International Journal of Computational and Applied Mathematics.
ISSN 1819-4966 Volume 12, Number 2 (2017), pp. 399–410

http://www.ripublication.com/

Convergence of a Third-order family of methods in Banach spaces

Gagandeep
*Department of Mathematics,
Hans Raj Mahila Mahavidyalaya,
Jalandhar-144008, India.*

R. Sharma[1]
*Department of Applied Sciences,
DAV Institute of Engineering and Technology,
Jalandhar-144008, India.*

Abstract

In this paper, we study the semilocal convergence of a family of third-order methods for solving nonlinear equations in Banach spaces using recurrence relations. Recurrence relations for the family of methods are derived and finally an existence-uniqueness theorem is derived along with a priori error bounds.

AMS subject classification: 65D99.
Keywords: Nonlinear equations, Banach spaces, Recurrence relations, Semilocal convergence, Iterative methods.

1. Introduction

The most well known Newton's method and its variants are used to solve nonlinear operator equations $F(x) = 0$. The convergence of these second order methods was established by Kantorovich Theorem ([15], [16]). The convergence of sequences obtained by these methods is derived from convergence of majorizing sequences [21]. Rall in [18] established the convergence of Newton's method by using recurrence relations. With the same approach, various researchers established semilocal convergence of higher order

[1]Corresponding author.

methods in Banach spaces (see [3, 2, 13, 12, 7, 11, 17, 14, 10, 9, 20, 5, 22, 8, 1, 4, 6] and references there in). In this paper, we shall use recurrence relations to establish the semilocal convergence of a family of third-order methods [19] in Banach spaces. Based on these recurrence relations, an existence-uniqueness theorem is given and a priori error bounds are obtained for the method.

2. Recurrence relations

In this section, we discuss a third order method [19] for solving nonlinear operator equations

$$F(x) = 0,$$

where $F : \Omega \subseteq X \to Y$ is a nonlinear operator on an open convex subset Ω of a Banach space X with values in a Banach space Y. The third order method is defined as follows:

$$\begin{aligned} y_n &= x_n - \theta F'(x_n)^{-1} F(x_n), \\ x_{n+1} &= x_n - \left[\left(1 + \frac{1}{2\theta}\right) I - \frac{1}{2\theta} F'(x_n)^{-1} F'(y_n)\right] F'(x_n)^{-1} F'(x_n). \end{aligned} \tag{2.1}$$

Let F be a twice Fréchet differentiable in Ω and $BL(Y, X)$ be the set of bounded linear operators from Y into X. Let us assume that $\Gamma_0 = F'(x_0)^{-1} \in BL(Y, X)$ exists at some $x_0 \in \Omega_0$ and the following conditions hold:

(1) $\|F'(x) - F'(y)\| \le K\|x - y\|, x, y \in \Omega,$

(2) $\|F''(x)\| \le M, x \in \Omega$

(3) $\|\Gamma_0\| \le \beta,$

(4) $\|\Gamma_0 F(x_0)\| \le \eta.$

Let us denote

$$a = K\beta\eta. \tag{2.2}$$

Now, we define the sequences

$$\begin{aligned} a_0 = b_0 = 1, d_0 &= 1 + \frac{a}{2}, \\ a_{n+1} &= \frac{a_n}{1 - a a_n d_n}, \\ b_{n+1} &= a_{n+1}\beta\eta C_n, \\ d_{n+1} &= \left(1 + \frac{1}{2} a a_{n+1} b_{n+1}\right) b_{n+1}, \end{aligned} \tag{2.3}$$

where

$$C_n = \frac{M}{2}K_n^2 + K|\theta|b_n K_n + \left(\frac{M\theta^2 + K}{2}\right)b_n^2, \tag{2.4}$$

with

$$K_n = \left(|1-\theta| + \frac{aa_n b_n}{2}\right)b_n. \tag{2.5}$$

The polynomials C_n and K_n can be written as

$$C_n = (P_0 + P_1 a_n b_n + P_2 a_n^2 b_n^2)b_n^2,$$

$$K_n = (Q_0 + Q_1 a_n b_n)b_n.$$

Lemma 2.1. Under the previous assumptions, we prove the following:

(I_n) $\|\Gamma_n\| = \|F'(x_n)^{-1}\| \le a_n\beta$,

(II_n) $\|\Gamma_n F(x_n)\| \le b_n\eta$,

$(IIIn)$ $\|x_{n+1} - x_n\| \le d_n\eta$,

(IV_n) $\|x_{n+1} - y_n\| \le K_n\eta$.

