

Geometry of MHD Flows Using Some Types of Curvilinear Coordinates

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Abstract

Taking streamlines and magnetic lines as an orthogonal curvilinear coordinate system the governing equations of a steady plane, viscous, infinitely conducting (electrically), incompressible Magnetohydrodynamic flow are represented using the fundamental coefficients for a plane surface. Streamlines of the fluid flow are taken as one set of coordinate lines in a curvilinear coordinate system and magnetic lines are considered as the other coordinate lines. Establishing the validity of the transformation by proving that Jacobian is a non – zero constant, geometry of magnetic lines and streamlines are studied for some specific flows. Also the Gauss characteristic equation for solution of the flow is verified.

Keywords: streamlines, MHD, curvilinear coordinate system, first fundamental forms

1. INTRODUCTION:

M. H. Martin [1] has given a new method of writing the governing equations of the plane flow of viscous, incompressible, non – conducting fluids by introducing the curvilinear coordinates ϕ , ψ in the plane of flow in which the coordinate lines $\psi = \text{constant}$ are the stream lines of the flow and the coordinate lines $\phi = \text{constant}$ are left arbitrary. Taking Martin's work further, Nath and Chandna [2] used the approach to study MHD flows. They studied certain properties of plane Magnetohydrodynamic flows of viscous and incompressible fluids with orthogonal magnetic and velocity fields. Considering ψ and ϕ as independent variables, they proved that if the streamlines are straight lines but not parallel, then they must be concurrent and if the streamlines are involutes of a curve, then streamlines are concentric circles. O. P. Chandna and M. R. Garg [3] investigated the

geometries of steady plane magnetohydrodynamic flows of a steady viscous incompressible fluid when the streamlines and magnetic lines form an isometric net and when magnetic force vanishes. They [4] also investigated these flows with mutually orthogonal magnetic and velocity fields using Hodograph transformation method. Kingston and Talbot [5] classified the corresponding flows of an inviscid incompressible fluid.

Following Martin's approach, we obtain a system of partial differential equations for the fundamental coefficients E, F, G for a plane surface, as functions of φ, ψ , to study steady plane Magnetohydrodynamic flows of a viscous incompressible fluid, having infinite electrical conductivity, when the magnetic fields and velocity fields are mutually orthogonal. This approach is further illustrated by considering orthogonal curvilinear coordinate system with its respective first fundamental forms satisfying the condition that the corresponding Gaussian curvature is zero. Moreover, we calculate expressions for velocity and vorticity of the fluid flow, obtain the magnetic field and current density.

2. Flow Equations: Governing equations of a steady plane Magnetohydrodynamic flow of a viscous incompressible fluid of infinite electrical conductivity, in the absence of heat conduction, is given by

$$\operatorname{div} \vec{v} = 0 \quad (2.1)$$

$$\rho \left[(\vec{v} \cdot \operatorname{grad}) \vec{v} \right] + \operatorname{grad} p = \mu \operatorname{curl} \vec{H} \times \vec{H} + \eta \nabla^2 \vec{v} \quad (2.2)$$

$$\operatorname{curl} (\vec{v} \times \vec{H}) = \vec{0} \quad (2.3)$$

$$\operatorname{div} \vec{H} = 0 \quad (2.4)$$

where $\vec{v} = (v_1, v_2)$ denotes the velocity vector with $v = |\vec{v}| = \sqrt{v_1^2 + v_2^2}$, $\vec{H} = (H_1, H_2)$ the magnetic field vector, ρ the constant fluid density, p the fluid pressure, η the constant coefficient of viscosity, μ the constant magnetic permeability of flow.

Also the vorticity function $\omega(x, y)$, current density function $\delta(x, y)$ and the energy function $h(x, y)$ are given by [7]

$$\omega = \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \quad (2.5)$$

$$\delta = \frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y} \quad (2.6)$$

$$h = \frac{1}{2} \rho v^2 + p \quad \text{or} \quad h = \frac{1}{2} \rho (v_1^2 + v_2^2) + p \quad (2.7)$$

Now (2.1) becomes

$$\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} = 0 \tag{2.8}$$

Using (2.5) and (2.7), (2.2) becomes

$$\eta \frac{\partial \omega}{\partial y} - \rho \omega v_2 + \mu \delta H_2 = -\frac{\partial h}{\partial x} \tag{2.9}$$

$$\eta \frac{\partial \omega}{\partial x} - \rho \omega v_1 + \mu \delta H_1 = \frac{\partial h}{\partial y} \tag{2.10}$$

It is assumed in this work that streamlines are nowhere parallel to the magnetic lines, so from (2.3) we get

$$v_1 H_2 - v_2 H_1 = k \tag{2.11}$$

where k is an arbitrary non – zero constant.

