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A New Method of Generating Magic Rectangle

K. Mani[*] and M. Viswambari[#]

Department of Computer Science
Nehru Memorial College, Puthanampatti,
Trichy, Tamil Nadu, India.

Abstract

A magic rectangle can be formed in various ways. In the proposed method, to construct a magic rectangle of order m×n, a template is created in such a way that the sum of all the positive and negative numbers in each row and column are zero without considering the starting number and last number where these two numbers are also occurring in the magic rectangle. The novelty used in this paper is that a template for magic rectangle of order m×n has been created where m and n are even numbers. For that the magic sum and the starting number are accepted as input. The magic sum is considered as the column sum for the rectangle but the row sum is calculated. The row sum and column sum of magic rectangle are not calculated as in conventional magic rectangle but they are calculated from magic sum. Also the starting number for the magic rectangle is any number which differs in the conventional rectangle and it is always considered as one.

Keywords: Magic Rectangle, Magic Sum, Starting Number, Cryptography.

I. INTRODUCTION

A magic rectangle is a $m \times n$ array of the positive integers from 1 to $m \times n$ such that the sum of the integers in each row have a constant and sum of integers in each column is also a constant but the row sum and column sum is different. If the row sum and the column sum are same then it is called magic square. A magic rectangle is also defined similarly, except that there are no diagonals and only row and column sum are constraints. Magic rectangles are well-known for their very interesting and entertaining combinatorics which is used in designing experiments. In a conventional magic rectangle, the integers 1 to m*n are arranged in an array of m rows and n columns so that each row adds to the same total M and each column to the same total N.

Efforts to construct magic rectangles are far greater but the method invented for the same is somewhat less than those for their square counterparts. Various authors have introduced ideas for constructing the magic rectangles. Michael Springfield, Wayne Goddard [1] have showed all possible magic rectangles with arithmetic constraints and enumerated the 4×4 domino magic squares in which rotations and reflections are considered. In [2], Chand K. Midha,et.al. have provided a simple and systematic method for constructing any m × m + 2 magic rectangle for m odd and also designed an algorithm.

Nithiya Devi.G et.al. [3], have proposed a strong algorithm for data security by combining new modified magic rectangle and Iterative Fisher Yates Shuffle Algorithm (IFYS). The repetition of character in data was overcome through Newly Modified Magic Rectangle (NMMR) and complexity of the data was increased by introducing IFYS algorithm along with NMMR and proved that the data was hard to retrieve without the knowledge of magic pattern and shuffling order. In [4], Omar A. Dawood and et.al., have developed a new method for constructing magic cube using the folded magic square technique. It considered a new step towards the magic cube construction that applied a good insight and provided an easy generalized technique. The method generalized the design of magic cube with N order regardless the type of magic square whether odd order, singly even order or doubly even order.

J. P. De Los Reyes et.al. [5], provided a new systematic method for constructing any even by even magic rectangle. The method proposed was extremely simple as it allows one to arrive at the magic rectangles by simply carrying out some matrix operations. Also the magic rectangles of lower orders are embedded in a magic rectangle of higher order. Dalibor Froncek [6], has proved the existence of Magic Rectangle Sets (MRS) (a,b,c) for all admissible triples of odd numbers a,b,c. In [7], John Lorch has introduced a linear- algebraic construction for Magic Rectangle of size $p^r \times p^s$ to produce many different MRs with non-coprime dimensions.

But this paper proposes a different method in constructing the Magic Rectangle. It is illustrated in section 3. The main difference between conventional Magic Rectangle and the proposed Magic Rectangle is that in conventional Magic Rectangle the numbers are from 1 to m* n. But in the proposed Magic Rectangle the numbers are not necessarily consecutive and they are generated from magic sum and starting number which are accepted as input.

