_i + 2s, Q_i + 2s, \tau_i, r} \left[z \prod_{j=1}^{s} \left(t_j \right)^{\eta_j} \left(1 - t_j \right)^{\lambda_j} \middle| \begin{matrix} C^* \\ D^* \end{matrix} \right] \quad (20)$$

Where

$$C^* = \left(a_j, A_j \right)_{1,N}, \left[\tau_i \left(a_{ji}, A_{ji} \right) \right]_{N+1, P_i}, \left(-\rho_j - \rho'_j - e_j R_j - e'_j R'_j - \eta'_j S_{\nu,g}; \eta_j \right)_{1,s},$$

$$\left(1 - \sigma_j - \rho_j - \rho'_j - \sigma'_j - \left(e_j + f_j \right) R_j - \left(e'_j + f'_j \right) R'_j + \left(\eta'_j + \lambda'_j \right) S_{\nu,g}; \left(\eta_j + \lambda_j \right) \right)_{1,s}$$

$$D^{*}=\left(b_j,B_j\right)_{1,M},\left(-\rho_j'-n-\rho_j-\sigma_j'-e_jR_j-\left(e_j'+f_j'\right)R_j'-\left(\lambda_j'+\eta_j'\right)S_{\nu,g};\eta_j\right)_{1,s},$$

$$\left(1-\rho_j-\rho_j'-\sigma_j-\sigma_j'-\left(e_j+f_j\right)R_j-\left(e_j'+f_j'\right)R_j'+\left(\eta_j'+\lambda_j'\right)S_{\nu,g};\left(\lambda_j+\eta_j\right)\right)_{1,s},\left[\tau_i\left(b_{ji},B_{ji}\right)\right]_{M+1,Q_i}$$

Here, it is pretended that the composition operator defined by the LHS of (19) exists, $f(t_1,...,t_s)\in A$ and $G'(t_1,...,t_s)$ can be written from $G(t_1,...,t_s)$ from (17) by interchanging the parameters with dashes with those without dashes also $\theta\left(S_{\nu,g}\right)$ and $S_{\nu,g}$ are given by (13) and the conditions are follows:

$$\min_{1\le k\le M}\operatorname{Re}\left[1+\rho_j+\eta_j\frac{b_k}{\beta_k}\right]>0\ ,\ \min_{1\le k\le M}\operatorname{Re}\left[1+\rho_j'+U_j+\eta_j'\frac{b_k}{\beta_k}\right]>0$$

$$\operatorname{Re}\left(W_j\right)>0\ \text{Or}\quad \operatorname{Re}\left(W_j\right)=0\ ,\ \min_{1\le k\le M}\operatorname{Re}\left[1+\rho_j++\sigma_j+V_j+\eta_j\frac{b_k}{\beta_k}\right]>0\,.$$

Proof- To Prove results 1,by using the equation (8), we express the I-operator present in the left hand side of (16) in the integral form, we have

$$I_{x:U,V;Z}^{\rho,\sigma;e,f;\eta,\lambda}\left\{I_{y:U',V';Z'}^{\rho',\sigma';e',f';\eta',\lambda'}\left[f\left(t_1,...,t_s\right)\right]\right\}$$

$$=\left(\prod_{j=1}^{s}x_j^{-\rho_j-\sigma_j}\right)\int_0^{x_1}...\int_0^{x_s}\left(\prod_{j=1}^{s}t_j^{\rho_j'}\right)f\left(t_1,...,t_s\right)\Delta\,dt_1...dt_s \tag{21}$$

Where

$$\Delta=\int_{t_1}^{x_1}...\int_{t_s}^{x_s}\left[\prod_{j=1}^{s}y_j^{\rho_j-\rho_j'-\sigma_j'}\left(x_j-y_j\right)^{\sigma_j-1}\left(y_j-t_j\right)^{\sigma_j'-1}\right]$$

$$\times S_V^{U_1,...,U_s}\left[E_1\left(\frac{y_1}{x_1}\right)^{e_1}\left(1-\frac{y_1}{x_1}\right)^{f_1},...,E_s\left(\frac{y_s}{x_s}\right)^{e_s}\left(1-\frac{y_s}{x_s}\right)^{f_s}\right]$$