Finally (2.4) gives

$$\frac{\partial H_1}{\partial x} + \frac{\partial H_2}{\partial y} = 0. \tag{2.12}$$

We shall determine the seven unknown functions $v_1, v_2, H_1, H_2, \omega, \delta$ and h .

3. CONCEPTS OF DIFFERENTIAL GEOMETRY:

Let

$$x = x(\phi, \psi) \text{ and } y = y(\phi, \psi)$$

(3.1) define a system of curvilinear coordinate system (ϕ, ψ) in the (x, y) plane. The squared element of arc length in the curvilinear coordinate system (ϕ, ψ) is given by [6]

$$ds^2 = E d\phi^2 + 2F d\phi d\psi + G d\psi^2 \tag{3.2}$$

where

$$E = \left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2, F = \frac{\partial x}{\partial \phi} \frac{\partial x}{\partial \psi} + \frac{\partial y}{\partial \phi} \frac{\partial y}{\partial \psi}, G = \left(\frac{\partial x}{\partial \psi}\right)^2 + \left(\frac{\partial y}{\partial \psi}\right)^2 \text{ and } W^2 = EG - F^2$$

Using (3.1) we can obtain $\phi = \phi(x, y)$ and $\psi = \psi(x, y)$ such that

$$\frac{\partial x}{\partial \phi} = J \frac{\partial \psi}{\partial y}, \quad \frac{\partial y}{\partial \phi} = -J \frac{\partial \psi}{\partial x}, \quad \frac{\partial x}{\partial \psi} = -J \frac{\partial \phi}{\partial y}, \quad \frac{\partial y}{\partial \psi} = J \frac{\partial \phi}{\partial x} \tag{3.3}$$

provided $0 < |J| < \infty$ where J , the Jacobian is given by [8]

$$J = \frac{\partial(x, y)}{\partial(\psi, \phi)} = \begin{vmatrix} \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \psi} \\ \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \psi} \end{vmatrix} = \frac{\partial x}{\partial \phi} \frac{\partial y}{\partial \psi} - \frac{\partial x}{\partial \psi} \frac{\partial y}{\partial \phi} \quad (3.4)$$

and $W^2 = EG - F^2 = J^2 \Rightarrow J = \pm W$

The incompressibility constraint equation (2.8) implies the existence of a stream function $\psi(x, y)$ and the solenoidal equation (2.12) implies the existence of a magnetic flux function $\phi(x, y)$ such that

$$\frac{\partial \psi}{\partial x} = -v_2, \quad \frac{\partial \psi}{\partial y} = v_1, \quad \frac{\partial \phi}{\partial x} = H_2, \quad \frac{\partial \phi}{\partial y} = -H_1 \quad (3.5)$$

We assume that the curves $\psi = \text{constant}$ and the curves $\phi = \text{constant}$ form the curvilinear coordinate system.

Using (3.4) and (3.5) we get from (2.11)

$$\frac{\partial \psi}{\partial y} \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial \phi}{\partial y} = \frac{\partial(\psi, \phi)}{\partial(x, y)} = \frac{1}{J} = k \neq 0 \quad (3.6)$$

\therefore By inverse theorem of differential calculus, when x and y are known as functions of ϕ and ψ , then ϕ and ψ can be obtained as functions of x and y .

Moreover, if α is the angle made by the tangent to the coordinate line $\psi = \text{constant}$, directed in the sense of increasing ϕ , then writing $\frac{\partial x}{\partial \phi} = \sqrt{E} \cos \alpha$ and $\frac{\partial y}{\partial \phi} = \sqrt{E} \sin \alpha$

we find that

$$K = \frac{1}{W} \left[\left\{ \frac{W}{E} \left(\frac{-FE_\phi + 2EF_\phi - EE_\psi}{2W^2} \right) \right\}_\psi - \left\{ \frac{W}{E} \left(\frac{EG_\phi - FE_\psi}{2W^2} \right) \right\}_\phi \right] = 0 \quad (3.7)$$

Thus we can rewrite the flow equation in a new form to obtain the solution to determine the seven unknown functions $v_1, v_2, H_1, H_2, \omega, \delta$ and h in terms of ϕ and ψ .