II. PRELIMINARIES

A magic rectangle of order m×n ($MR_{m\times n}$) is an arrangement of integer in an m×n matrix, such that the sum of all the elements in every row is equal and also every column is equal. But the row sum is not equal to the column sum. MRs are a generalization of MSs. A MR = $[a_{ij}]_{m\times n}$ of size m×n is an m×n array whose entries are {1,2, …, mn} each appearing once, with all its row sums and all its column sums are equal. A normal MR contain the integers from 1 to mn. It exists for all orders m,n ≥1 and m≠n. The row sum and column sum of MR of order m×n is calculated using the sum of all entries in the array is $\frac{1}{2}\, mn\,(mn+1)$. The row sum and column sum are calculated using (1) and (2) respectively.

$$\sum_{i=1}^{m} aij = \frac{1}{2}\, n\,(mn+1), all\ j, \quad \text{...(1)}$$

$$\sum_{j=1}^{n} aij = \frac{1}{2}\, m\,(mn+1), all\ j, \quad \text{...(2)}$$

Thus m and n must be even or both be odd.

A. NOTATIONS

While generating the magic rectangle of order m×n where m and n should be taken as m ≡ 0 (mod 2) and n ≡ 0 (mod m+2) respectively and m > 2. While constructing the Magic Rectangle the following notations are used in this paper.

MR	: magic rectangle
m	: number of rows used in MR
n	: number of columns used in MR
MRrsum	: MR row sum
MRcsum	: MR column sum
MS	: magic square
MRS	: starting number for MR
MRL	: last number for MR
No_MR	: number of MR
MRm×n	: MR of order m×n
MSp	: MS of order p
MRSm×n	: MR sum of order m×n
MSSm	: MS sum of order m

III. PROPOSED METHODOLOGY FOR CONSTRUCTING THE MR

In order to construct the MR using the proposed method, the number of rows taken for MR is m then the number of column taken for n is m+2 and the magic sum is treated as MSS_m as well as $MR_{m\times n}$csum. But for finding $MR_{m\times n}$ rsum in the proposed methodology, divide and conquer strategy is used in this paper. For that different possible sub MSs and their corresponding sums are calculated using (3) and (4). The magic sum accepted as input is treated as tentatively MSSp. From that MRrsum and MRcsum are calculated. The process is terminated when MSS_2 [8] is reached.

$$p_k = p/2^k, \ k = 1,2,3,\ldots, l\text{-}1 \quad \text{...(3)}$$

where *l* is an integer. It is found in such a way that when it is substituted in (3), it will produce MS_2.

$$MSSp_k = MSSp/p_k \qquad \ldots(4)$$

Thus, $MR_{m\times n}$ csum = Magic Sum; $MR_{m\times n}$ rsum = $MR_{m\times n}$ csum $+MSS_2$

A template is created for $MR_{m\times n}$ where m=2i, i=1,2,…,k and n are also even number i.e., n=m+2. In order to fill the numbers in $MR_{m\times n}$, a template is created with some range of numbers where the range of numbers are calculated using (5) and (6).

$$Value = \frac{m \times n - 2}{4} \qquad \ldots(5)$$

range = [-value, + value] after omitting zero, MRS and MRL …(6)

where MRL = MSS_2 - MRS

After obtaining the range, find the individual numbers (t_{ij}) starting with negative value [-value] , incremented by 0.5 until positive value [+value] is reached. Then (t_{ij}) are filled in the template in such a way that the sum of all positive (t_{ij}) and negative ($-t_{ij}$) numbers in each row and column of the template is zero after omitting MRS and MRL. Mathematically, it is represented using (7), (8) and (9).

$$\sum_{j=1}^{n} t_{ij} + \sum_{j=1}^{n} (-t_{ij}) = 0 \; for\ each\ row\ i \qquad \ldots(7)$$

$$\sum_{i=1}^{m} t_{ij} + \sum_{i=1}^{m} (-t_{ij}) = 0 \; for\ each\ column\ \mathrm{j} \qquad \ldots(8)$$

$$\sum_{\substack{1\le i\le m \\ 1<j<n}} t_{ij} = 0 \qquad \ldots(9)$$

Thus, to generate the MR using the template shown in fig. 1, (t_{ij}) represents the number in the concerned cell are filled by MRS is multiplied by (t_{ij}) times and added to MRS. Similarly, ($-t_{ij}$) represents MRS is multiplied by (t_{ij}) and subtracted from MRL. For example, a template for $MR_{16\times 18}$ is shown in fig.1. It is noted that the template shown in fig.1 is treated as a base template, because from it other MR templates viz., 4×6, 6×8,… can be taken. The required templates of $MR_{4\times 6}$ with MRS is even number and $MR_{6\times 8}$ with MRS is even number are generated from $MR_{16\times 18}$ and they are shown in fig. 3 and fig. 4 respectively. It is noted that in the proposed methodology to generate MR, top down approach is used. That is a template is created by starting with some large value of m and n and decremented by 2 from m and n. The process is stopped when it reaches m=4 and n=6.