$$\times S_V^{U_1',\ldots,U_s'}\left[E_1'\left(\frac{t_1}{y_1}\right)^{e_1'}\left(1-\frac{t_1}{y_1}\right)^{f_1'},\ldots,E_s\left(\frac{t_s}{y_s}\right)^{e_s'}\left(1-\frac{t_s}{y_s}\right)^{f_s'}\right]$$

$$\times \aleph\left[z\prod_{j=1}^{s}\left(\frac{y_j}{x_j}\right)^{\eta_j}\left(1-\frac{y_j}{x_j}\right)^{\lambda_j}\right]\aleph'\left[z'\prod_{j=1}^{s}\left(\frac{y_j}{x_j}\right)^{\eta_j'}\left(1-\frac{y_j}{x_j}\right)^{\lambda_j'}\right]dy_1,\ldots,dy_s \quad (22)$$

To find Δ, very first we express both the multivariable polynomials $S_V^{U_1,\ldots,U_s}$, $S_V^{U_1',\ldots,U_s'}$ and Aleph $\aleph'$ -function involved in terms of their respective series with the help of equation (8) and (9) respectively, and express the Aleph $\aleph$-function in terms of Mellin –Barnes type contour integral by using (1). Then replacing y_j -integrals at the place of the order of summations and Mellin- Barnes contour integral, we get

$$\Delta=\sum_{R_1,\ldots,R_s}^{\sum_{j=1}^{s}U_jR_j\le V}(-V)_{\sum_{j=1}^{s}U_jR_j}A\left(V,R_1,\ldots,R_s\right)\frac{E_1^{R_1}}{R_1!}\cdots\frac{E_s^{R_s}}{R_s!}$$

$$\times\sum_{R_1',\ldots,R_s'}^{\sum_{j=1}^{s}U_j'R_j'\le V'}\left(-V'\right)_{\sum_{j=1}^{s}U_j'R_j'}A\left(V',R_1',\ldots,R_s'\right)\frac{E_1'^{R_1'}}{R_1'!}\cdots\frac{E_s'^{R_s'}}{R_s'!}$$

$$\times\sum_{\nu=1}^{M'}\sum_{g=0}^{\infty}\theta\left(S_{\nu,g}\right)z'^{-S_{\nu,g}}\frac{1}{2\pi i}\int_L\varphi(\xi)z^{-\xi}d\xi\; x^{-e_jR_j-f_jR_j+\left(\eta_j+\lambda_j\right)\xi}$$

$$\int_{t_1}^{x_1}\cdots\int_{t_s}^{x_s}\left[\prod_{j=1}^{s} y_j^{\rho_j - \rho'_j - \sigma'_j + e_j R_j - \left(e'_j + f'_j\right)R'_j + \left(\eta'_j + \lambda'_j\right)S_{\nu,g} - \eta_j\xi}\right. \tag{23}$$

$$\left.\left(x_j - y_j\right)^{\sigma_j + f_j R_j - \lambda_j \xi - 1}\left(y_j - t_j\right)^{\sigma'_j + f'_j R'_j - \lambda'_j S_{\nu,g} - 1} dy_1 \ldots dy_s\right] d\xi$$

Now we Put $\frac{\left(x_j - y_j\right)}{\left(x_j - t_j\right)} = u_j$ in (23) and calculate the u_j integrals thus obtained by using the following result [5, p.287.eq.3.197 (8)]

$$\int_0^1 x^{\delta-1}(x+a)^{\lambda}(1-x)^{\mu-1}dx = a^{\lambda}\ \beta(\delta,\mu)\ {}_2F_1\left[-\lambda,\delta;\delta+\mu;-\frac{1}{a}\right] \tag{24}$$

Now, we get the results (16) by rearranging the result which is obtained in term of the Aleph-function and also by substituting the values in (23), after little arrangements.

On the Similar basis, the Proof of the results 2 might be developing, so skip the details.