4. New form of the governing equations: The set of seven partial differential equations (2.8) – (2.12) and (2.5) – (2.6) for $v_1, v_2, H_1, H_2, \omega, \delta$ and h as functions of x, y can be replaced by the system

$$-F \left(\left(\frac{\partial h}{\partial \phi} \right) + \mu \delta \right) + E \left(\left(\frac{\partial h}{\partial \psi} \right) + \rho \omega \right) = \eta J \left(\frac{\partial \omega}{\partial \phi} \right) \quad (4.1)$$

$$G \left(\left(\frac{\partial h}{\partial \phi} \right) + \mu \delta \right) - F \left(\left(\frac{\partial h}{\partial \psi} \right) + \rho \omega \right) = -\eta J \left(\frac{\partial \omega}{\partial \psi} \right) \quad (4.2)$$

$$\frac{\partial}{\partial \psi} \left\{ \frac{W}{E} \left(\frac{-FE_\phi + 2EF_\phi - EE_\psi}{2W^2} \right) \right\} - \frac{\partial}{\partial \phi} \left\{ \frac{W}{E} \left(\frac{EG_\phi - FE_\psi}{2W^2} \right) \right\} = 0 \quad (4.3)$$

$$\omega = \frac{1}{W} \left[\frac{\partial}{\partial \phi} \left(\frac{F}{W} \right) - \frac{\partial}{\partial \psi} \left(\frac{E}{W} \right) \right] \quad (4.4)$$

$$\delta = \frac{1}{W} \left[\frac{\partial}{\partial \phi} \left(\frac{G}{W} \right) - \frac{\partial}{\partial \psi} \left(\frac{F}{W} \right) \right] \quad (4.5)$$

$$W^2 = J^2 = EG - F^2 = \frac{1}{k^2} \quad (4.6)$$

of six partial differential equations for E, F, G, ω, δ and h as functions of ϕ and ψ . The Jacobian J is positive or negative as the parameter of ψ is increasing or decreasing in the direction of the magnetic field \vec{H} .

Given a solution

$$\begin{aligned} E &= E(\phi, \psi) & F &= F(\phi, \psi) & G &= G(\phi, \psi) \\ \omega &= \omega(\phi, \psi) & \delta &= \delta(\phi, \psi) & h &= h(\phi, \psi) \end{aligned}$$

of the system of partial differential equations (4.1) – (4.6), we can find x and y as functions of ϕ and ψ from

$$z = x + iy = \int \frac{e^{i\alpha}}{\sqrt{E}} \{ E d\phi + (F + iJ) d\psi \} \quad \text{where} \quad \alpha = \int \frac{J}{E} (\Gamma_{11}^2 d\phi + \Gamma_{12}^2 d\psi) \text{ and thus}$$

obtain E, F, G, ω, δ and h as functions of x and y , since $0 < |J| < \infty$. Once we obtain $E, F, G,$ and h as functions of ϕ and ψ , then H_1, H_2, v_1, v_2 and p as functions of x and y are given by

$$WV = \sqrt{E} \text{ and } v_1 + iv_2 = \frac{\sqrt{E}}{J} e^{i\alpha} \quad (4.7)$$

where α is the angle between the tangent to the co-ordinate line $\psi = \text{constant}$, directed in the sense of increasing ϕ with the $x - \text{axis}$

$$WH = \sqrt{G} \text{ and } H_1 + iH_2 = \sqrt{\frac{G}{J}} e^{i\beta} \quad (4.8)$$

where β is the angle between the tangent to the co-ordinate line and $\phi = \text{constant}$, directed in the sense of increasing ψ with $x - \text{axis}$.

$$p = h - \frac{\rho}{2} \frac{E}{W^2} \quad (4.9)$$

Eliminating $\partial h / \partial \phi$ and $\partial h / \partial \psi$ respectively from equations (4.1) and (4.2), we get

$$\frac{\partial h}{\partial \phi} = \frac{\eta}{J} \left(F \frac{\partial \omega}{\partial \phi} - E \frac{\partial \omega}{\partial \psi} \right) - \mu \delta \quad (4.10)$$

$$\frac{\partial h}{\partial \psi} = \frac{\eta}{J} \left(G \frac{\partial \omega}{\partial \phi} - F \frac{\partial \omega}{\partial \psi} \right) - \rho \omega \quad (4.11)$$

Differentiating (4.9) w. r. t. ψ , (4.10) w.r.t., ϕ and then using the condition that the second – order mixed derivative of h with respect to ϕ and ψ is independent of the order of differentiation, we find

$$\eta \left[\frac{\partial}{\partial \phi} \left\{ \frac{1}{J} \left(G \frac{\partial \omega}{\partial \phi} - F \frac{\partial \omega}{\partial \psi} \right) \right\} - \frac{\partial}{\partial \psi} \left\{ \frac{1}{J} \left(F \frac{\partial \omega}{\partial \phi} - E \frac{\partial \omega}{\partial \psi} \right) \right\} \right] + \mu \frac{\partial \delta}{\partial \psi} - \rho \frac{\partial \omega}{\partial \phi} = 0 \quad (4.12)$$

Thus we conclude that when streamlines, $\psi = \text{constant}$ and the magnetic lines $\phi = \text{constant}$ of steady plane viscous Magneto hydrodynamic flows are taken as curvilinear co-ordinates system (ϕ, ψ) in the physical plane then the set of six of partial differential equations (4.1) - (4.6) for $v_1, v_2, H_1, H_2, \omega, \delta$ and h as functions of x and y is replaced by the system of five partial differential equations (4.3) – (4.6) and (4.12), in five dependent variables E, F, G, ω and δ . If the solutions to these equations are given we can find $h = h(x, y)$.