$$\begin{pmatrix} MRS & 3 & 5 & -1 & -2 & -5 & -10 & 10 & -19.5 & 19.5 & 29.5 & -29.5 & 39.5 & -39.5 & 49.5 & -49.5 & 69.5 & -69.5 \\ 3.5 & 0.5 & -4 & -2.5 & -1.5 & 4 & 9 & -9 & -15.5 & 15.5 & 25.5 & -25.5 & 35.5 & -35.5 & 45.5 & -45.5 & 65.5 & -65.5 \\ -3 & MRL & 4.5 & 2 & 1 & -4.5 & 9.5 & -9.5 & -13 & 13 & 22 & -22 & 33 & -33 & 43 & -43 & 63 & -63 \\ -0.5 & -3.5 & -5.5 & 1.5 & 2.5 & 5.5 & 8 & -8 & 16 & -16 & 21 & -21 & 31 & -31 & 48 & -48.5 & 61 & -61 \\ 11 & 8.5 & 6.5 & -7 & -7.5 & -11.5 & -6 & 6 & 17.5 & -17.5 & -26 & 26 & 38.5 & -38.5 & 47 & -47 & 68.5 & -68.5 \\ -11 & -8.5 & -6.5 & 7 & 7.5 & 11.5 & -10.5 & 10.5 & 14.5 & -14.5 & -20 & 20 & 37 & -37 & 52 & -52 & 67 & -67 \\ 19 & 16.5 & 15 & 14 & 13.5 & -18 & -18.5 & -12.5 & -12 & -17 & -27.5 & 27.5 & -40 & 40 & -50 & 50 & 57.5 & -57.5 \\ -19 & -16.5 & -15 & -14 & -13.5 & 18 & 18.5 & 12.5 & 12 & 17 & -24.5 & 24.5 & -35 & 35 & -45 & 45 & 56 & -56 \\ 23.5 & 26.5 & 21.5 & 27 & 28.5 & 22.5 & -29 & -28 & -25 & -24 & -23 & -20.5 & -31.5 & 31.5 & -42.5 & 42.5 & -70 & 70 \\ -23.5 & -26.5 & -21.5 & -27 & -28.5 & -22.5 & 29 & 28 & 25 & 24 & 23 & 20.5 & -32.5 & 32.5 & -48 & 48 & -65 & 65 \\ 41.5 & 39 & 30.5 & 33.5 & 34 & 36 & -40.5 & -41 & -30 & -32 & -34.5 & -36.5 & -38 & 38 & -47.5 & 47.5 & -61.5 & 61.5 \\ -41.5 & 39 & 30.5 & -33.5 & -34 & -36 & 40.5 & 41 & 30 & 32 & 34.5 & 36.5 & -37.5 & 37.5 & -52.5 & 52.5 & -62.5 & 62.5 \\ 51.5 & 49 & 43.5 & 44 & 46 & 55.5 & 53.5 & 54 & -50.5 & -51 & -42 & -44.5 & -46.5 & -54.5 & -53 & -55 & -68 & 68 \\ -51.5 & -49 & -43.5 & -44 & -46 & -55.5 & -53.5 & -54 & 50.5 & 51 & 42 & 44.5 & 46.5 & 54.5 & 53 & 55 & -67.5 & 67.5 \\ 71.5 & 69 & 60.5 & 64 & 66 & 63.5 & 59.5 & 58 & -70.5 & -71 & -60 & -62 & -64.5 & -66.5 & -59 & -58.5 & -57 & 57 \\ -71.5 & -69 & -60.5 & -64 & -66 & -63.5 & -59.5 & -58 & 70.5 & 71 & 60 & 62 & 64.5 & 66.5 & 59 & 58.5 & -56.5 & 56.5 \end{pmatrix}$$