Proof. We use induction to prove the above claims. Notice that (I_0) and (II_0) follow immediately from the assumptions. To prove (III_0), we consider (2.1). Using the assumptions, it follows that

$$\begin{aligned}\|x_1 - x_0\| &\le \left[1 + \frac{1}{2|\theta|}K\beta\|y_0 - x_0\|\right]\|\Gamma_0 F(x_0)\| \\ &\le \left[1 + \frac{a}{2}\right]\eta = d_0\eta,\end{aligned} \tag{2.6}$$

and (III_0) holds. We have

$$x_1 - y_0 = -\left[(1-\theta)I + \frac{1}{2\theta}F'(x_0)^{-1}(F'(x_0) - F'(y_0))\right]F'(x_0)^{-1}F'(x_0), \tag{2.7}$$

so that

$$\begin{aligned}\|x_1 - y_0\| &\le \left[|1-\theta| + \frac{1}{2|\theta|}K\beta\|y_0 - x_0\|\right]\|\Gamma_0 F(x_0)\| \\ &\le \left[|1-\theta| + \frac{a}{2}\right]\eta = K_0\eta,\end{aligned} \tag{2.8}$$

and (IV_0) also holds. Following an inductive procedure and assuming that $x_n \in \Omega$ and $aa_n d_n < 1$, if $x_{n+1} \in \Omega$, we have

$$\|I - \Gamma_n F'(x_{n+1})\| \le \|\Gamma_n\|\|F'(x_n) - F'(x_{n+1})\| \le aa_n d_n < 1. \tag{2.9}$$

Then, from Banach lemma, Γ_{n+1} exists and

$$\|\Gamma_{n+1}\| \leq \frac{\|\Gamma_n\|}{1-\|\Gamma_n\|\|F'(x_n)-F'(x_{n+1})\|} \leq \frac{a_n\beta}{1-aa_nd_n} = a_{n+1}\beta. \tag{2.10}$$

Hence, by induction (2.10) holds for all n. This proves condition (I_n).
Using the first step of (1), we have

$$\begin{aligned} F(y_n) &= F(y_n) - \theta F(x_n) - F'(x_n)(y_n - x_n) \\ &= (1-\theta)F(x_n) + F(y_n) - F(x_n) - F'(x_n)(y_n - x_n) \\ &= (1-\theta)F(x_n) + \int_0^1 F''(x_n + t(y_n - x_n))(1-t)dt(y_n - x_n)^2. \end{aligned} \tag{2.11}$$

Now subtract first step of (2.1) from second, we get

$$x_{n+1} - y_n = -\left[(1-\theta)I + \frac{1}{2\theta}F'(x_n)^{-1}(F'(x_n) - F'(y_n))\right]F'(x_n)^{-1}F'(x_n). \tag{2.12}$$

so that

$$\begin{aligned} \|x_{n+1} - y_n\| &\leq \left[|1-\theta| + \frac{1}{2|\theta|}a_n\beta K\|y_n - x_n\|\right]\|\Gamma_n F(x_n)\| \\ &\leq \left[|1-\theta| + \frac{aa_nb_n}{2}\right]b_n\eta = K_n\eta. \end{aligned} \tag{2.13}$$

Using Taylor's formula, we have

$$\begin{aligned} F(x_{n+1}) &= F(y_n) + F'(y_n)(x_{n+1} - x_n) \\ &\quad + \int_0^1 F''(y_n + t(x_{n+1} - y_n))(1-t)dt(x_{n+1} - y_n)^2 \\ &= (1-\theta)F(x_n) + F'(x_n)(x_{n+1} - x_n) \\ &\quad + \int_0^1 F''(x_n + t(y_n - x_n))(1-t)dt(y_n - x_n)^2 \\ &\quad\quad + \int_0^1 F''(y_n + t(x_{n+1} - y_n))(1-t)dt(x_{n+1} - y_n)^2 \\ &\quad + (F'(y_n) - F'(x_n))(x_{n+1} - y_n) \\ &= F'(x_n)\frac{1}{2\theta}F'(x_n)^{-1}(F'(y_n) - F'(x_n))\Gamma_n F(x_n) + (F'(y_n) - F'(x_n))(x_{n+1} - y_n) \\ &\quad\quad + \int_0^1 F''(y_n + t(x_{n+1} - y_n))(1-t)dt(x_{n+1} - y_n)^2 \\ &\quad + \int_0^1 F''(x_n + t(y_n - x_n))(1-t)dt(y_n - x_n)^2. \end{aligned} \tag{2.14}$$