Proof the results 3 - very First we write I and J- Multidimensional fractional integral operators involved in the LHS of the (19), in the integral form by using the definition of I and J-fractional integral operators (8) and (9) respectively, then we get the integral given below:

$$I_{x:U,V;Z}^{\rho,\sigma;e,f;\eta,\lambda}\left\{J_{y:U',V';Z'}^{\rho',\sigma';e',f';\eta',\lambda'}\left[f\left(t_1,\ldots,t_s\right)\right]\right\}$$

$$=\left(\prod_{j=1}^{s} x_j^{-\rho_j-\sigma_j}\right)\int_0^{x_1}\cdots\int_0^{x_s}\left[\prod_{j=1}^{s} y_j^{\rho_j}\left(x_j - y_j\right)^{\sigma_j - 1}\right]$$

$$\times S_V^{U_1,\dots,U_s}\left[E_1\left(\frac{y_1}{x_1}\right)^{e_1}\left(1-\frac{y_1}{x_1}\right)^{f_1},\dots,E_s\left(\frac{y_s}{x_s}\right)^{e_s}\left(1-\frac{y_s}{x_s}\right)^{f_s}\right]$$

$$\times \aleph\left[z\prod_{j=1}^{s}\left(\frac{y_j}{x_j}\right)^{\eta_j}\left(1-\frac{y_j}{x_j}\right)^{\lambda_j}\right]\left(\prod_{j=1}^{s} y_j^{\rho_j'}\right)\int_{y_1}^{\infty}\dots\int_{y_s}^{\infty}\left[\prod_{j=1}^{s} t_j^{-\rho_j'-\sigma_j'}\left(t_j-y_j\right)^{\sigma_j'-1}\right]$$

$$\times S_V^{U_1',\dots,U_s'}\left[E_1'\left(\frac{y_1}{t_1}\right)^{e_1'}\left(1-\frac{y_1}{t_1}\right)^{f_1'},\dots,E_s\left(\frac{y_s}{t_s}\right)^{e_s'}\left(1-\frac{y_s}{t_s}\right)^{f_s'}\right]$$

$$\times \aleph'\left[z'\prod_{j=1}^{s}\left(\frac{y_j}{t_j}\right)^{\eta_j'}\left(1-\frac{y_j}{t_j}\right)^{\lambda_j'}\right] f\left(t_1,\dots,t_s\right)dt_1,\dots,dt_s dy_1,\dots,dy_s \tag{25}$$

After this, by exchanging the order of t_j and y_j integrals (which is permissible under the conditions stated) we get

$$I_{x:U,V;Z}^{\rho,\sigma;e,f;\eta,\lambda}\left\{J_{y:U',V';Z'}^{\rho',\sigma';e',f';\eta',\lambda'}\left[f\left(t_1,\dots,t_s\right)\right]\right\}$$

$$=\int_0^{x_1}\dots\int_0^{x_s}\left\{\int_0^{t_1}\dots\int_0^{t_s}\Lambda\, dy_1\dots dy_s\right\}f\left(t_1,\dots,t_s\right)dt_1\dots dt_s$$

$$+\int_{x_1}^{\infty}\dots\int_{x_s}^{\infty}\left\{\int_0^{x_1}\dots\int_0^{x_s}\Lambda\, dy_1\dots dy_s\right\}f\left(t_1,\dots,t_s\right)dt_1\dots dt_s$$

$$=\int_0^{x_1}\dots\int_0^{x_s} I_1\, f\left(t_1,\dots,t_s\right)dt_1\dots dt_s+\int_{x_1}^{\infty}\dots\int_{x_s}^{\infty} I_2\, f\left(t_1,\dots,t_s\right)dt_1\dots dt_s \tag{26}$$

Where

$$\Lambda = \left(\prod_{j=1}^{s} x_j^{-\rho_j - \sigma_j} \, t_j^{-\rho'_j - \sigma'_j} \, y_j^{\rho_j + \rho'_j} \left(x_j - y_j\right)^{\sigma_j - 1} \left(t_j - y_j\right)^{\sigma'_j - 1} \right)$$

$$S_V^{U_1,\ldots,U_s}\left[E_1 \left(\frac{y_1}{x_1}\right)^{e_1} \left(1 - \frac{y_1}{x_1}\right)^{f_1}, \ldots, E_s \left(\frac{y_s}{x_s}\right)^{e_s} \left(1 - \frac{y_s}{x_s}\right)^{f_s} \right]$$