Fig.1 Template for $MR_{16\times18}$ if MRS is even

$$\begin{pmatrix} MRS & 3 & 5 & -1 & -2 & -5 \\ 3.5 & 0.5 & -4 & -2.5 & -1.5 & 4 \\ -3 & MRL & 4.5 & 2 & 1 & -4.5 \\ -0.5 & -3.5 & -5.5 & 1.5 & 2.5 & 5.5 \end{pmatrix}$$

Fig.2 Template for $MR_{4\times6}$ taken from $MR_{16\times18}$ if MRS is even

$$\begin{pmatrix} MRS & 3 & 5 & -1 & -2 & -5 & -10 & 10 \\ 3.5 & 0.5 & -4 & -2.5 & -1.5 & 4 & 9 & -9 \\ -3 & MRL & 4.5 & 2 & 1 & -4.5 & 9.5 & -9.5 \\ -0.5 & -3.5 & -5.5 & 1.5 & 2.5 & 5.5 & 8 & -8 \\ 11 & 8.5 & 6.5 & -7 & -7.5 & -11.5 & -6 & 6 \\ -11 & -8.5 & -6.5 & 7 & 7.5 & 11.5 & -10.5 & 10.5 \end{pmatrix}$$

Fig. 3 Template for $MR_{6\times8}$ taken from $MR_{16\times18}$ if MRS is even

Suppose, MRS is odd, even if the base $MR_{m\times n}$ template is created, it is not possible to generate the MR numbers because as per the proposed methodology the MRS is multiplied by (t_{ij}) of times which results in getting the fractional number. For example, in fig.1, the number present in second row and first column is 3.5. To generate the MR number in the concerned cell is calculated as 3.5×5+5= 22.5 (here 5 is MRS). It is a fractional value which violates the number to be filled in MR because the all the numbers in MR should be integer in this case. Thus to obtain the integer values of MR all the numbers in the base templates are multiplied by two if MRS is odd. Fig. 3 shows a Template for $MR_{6\times8}$ if the starting number is even and all the numbers in fig. 3 is multiplied by two. It is shown in fig. 4 which is a Template for $MR_{6\times8}$ if the starting number is odd.

$$\begin{pmatrix} MRS & 6 & 10 & -2 & -4 & -10 & -20 & 20 \\ 7 & 1 & -8 & -5 & -3 & 8 & 18 & -18 \\ -6 & MRL & 9 & 4 & 2 & -9 & 19 & -19 \\ -1 & -7 & -11 & 3 & 5 & 11 & 16 & -16 \\ 22 & 17 & 13 & -14 & -15 & -23 & -12 & 12 \\ -22 & -17 & -13 & 14 & 15 & 23 & -21 & 21 \end{pmatrix}$$

Fig.4 Template for $MR_{6\times8}$ taken from $MR_{16\times18}$ if MRS is odd

$$\begin{pmatrix} MRS & 3 & 5 & -1 & -2 & -5 & -10 & 10 & -19.5 & 19.5 \\ 3.5 & 0.5 & -4 & -2.5 & -1.5 & 4 & 9 & -9 & -15.5 & 15.5 \\ -3 & MRL & 4.5 & 2 & 1 & -4.5 & 9.5 & -9.5 & -13 & 13 \\ -0.5 & -3.5 & -5.5 & 1.5 & 2.5 & 5.5 & 8 & -8 & 16 & -16 \\ 11 & 8.5 & 6.5 & -7 & -7.5 & -11.5 & -6 & 6 & 17.5 & -17.5 \\ -11 & -8.5 & -6.5 & 7 & 7.5 & 11.5 & -10.5 & 10.5 & 14.5 & -14.5 \\ 19 & 16.5 & 15 & 14 & 13.5 & -18 & -18.5 & -12.5 & -12 & -17 \\ -19 & -16.5 & -15 & -14 & -13.5 & 18 & 18.5 & 12.5 & 12 & 17 \end{pmatrix}$$