Hence using (2.13) in (2.14), we have

$$\begin{aligned}\|F(x_{n+1})\| &\le \frac{K}{2\theta^2}\|y_n - x_n\|^2 + K\|y_n - x_n\|\|x_{n+1} - y_n\| \\ &\quad + \frac{M}{2}\|y_n - x_n\|^2 + \frac{M}{2}\|x_{n+1} - y_n\|^2 \\ &\le \left[\frac{M}{2}K_n^2 + K|\theta|b_n K_n + \left(\frac{M\theta^2 + K}{2}\right)b_n^2\right]\eta^2 \\ &= C_n\eta^2. \end{aligned} \tag{2.15}$$

Therefore

$$\begin{aligned}\|\Gamma_{n+1}F(x_{n+1})\| &\le \|\Gamma_{n+1}\|\|F(x_{n+1})\| \\ &\le a_{n+1}\beta C_n\eta^2 = b_{n+1}\eta, \end{aligned} \tag{2.16}$$

So, by induction condition (II_n) holds for all n. Using (2.16), we have

$$\begin{aligned}\|x_{n+2} - x_{n+1}\| &\le \left[1 + \frac{1}{2|\theta|}\|F'(x_{n+1})^{-1}\|\|F'(x_{n+1})^{-1} - F'(y_{n+1})^{-1}\|\right]\|\Gamma_{n+1}F(x_{n+1})\| \\ &\le \left[1 + \frac{1}{2}a_{n+1}\beta K b_{n+1}\eta\right]b_{n+1}\eta \\ &= \left[1 + \frac{aa_{n+1}b_{n+1}}{2}\right]b_{n+1}\eta = d_{n+1}\eta. \end{aligned} \tag{2.17}$$

Hence, by induction, this inequality holds for all n. This proves condition (III_n). We have from (2.13) and (2.16) that

$$\begin{aligned}\|x_{n+2} - y_{n+1}\| &\le \left[|1-\theta| + \frac{1}{2|\theta|}a_{n+1}\beta K\|y_{n+1} - x_{n+1}\|\right]\|\Gamma_n F(x_n)\| \\ &\le \left[|1-\theta| + \frac{aa_{n+1}b_{n+1}}{2}\right]b_{n+1}\eta = K_{n+1}\eta. \end{aligned} \tag{2.18}$$

Hence, by induction, this inequality holds for all n. This proves condition (IV_n). ■

3. Convergence Analysis

In this section, we establish the convergence of our third-order method (2.1). To this end, we have to prove the convergence of the sequence x_n defined in a Banach space or, which is same, to prove that d_n is a Cauchy sequence and that the following assumptions hold:

(1) $x_n \in \Omega$,

(2) $aa_nd_n < 1, n \in \mathbb{N}$.

The next two lemmas will show the Cauchy property for the sequence d_n.

Lemma 3.1. Assume that x_0 is chosen so as to satisfy $0 < d_0 < \frac{1}{a}$, that is, $a \in (0, \sqrt{3}-1)$. Then, the sequence $a_n > 0$ is increasing, as n increases.

Proof. We show now that all the involved sequences are positive. Under the imposed conditions, we see that a_0, b_0, d_0, C_0, K_0 are all positive, and also that $1 - aa_0d_0 > 0$. Assume, now, that all a_i, b_i, d_i, C_i, K_i, and $1-aa_id_i$ are positive, for $i = 0, 1, 2, \ldots i, n$

Since $C_n > 0$ and $b_{n+1} = a_{n+1}\beta\eta C_n$, it follows that a_{n+1}, b_{n+1} have same sign, and so $a_{n+1}b_{n+1} > 0$. Further, from $d_{n+1} = (1 + \frac{1}{2}aa_{n+1}b_{n+1})b_{n+1}$, we get that d_{n+1} has same sign as that of b_{n+1}, and so, all the three terms $a_{n+1}, b_{n+1}, d_{n+1}$ share the same sign.