$$\times S_V^{U'_1,\ldots,U'_s}\left[E'_1 \left(\frac{y_1}{t_1}\right)^{e'_1} \left(1 - \frac{y_1}{t_1}\right)^{f'_1}, \ldots, E_s \left(\frac{y_s}{t_s}\right)^{e'_s} \left(1 - \frac{y_s}{t_s}\right)^{f'_s} \right]$$

$$\times \aleph\left[z \prod_{j=1}^{s} \left(\frac{y_j}{x_j}\right)^{\eta_j} \left(1 - \frac{y_j}{x_j}\right)^{\lambda_j} \right] \aleph'\left[z' \prod_{j=1}^{s} \left(\frac{y_j}{t_j}\right)^{\eta'_j} \left(1 - \frac{y_j}{t_j}\right)^{\lambda'_j} \right] dy_1, \ldots, dy_s \qquad (27)$$

$$\text{And } I_1 = \left\{ \int_0^{t_1} \cdots \int_0^{t_s} \Lambda \, dy_1 \ldots dy_s \right\} \text{and } I_2 = \left\{ \int_0^{x_1} \cdots \int_0^{x_s} \Lambda \, dy_1 \ldots dy_s \right\} \qquad (28)$$

Now to find I_1, involving in the integral on the RHS of (26), we write the multivariable polynomials $S_V^{U_1,\ldots,U_s}$, $S_V^{U'_1,\ldots,U'_s}$ and Aleph $\aleph'$ -function involved in terms of their respective series with the help of equation (8) and (12) respectively , and express the Aleph $\aleph$-function in terms of Mellin –Barnes type contour integral by using (2). The interchanging the order of summations and Mellin-Barnes contour integral with y_j -integrals and further, solving the y_j -integrals, we have

$$I_1 = \left\{ \int_0^{t_1} \cdots \int_0^{t_s} \Lambda \, dy_1 \ldots dy_s \right\} = \sum_{\substack{R_1,\ldots,R_s \\ \sum_{j=1}^{s} U_j R_j \le V}} (-V)_{\sum_{j=1}^{s} U_j R_j} \, A\left(V, R_1, \ldots, R_s\right) \frac{E_1^{R_1}}{R_1!} \cdots \frac{E_s^{R_s}}{R_s!}$$

$$\times \sum_{\substack{R'_1,\ldots,R'_s \\ \sum_{j=1}^{s} U'_j R'_j \le V'}} \left(-V'\right)^{\sum_{j=1}^{s} U'_j R'_j} A\left(V', R'_1,\ldots,R'_s\right) \frac{E_1'^{\,R'_1}}{R'_1!} \cdots \frac{E_s'^{\,R'_s}}{R'_s!}$$

$$\times \sum_{\nu=1}^{M'} \sum_{g=0}^{\infty} \theta\left(S_{\nu,g}\right) z'^{\,-S_{\nu,g}} \frac{1}{2\pi i} \int_L \varphi(\xi)\, z^{-\xi}\, d\xi$$

$$\left(\prod_{j=1}^{s} x_j^{-\rho_j-\sigma_j-e_jR_j-f_jR_j+\left(\eta_j+\lambda_j\right)\xi}\; t_j^{-\rho'_j-\sigma'_j-\left(e'_j+f'_j\right)R'_j+\left(\eta'_j+\lambda'_j\right)S_{\nu,g}} \right)$$

$$\int_0^{t_1} \cdots \int_0^{t_s} \left[\prod_{j=1}^{s} y_j^{\rho_j+\rho'_j+e_jR_j+e'_jR'_j-\eta'_j S_{\nu,g}-\eta_j\xi} \left(x_j-y_j\right)^{\sigma_j+f_jR_j-\lambda_j\xi-1} \left(t_j-y_j\right)^{\sigma'_j+f'_jR'_j-\lambda'_j S_{\nu,g}-1} \right] dy_1 \ldots dy_s \tag{29}$$