Fig.5 Template for $MR_{8\times10}$ taken from $MR_{16\times18}$ if MRS is odd

Further, while generating the $MR_{m\times n}$, MRS is incremented by m×n and MRL is decremented by m×n but at one stage the numbers may coincide with each other which violates the concept of MR. To avoid it, the MS of order p is determined where p is the largest integer such that $p^2 > m\times n$. After determining p, the magic sum accepted as input is treated as MSSp where p=4k, k=1, 2, 3,…,s. where s is finite number. Then MSS_2 is calculated from MSSp using divide and conquer strategy and it must satisfy the relation using (10)

$$MSS_2 \geq (m\times n) \times 2 + MRS \qquad \ldots(10)$$

A. PROPOSED METHODOLOGY FOR CONSTRUCTING MR – AN - EXAMPLE

In order to understand the relevance of the work, let m=16, n=18, magic sum =5000 and MRS is 4. To generate the $MR_{16\times 18}$, first p is computed as p=16 because $16^2 > 288$ and MSS_{16}= 5000. Using (3), *l* is computed as 3 and k= 2. Thus, p_1 = 8, p_2= 4 correspondingly, MSS_8 = 5000/2 = 2500; MSS_4 = 5000/4 = 1250; MSS_2 = 5000/8 = 625. Using (9), 625≥ 580. Thus, the magic sum accepted for $MR_{16\times 18}$ is valid and MRL= 625-4=621. Using fig.1, the $MR_{16\times 18}$ is generated and it is shown in fig. 6. Using (4) and (5), the $MR_{16\times 18}$rsum and $MR_{16\times 18}$csum is calculated as 5625 and 5000 respectively.

617	4	613	16	601	24	28	593	589	40	577	48	565	52	569	64	553	72
611	18	615	6	20	605	42	587	591	30	44	581	567	66	563	54	68	557
12	609	8	621	603	22	585	36	3332	597	579	46	56	561	60	573	555	70
10	619	14	607	26	599	595	34	38	583	50	575	62	571	58	559	74	551
545	88	541	76	529	96	521	100	505	112	517	120	497	124	493	144	136	481
539	78	543	90	92	533	515	114	116	102	519	509	491	138	495	485	126	140
84	549	80	537	531	94	108	513	507	525	104	118	132	489	128	142	501	483
82	535	86	547	98	527	106	523	122	511	110	503	130	499	134	479	487	146
469	148	473	160	457	168	449	184	445	172	433	192	425	196	409	208	421	216
471	162	467	150	164	461	443	174	447	186	188	437	419	210	212	198	423	413
152	465	156	477	459	166	180	453	176	441	435	190	204	417	411	429	200	214
158	475	154	463	170	455	178	439	182	451	194	431	202	427	218	415	206	407
401	220	397	240	385	232	244	377	373	256	361	264	353	349	268	280	337	288
395	234	399	389	236	222	258	371	375	246	260	365	347	351	282	270	284	341
228	393	224	238	387	405	369	252	248	381	363	262	276	272	345	357	339	286
226	403	230	383	242	391	379	250	254	367	266	359	274	278	355	343	290	335

Fig.6 $MR_{16\times 18}$ with MRS= 4

It is noted that from $MR_{16\times 18}$, the other MRs viz., $MR_{4\times 6}$, $MR_{6\times 8}$, $MR_{8\times 10}$ are generated and are shown in fig. 7, fig. 8 and fig. 9 respectively.