By absurd, we suppose that the implied sign is negative. Then $d_n+d_{n+1} < d_n$, and so, $1-aa_n(d_n+d_{n+1}) > 1-aa_nd_n$, which renders $1-aa_{n+1}d_{n+1} = \frac{1 - aa_n(d_n + d_{n+1})}{1 - aa_nd_n} > 1$, which implies $aa_{n+1}d_{n+1} < 0$, but that is impossible since a_{n+1}, d_{n+1} have the same sign and $a > 0$.

Next, since $a_{n+1} = \frac{a_n}{1 - aa_nd_n}$, then

$$d_n = \frac{1}{a}\left(\frac{1}{a_n} - \frac{1}{a_{n+1}}\right),$$

and so, by telescoping, we get $\sum_{i=0}^{n-1} d_i = \frac{1}{a}\left(\frac{1}{a_0} - \frac{1}{a_n}\right)$, where $a_0 = 1$.

This will render $a_n = \frac{1}{1 - a\sum_{i=0}^{n-1} d_i}$. Certainly, since $a > 0, d_i > 0$, for all $i \geq 0$, then $a\sum_{i=0}^{n-1} d_i$ increases as n increases and so, $1 - a\sum_{n=0}^{n-1} d_i$ decreases as n increases, which implies that the reciprocal, a_n is an increasing sequence, and consequently, $a_n \geq a_0 = 1$.

■

We define the sequence $c_n = a_n b_n$. Then the sequences $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$ can be rewritten as

$$\begin{aligned}
a_{n+1} &= \frac{a_n}{1 - a a_n d_n} = \frac{a_n}{1 - a(c_n + \frac{a}{2}c_n^2)},\\
b_{n+1} = a_{n+1}\beta\eta C_n &= \frac{\beta\eta b_n c_n (P_0 + P_1 c_n + P_2 c_n^2)}{1 - a(c_n + \frac{a}{2}c_n^2)},\\
c_{n+1} = a_{n+1}b_{n+1} &= \frac{\beta\eta c_n^2 (P_0 + P_1 c_n + P_2 c_n^2)}{[1 - a(c_n + \frac{a}{2}c_n^2)]^2},\\
d_{n+1} &= \left(1 + \frac{1}{2} a a_{n+1} b_{n+1}\right) b_{n+1}\\
&= \frac{\beta\eta b_n c_n (P_0 + P_1 c_n + P_2 c_n^2)}{1 - a(c_n + \frac{a}{2}c_n^2)} \left(1 + \frac{a}{2} c_{n+1}\right).
\end{aligned}$$

That the sequence $\{c_n\}$ is a decreasing sequence under the assumption that $a_1 b_1 < 1$ can be proved by using the mathematical induction. It is obvious that $c_1 = a_1 b_1 < 1 = c_0$. Assuming that $c_n < c_{n-1}$ for some $n > 0$, we have

$$\begin{aligned}
c_{n+1} &= \frac{\beta\eta c_n^2 (P_0 + P_1 c_n + P_2 c_n^2)}{[1 - a(c_n + \frac{a}{2}c_n^2)]^2}\\
&< \frac{\beta\eta c_{n-1}^2 (P_0 + P_1 c_{n-1} + P_2 c_{n-1}^2)}{[1 - a(c_{n-1} + \frac{a}{2}c_{n-1}^2)]^2} = c_n.
\end{aligned}$$

Therefore the sequence $\{c_n\}$ becomes a decreasing sequence with $c_n < 1$ for all n. If $0 < s < 1$ and $c_n \leq s c_{n-1}$, then

$$\begin{aligned}
c_{n+1} &= \frac{\beta\eta c_n^2 (P_0 + P_1 c_n + P_2 c_n^2)}{[1 - a(c_n + \frac{a}{2}c_n^2)]^2}\\
&\leq s^2 \frac{\beta\eta c_{n-1}^2 (P_0 + P_1 s c_{n-1} + P_2 s^2 c_{n-1}^2)}{[1 - a(s c_{n-1} + \frac{a}{2}s^2 c_{n-1}^2)]^2}\\
&\leq s^2 \frac{\beta\eta c_{n-1}^2 (P_0 + P_1 c_{n-1} + P_2 c_{n-1}^2)}{[1 - a(c_{n-1} + \frac{a}{2}c_{n-1}^2)]^2} = s^2 c_n.
\end{aligned}$$