5. APPLICATION OF THE FUNDAMENTAL FORM IN CURVILINEAR COORDINATE SYSTEM:

We now study an examples in which the curves $\psi = \text{constant}$ and the curves $\phi = \text{constant}$ form an orthogonal curvilinear coordinate system. The streamlines are considered along the curve $\psi = \text{constant}$ and the magnetic field is along the curve $\phi = \text{constant}$.

Investigating the flows for which

$$\phi = \phi(r) \text{ and } \psi = \psi(\theta) \quad (5.1)$$

where (r, θ) is the polar coordinate, the square of the element of arc length ds in this orthogonal curvilinear coordinate system is given by

$$ds^2 = (1+a^2)dr^2 + a^2r^2d\theta^2 \quad (5.2)$$

where a is any constant. Comparing (5.2) with (3.2) we find,

$$\begin{aligned} E &= \frac{1+a^2}{\phi'^2}, & F &= 0, & G &= \frac{a^2r^2}{\psi'^2} \\ W^2 &= EG = \left(\frac{1+a^2}{\phi'^2} \right) \left(\frac{a^2r^2}{\psi'^2} \right) \Rightarrow W = \frac{ar\sqrt{1+a^2}}{\phi'\psi'} \end{aligned} \quad (5.3)$$

Substituting the values of E, F, G and W from (5.3) in (3.7), the Gauss characteristic equation becomes

$$K = \frac{1}{W} \left[-\frac{\partial}{\partial \phi} \left(\frac{W}{E} \frac{EG_\phi}{2W^2} \right) \right] = -\frac{1}{W} \left[\frac{\partial}{\partial r} \left(\frac{a}{\sqrt{1+a^2}} \psi' \right) \frac{1}{\phi'} \right] = 0$$

Using (5.3) in (4.4) and (4.5), we find the vorticity function and the density function respectively, given by, $\omega = -\frac{1}{W} \frac{\partial}{\partial \psi} \left(\frac{E}{W} \right) = -\frac{\psi''}{a^2 r^2}$ and $\delta = \frac{1}{W} \frac{\partial}{\partial \phi} \left(\frac{G}{W} \right) = \frac{\phi' + r\phi''}{r(1+a^2)}$.

From (4.12) we obtain using (5.2) and (5.3),

$$\eta \left[\frac{\partial}{\partial \phi} \left(\frac{G}{J} \frac{\partial \omega}{\partial \phi} \right) + \frac{\partial}{\partial \psi} \left(\frac{E}{J} \frac{\partial \omega}{\partial \psi} \right) \right] - \rho \frac{\partial \omega}{\partial \phi} = 0$$

$$\Rightarrow \eta(1+a^2)\psi^{iv} + (4a^2\eta + 2\rho a\sqrt{1+a^2}\psi')\psi'' = 0 \tag{5.4}$$

For, if we suppose that $4a^2\eta + 2\rho a\sqrt{1+a^2}\psi' = 0$ in (5.4), then $\psi' = -\frac{2a\eta}{\rho\sqrt{1+a^2}}$ and is a constant, say k_1 . Therefore, $\psi'' = 0, \psi^{iv} = 0$ and hence we have (5.4). That is, whenever $\eta = c\rho$ with $c = \frac{-k_1\sqrt{1+a^2}}{2a}$, we can conclude that the streamlines in this two dimensional flow of a plane viscous MHD fluid are given by $\psi(\theta) = k_1\theta + k_2$, where k_2 is an arbitrary constant.

Now from (4.6) and (5.3), we have

$$\psi' = \frac{k a r \sqrt{1+a^2}}{\phi'} = k_1 \tag{5.5}$$

Using (5.3) and (5.5) we get from (4.7),

$$V = \frac{k_1}{a r} = \frac{k'_1}{r} \text{ where } k'_1 = \frac{k_1}{a} \tag{5.6}$$

Using (5.3) and (5.5) we get from (4.8),

$$H = \frac{k a r}{k_1} = k''_1 r \tag{5.7}$$

where, $k''_1 = \frac{ka}{k_1}$. Using (5.3) and (5.5) we get from (4.5) and (4.6),

$$\omega = 0, \quad \delta = \frac{2k_2''}{\sqrt{1+a^2}} \quad (5.8)$$

Finally from (4.10) and (4.11) we get using (5.8), $h = -\mu(k_2''r)^2 + B$ where B is an arbitrary constant and thus we obtain $p = -\mu(k_2''r)^2 - \frac{\rho k_1^2}{2r^2} + B$

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