Then, we put $y_j = t_j u_j$ in (29) and integrate it with help of the results [11, p.47, Th. 1.6] we obtain the equation that is given below:

$$I_1 = \left\{ \int_0^{t_1} \cdots \int_0^{t_s} \Lambda\, dy_1 \ldots dy_s \right\} = \sum_{\substack{R_1,\ldots,R_s \\ \sum_{j=1}^{s} U_j R_j \le V}} \left(-V\right)^{\sum_{j=1}^{s} U_j R_j} A\left(V, R_1,\ldots,R_s\right) \frac{E_1^{R_1}}{R_1!} \cdots \frac{E_s^{R_s}}{R_s!}$$

$$\times \sum_{\substack{R'_1,\ldots,R'_s \\ \sum_{j=1}^{s} U'_j R'_j \le V'}} \left(-V'\right)^{\sum_{j=1}^{s} U'_j R'_j} A\left(V', R'_1,\ldots,R'_s\right) \frac{E_1'^{\,R'_1}}{R'_1!} \cdots \frac{E_s'^{\,R'_s}}{R'_s!}$$

$$\times \sum_{\nu=1}^{M'} \sum_{g=0}^{\infty} \theta\left(S_{\nu,g}\right) z'^{\,-S_{\nu,g}} \frac{1}{2\pi i} \int_L \varphi(\xi)\, z^{-\xi}\, d\xi$$

$$\left(\prod_{j=1}^{s} x_j^{-\rho_j - e_j R_j + \eta_j \xi}\, t_j^{\rho_j + e_j R_j - \eta_j \xi}\right)$$

$$\beta\left(\sigma'_j + f'_j R'_j - \lambda'_j S_{\nu,g}, \rho_j + \rho'_j + e_j R_j + e'_j R'_j - \eta'_j S_{\nu,g} - \eta_j \xi + 1\right)$$

$$ {}_2F_1\left[\begin{matrix} -\left(\sigma_j + f_j R_j - \lambda_j \xi - 1\right); 1 + \rho_j + \rho'_j + e_j R_j + e'_j R'_j - \eta'_j S_{\nu,g} - \eta_j \xi \\ 1 + \rho_j + \rho'_j + \sigma'_j + e_j R_j + \left(e'_j + f'_j\right) R'_j - \left(\eta'_j + \lambda'_j\right) S_{\nu,g} - \eta_j \xi \end{matrix}; \left(\frac{t_j}{x_j}\right)\right]$$

$$\int_0^{t_1} \cdots \int_0^{t_s} \left[\prod_{j=1}^{s} y_j^{\rho_j + \rho'_j + e_j R_j + e'_j R'_j - \eta'_j S_{\nu,g} - \eta_j \xi}\right.$$

$$\left.\left(x_j - y_j\right)^{\sigma_j + f_j R_j - \lambda_j \xi - 1} \left(t_j - y_j\right)^{\sigma'_j + f'_j R'_j - \lambda'_j S_{\nu,g} - 1}\right] dy_1 \ldots dy_s \qquad (30)$$

Finally, using the following result [9, p.60, Eq. (5)], we transform right hand side of (30),

$$ {}_2F_1[a, b; c; z] = (1 - z)^{c - a - b}\, {}_2F_1[c - a, c - b; c; z]$$

And expand the ${}_2F_1$ thus deduced in the series form and rearranging the result in terms of Aleph function we obtain the solution of I_1.

To find $I_2 = \int_0^{x_1} \ldots \int_0^{x_s} \Lambda\, dy_1 \ldots dy_s$, we follow procedure as it is mentioned above with the only difference that we substitute $y_j = x_j u_j$ in the corresponding expression to (29). By writing the values of I_1 and I_2 in (26), we get the required result (19).

SPECIAL CASE OF COMPOSITION FORMULAE

We now present a two dimensional analogue of our first composition formula. By Putting

$s = 2$ and pretending the generalized class of polynomials as unity, we get

$$I_{x,y;Z}^{\rho,\sigma;\eta,\lambda}\left\{I_{s,t;Z'}^{\rho',\sigma';\eta',\lambda'}\left[f(u,v)\right]\right\}$$

$$=I_{x,y;Z}^{\rho,\sigma;\eta,\lambda}\Bigg\{s^{-\rho'-\sigma'}t^{-m'-n'}\int_0^s\int_0^t u^{\rho'}v^{m'}(s-u)^{\sigma'-1}(t-v)^{n'-1}$$

$$\times\aleph'\left[z'\left(\frac{u}{s}\right)^{\eta'}\left(\frac{v}{t}\right)^{\delta'}\left(1-\frac{u}{s}\right)^{\lambda'}\left(1-\frac{v}{t}\right)^{\mu'}\right]f(u,v)\,du\,dv\Bigg\}$$