Let $\zeta = \dfrac{c_1}{c_0} = c_1 = a_1 b_1$, then we have $0 < \zeta < 1$ and $c_1 \leq \zeta c_0 = \zeta$, so that

$$c_1 \leq \zeta c_0, \quad c_2 \leq \zeta^2 c_1, \quad c_3 \leq \zeta^{2^2} c_2, \quad \cdots \quad c_{n+1} \leq \zeta^{(2^n + 2^{n-1} + \cdots + 2^1 + 1)} c_0 = \zeta^{2^{n+1}} \cdot \frac{1}{\zeta}.$$

On other hand the sequence $\{d_n\}$ under the assumption that $a_1b_1 < 1$ we have

$$
\begin{aligned}
d_n = \frac{aa_nd_n}{aa_n} &= (c_n + \frac{a}{2}c_n^2)\frac{1}{a_n} \\
&\leq \left(c_n + \frac{a}{2}c_n^2\right)\frac{1}{a_0} = c_n + \frac{a}{2}c_n^2 \\
&\leq \left(1 + \frac{a}{2}\right)c_n \\
&\leq \left(1 + \frac{a}{2}\right)\zeta^{2^n}.\frac{1}{\zeta}
\end{aligned}
$$

Since $\{a_n\}$ is an increasing sequence, and $a_0 \geq 1$. We thus have proved the following estimates.

Lemma 3.2. We assume that $a_1b_1 < 1$. Then the sequence $\{c_n\}$ is a decreasing sequence and for all $n \in \mathbb{N}$ we have the following estimates

$$
\begin{aligned}
c_{n+1} &\leq \zeta^{2^{n+1}}.\frac{1}{\zeta}, \\
d_n &\leq \left(1 + \frac{a}{2}\right)\zeta^{2^n}.\frac{1}{\zeta}
\end{aligned}
$$

where $0 < \zeta = a_1b_1 < 1$.

Lemma 3.3. The sequence $d_n > 0$ is convergent sequence and its limit is 0.

Proof. Since $a_n \geq 1$ is an increasing, then $1/a_n \leq 1$ is a decreasing sequence and further $0 \leq 1/a_n \leq 1$. Therefore $1/a_n$ is convergent to a limit L. Since $d_n = \frac{1}{a}\left(\frac{1}{a_n} - \frac{1}{a_{n+1}}\right)$, then d_n is convergent to the limit $\frac{1}{a}(L - L) = 0$. ■

Remark 3.4. Clearly $\sum_{i=0}^{\infty} d_i < \infty$, since $\sum_{i=0}^{\infty} d_i = \lim_{i\to\infty}\sum_{i=1}^{n-1} d_i = \lim_{n\to\infty}\frac{1}{a}\left(1 - \frac{1}{a_n}\right) = \frac{1}{a}(1 - L)$, where L would be the finite limit of $1/a_n$.

Theorem 3.5. Let X, Y be Banach spaces and F be a twice Fréchet differentiable in an open convex domain Ω of Banach space X and $BL(Y, X)$ be set of bounded linear operators from Y into X. Let us assume that $\Gamma_0 = F'(x_0)^{-1} \in BL(Y, X)$ exists at some $x_0 \in \Omega_0$ and the following conditions hold:

(1) $\|F'(x) - F'(y)\| \leq K\|x - y\|, x, y \in \Omega$,

(2) $\|F''(x)\| \leq M$,

(3) $\|\Gamma_0\| \leq \beta$,

(4) $\|\Gamma_0 F(x_0)\| \leq \eta$.

Let us denote $a = K\beta\eta$. Suppose that x_0 is chosen so as to satisfy $a \in (0, \sqrt{3} - 1)$ and $a_1 b_1 < 1$. Then, if $\overline{B}(x_0, r\eta) \subset \Omega$, where $r = \sum_{n=0}^{\infty} d_n$, then the sequence $\{x_n\}$ defined by (1)and starting at x_0 converges to a solution x^* of the solution $F(x) = 0$. In this case, the solution x^* and the iterates x_n belong to $\overline{B}(x_0, r\eta)$, and x^* is the only solution of $F(x) = 0$ in $B(x_0, \frac{2}{K\beta} - r\eta) \cap \Omega$.