4	617	613	16	601	24
18	611	615	6	20	605
609	12	8	621	603	22
619	10	14	607	26	599

Fig.7 $MR_{4\times 6}$ taken from $MR_{16\times 18}$

with MRrsum= 2500 and MRcsum= 1875

4	16	24	617	613	601	581	44
18	6	605	611	615	20	40	585
609	621	22	12	8	603	42	583
619	607	599	10	14	26	36	589
48	38	30	593	591	575	597	28
577	587	595	32	34	50	579	46

Fig.8 $MR_{6\times 8}$ taken from $MR_{16\times 18}$

with MRrsum= 1875 and MRcsum= 1250

$$\begin{pmatrix} 4 & 16 & 24 & 617 & 613 & 601 & 581 & 44 & 543 & 82 \\ 18 & 6 & 605 & 611 & 615 & 20 & 40 & 585 & 559 & 66 \\ 609 & 621 & 22 & 12 & 8 & 603 & 42 & 583 & 569 & 56 \\ 619 & 607 & 599 & 10 & 14 & 26 & 36 & 589 & 68 & 557 \\ 48 & 38 & 30 & 593 & 591 & 575 & 597 & 28 & 74 & 551 \\ 577 & 587 & 595 & 32 & 34 & 50 & 579 & 46 & 62 & 563 \\ 80 & 70 & 64 & 60 & 58 & 549 & 547 & 571 & 573 & 553 \\ 545 & 555 & 561 & 565 & 567 & 76 & 78 & 54 & 52 & 72 \end{pmatrix}$$

Fig.9 $MR_{4\times6}$ taken from $MR_{16\times18}$

with MRrsum= 2500 and MRcsum= 1875

B. SOME BASIC PROPERTIES OF PROPOSED MAGIC RECTANGLE

1. The sum of two magic rectangle(MR) of the same order say m×n is also a MR. Suppose , A and B are both $MR_{m\times n}$, an σ(A)=a, σ(B)=b where σ(A) and σ(B) denotes the row sum of MR A and B representation. Then, for any row of A+B, σ(A+B)= σ(A)+ σ(B). The same is true for column also.
2. If A is a MR, then A^T is also a MR. That is in A^T, the MRrsum is MRcsum. Similarly in A^T, MRcsum is MRrsum.
3. If MRT is a MR, then MR′ can be obtained from MR by rigid transformation (i.e), MR can be rotated by either 90^0, 180^0, 270^0, etc.
4. If A is a MR, and each element of B is obtained by adding, subtracting, multiplying by the same number to the corresponding element of A, but division is applicable only if A is divided by the starting number, where the starting number is odd.
5. Number MR of order 2×4, 4×2 exists. A trivial 2×4 MR may not be constructed using 8 elements, because 2 elements viz., MRS, MRL are occupied into the cells but using the remaining cells, MR cannot be constructed by proposed method.

IV. APPLICATIONS TO CRYPTOGRAPHY

The efficiency of any cryptosystem depends not only on decreasing the encryption/decryption time but also it enhances the security. In many number theoretic public key algorithms like RSA, ElGamal, etc., to encrypt a message, the ASCII value of each character of message (plaintext) is taken. Suppose, if a character is repeated several times in a plaintext, when it is encrypted the same ciphertext is produced at all times. This may sometimes be easily cracked by the eavesdropper because he/she may identify the relationship between plaintext and ciphertext. In order to avoid, first ASCII of each character is taken and it is treated as a positional value say ‘p’. The number which occurs at ‘p’ in MR are taken as encoding value of that character which is then used for encryption using any one of the cryptographic algorithms like RSA, ElGamal, Rabin etc. It provides an additional layer of security for any cryptographic algorithms because to generate the MR, the MRS and the MSSm are accepted as input are known only to the sender and the receiver. Further, (n! + m!) MRs are generated for the same MRS and the MSSp and for each MR different orientations (by rotating it) are obtained and hence the eavesdropper may not easily identify the correct MR which was used for encryption. Thus this model acts as a wrapper to any cryptographic algorithms.

V. CONCLUSION

A novel method for generating the MR has been proposed in this paper. This method has a unique feature that the magic sum and starting number are accepted as input from which several MRs are generated. Further the numbers used in the MR are not necessarily consecutive numbers as used in conventional MR but the numbers used in the proposed method is purely depending on the magic sum and the starting number. Similarly, the template used in creating the MR is not unique because more number of templates is created for the same MR. Further, the numbers generated in the MR may be used for encoding any text, digits and special characters so that it will provide an additional level of security in any cryptographic algorithms. Because, for the same magic sum and starting number various MR are generated and hence in the cryptographic scenario the same may be implemented using bottom-up approach in future.

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