$$=\sum_{\nu=1}^{M'}\sum_{g=0}^{\infty}\theta\left(S_{\nu,g}\right)z'^{-S_{\nu,g}}\,x^{-\rho-\sigma-\rho'-\sigma'-k-\left(\lambda'_j+\mu'_j\right)S_{\nu,g}}$$

$$\times\,y^{-m-n-m'-n'-k-\left(\delta'_j+\mu'_j\right)S_{\nu,g}}\,\Gamma\left(\sigma'+\lambda'S_{\nu,g}\right)\Gamma\left(\eta'+\mu'S_{\nu,g}\right)$$

$$\int_0^x\cdots\int_0^y\left[(x-u)^{\sigma+\sigma'+k+\lambda' S_{\nu,g}-1}(y-v)^{\eta+\eta'+k+\mu'_j S_{\nu,g}-1}u^{\rho'+\eta' S_{\nu,g}}v^{m'+\delta' S_{\nu,g}}\right]$$

$$\times\,\aleph_{P_i+4,Q_i+4,\tau_i,r}^{M+2,N+2}\left[z\left(1-\frac{u}{x}\right)^{\lambda}\left(1-\frac{v}{y}\right)^{\mu}\,\middle|\,\begin{matrix}E^*\\F^*\end{matrix}\right]f(u,v)\,du\,dv \tag{31}$$

Where

$$E^*=\left(a_j,A_j\right)_{1,N},(1-k-\sigma;\lambda),(1-k-\eta;\mu),\left[\tau_i\left(a_{ji},A_{ji}\right)\right]_{N+1,P_i},$$

$$\left(\rho'+\sigma'-\left(\eta'+\lambda'\right)S_{\nu,g};\eta\right),\left(m'+n'-\left(\delta'+\mu'\right)S_{\nu,g};\delta\right)$$

$$F^*=\left(b_j,B_j\right)_{1,M},\left(\rho'+\sigma'-\left(\eta'+\lambda'\right)S_{\nu,g}+k;\eta\right),\left(m'+n'-\left(\delta'+\mu'\right)S_{\nu,g}+k;\delta\right),$$

$$\left[\tau_i\left(b_{ji},B_{ji}\right)\right]_{M+1,Q_i},\left(1-\sigma-\sigma'-k-\lambda'S_{\nu,g}+k;\delta\right),\left(1-\eta-\eta'-k-\mu'S_{\nu,g};\mu\right)$$

Where $\theta\left(S_{\nu,g}\right)$ and $S_{\nu,g}$ are given by (13). And following conditions are satisfied:

$\min_{1\le k\le M} \operatorname{Re}\left[1+\rho+m+(\eta+\delta)\left(\frac{b_k}{\beta_k}\right)\right]>0$, $\min_{1\le k\le M} \operatorname{Re}\left[\sigma+\eta+(\lambda+\mu)\left(\frac{b_k}{\beta_k}\right)\right]>0$

$\min_{1\le k\le M} \operatorname{Re}\left[1+\rho'+m'+\left(\eta'+\delta'\right)\left(\frac{b'_k}{\beta'_k}\right)\right]>0$,

$\min_{1\le k\le M} \operatorname{Re}\left[1+\sigma'+\eta'+\left(\lambda'+\mu'\right)\left(\frac{b'_k}{\beta'_k}\right)\right]>0$ In the same way two dimensional Formulae can be gained easily from result 2 and 3. These formulae can be further reduced to results given by Raina [8] by taking $\tau_i = 1$ and I-function to unity.

If in these composition formulas we reduce both the generalized class of polynomials to unity, we reach at the multidimensional analogue of the simpler results given by Erdelyi [1], Again reducing the generalized hyper geometric function, we arrive at the corresponding result given by Goyal and Jain [13, p.253, eq.(2.4),p. 254 , eq. (2.7); p.255,eq. (2.12)] after a little simplification. Further , if we reduce generalized class of polynomials to polynomials we arrive at the result which are in core the same as those obtained by Goyal , Jain and Gaur [3, p.404-405,eq. (2.1);p.406,eq. (2.7); p. 407-408,eq.(2.12)].

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