Furthermore, the error bound on x^* depends on the sequence $\{d_n\}$ given by

$$\|x_{n+1} - x^*\| \leq \sum_{k=n+1}^{\infty} d_k\eta \leq \frac{(1 + a/2)\eta}{\zeta} \sum_{k=n+1}^{\infty} \zeta^{2^k}, \zeta = a_1 b_1. \tag{3.1}$$

Proof. It is easy to see that the sequence $\{x_n\}$ is convergent. Hence, there exists a limit x^* such that $\lim_{n\to\infty} x_n = x^*$. The sequence $\{a_n\}$ is bounded above since

$$a_n = \frac{1}{1 - a\sum_{i=0}^{n-1} d_i} \leq \frac{1}{1 - a\sum_{i=0}^{\infty} d_i}.$$

Since $\lim_{n\to\infty} d_n = 0$, so we have $\lim_{n\to\infty} b_n = 0$. This indicates that $\lim_{n\to\infty} C_n = 0$. Thus by and by the continuity of F, we proved that

$$\|F(x^*)\| = 0.$$

Also,

$$\begin{aligned} \|x_{n+1} - x_0\| &\leq \|x_{n+1} - x_n\| + \|x_n - x_{n-1}\| + \cdots + \|x_1 - x_0\| \\ &\leq \sum_{k=0}^{n} d_k\eta \leq r\eta, \end{aligned} \tag{3.2}$$

where $r = \sum_{n=0}^{\infty} d_n$. We conclude that $\{x_n\}$ lies in $\overline{B}(x_0, r\eta)$ and taking limit as $n \to \infty$ we have $x^* \in \overline{B}(x_0, r\eta)$. To show the uniqueness of the solution, suppose that

$$y^* \in B(x_0, \frac{2}{K\beta} - R\eta) \cap \Omega_0$$

is another solution of $F(x) = 0$. Then,

$$0 = F(y^*) - F(x^*) = \int_0^1 F'(x^* + t(y^* - x^*))dt(y^* - x^*). \tag{3.3}$$

To show that $y^* = x^*$, we have to show that the operator $\int_0^1 (F'(x^* + t(y^* - x^*))dt$ is invertible. Now, for

$$\begin{aligned}
&\|\Gamma_0\| \int_0^1 \|F'(x^* + t(y^* - x^*)) - F'(x_0)\|dt \\
&\leq K\beta \int_0^1 \|x^* + t(y^* - x^*) - x_0\|dt \\
&\leq K\beta \int_0^1 ((1-t)\|x^* - x_0\| + t\|y^* - x_0\|)dt \\
&< \frac{K\beta}{2}(r\eta + \frac{2}{K\beta} - r\eta) = 1, \qquad (3.4)
\end{aligned}$$

it follows from Banach's Theorem [15] that the operator $\int_0^1 (F'(x^* + t(y^* - x^*))dt$ has an inverse, and consequently, $y^* = x^*$. For every $m \geq n+1$, we have

$$\begin{aligned}
\|x_m - x_{n+1}\| &\leq \|x_m - x_{m-1}\| + \|x_{m-1} - x_{m-2}\| + \cdots + \|x_{n+2} - x_{n+1}\| \\
&\leq \sum_{k=n+1}^{m-1} d_k\eta \leq r\eta. \qquad (3.5)
\end{aligned}$$

By taking $m \to \infty$, we get

$$\|x_{n+1} - x^*\| \leq \sum_{k=n+1}^{\infty} d_k\eta \leq r\eta. \qquad (3.6)$$

and from Lemma 3.1

$$\|x_{n+1} - x^*\| \leq \sum_{k=n+1}^{\infty} d_k\eta \leq \frac{(1+a/2)\eta}{\zeta} \sum_{k=n+1}^{\infty} \zeta^{2^k}, 0 < \zeta < 1. \qquad (3.7)$$

which shows that $\{x_n\}$ converges and completes the proof. ■

4. Conclusion

In this paper, the recurrence relations are developed for establishing the convergence of a family of third-order methods for solving $F(x) = 0$ in Banach spaces. Based on recurrence relations, we prove a semilocal convergence, which shows the existence-uniqueness theorem for this family of methods and a priori error bounds.

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