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Prey-Predator Model with Infection in Both the Populations

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Abstract

In the present paper, a four species Prey predator model is proposed and analyzed mathematically. It is assumed that both the prey and predator species are infected with some disease. Recovery from the disease is not incorporated. Consequently, SI model is taken for both the species. It is further assumed that the predator species (both susceptible and infected) consumes only the infected prey as they are less mobile due to infection. Based upon the above assumptions, a mathematical model is formed and solved with the help of mathematical tools. Conditions for the existence and local stability of all the feasible equilibrium points have been obtained. The epidemiological threshold quantities have been obtained using next generation approach for the above model system.

Keywords: Carrying capacity, Predator, Prey, Predation rate

I. INTRODUCTION

The importance of studying the behavior of ecological systems under the influence of various epidemiological factors has been focused by various researchers. Many of them tried to merge ecology and epidemiology and their efforts results in developing a new important branch of mathematical biology named eco-epidemiology [1], [2], [3]. Most of the works deal with the stability analysis of biological populations having infection in prey only [4], [5], [6]. Only few researchers consider the spread of disease in predator species [7], [8], [9,].

Many researchers modeled and studied the problem of Salton Sea in California [10], [11]. The waste water from Coachella and imperial valleys is continuously polluting the sea. Gradual build up of salts and nutrients in the Sea occurs due to lack of out-flowing streams, which resulted in causing massive algal blooms in the sea. These algae die as fast as they grow. When it dies, they consume the dissolved oxygen from the water. So due to unavailability of sufficient oxygen, millions of Tilapia dies off every year due to this vivrio infection. This disaster did not stop here. Pelican birds (predator of tilapia) who highly depends upon tilapia fishes for their food, also got infected when they consumes the sick tilapia. On August 12, 1999, almost 8 millions of the fish died and about 14000 water birds which include mainly white pelicans died after eating the infected tilapia. Also there are numerous examples of studies in which different models have been taken to stop this natural disaster, but none of them was able to solve the problem completely. In the present paper, a four species predator-prey model is proposed in which both the prey and predator are infected with disease. It is assumed that susceptible predator and prey once infected will never recover. Consequently SI model is considered both for the predator and prey species. Further it is assumed that the predator species (both susceptible and infected) consumes only the infected prey species, as they are less mobile as compared to the susceptible ones, and so are easily catchable.

2. MATHEMATICAL MODEL

To form the mathematical model, we take the following assumptions

1. The prey species grows with constant growth rate r.

2. Both prey and predator species are infected by a disease.

3. Mass action incidence rate for transmission of disease in both the species.

4. Predator species consumes only the infected prey.

5. Linear functional response for predator is considered.

Under above assumptions, following mathematical model is formed.

$$\frac{dx}{dt} = r - \beta xy - d_1 x$$

$$\frac{dy}{dt} = \beta xy - \alpha_1 yz - d_2 y - \alpha_2 yw$$

$$\frac{dz}{dt} = -d_3 z + c_1 yz - \lambda_1 zw$$

$$\frac{dw}{dt} = -d_4 w + c_2 yw + \lambda_1 zw$$

where 'r' is the constant recruitment rate into the prey population, d_1, d_2, d_3 and d_4 are the natural death rates of the susceptible prey, infected prey, susceptible predator and infected predator population respectively, c_1 and c_2 are respective conversion factors of susceptible and infected predator populations where $c_1 > c_2$, α_1 and α_2 are their respective search rates. λ_1 is the disease transmission coefficient in predator population.

3. EQUILIBRIUM POINTS

The system can have following different equilibriums

(a) An equilibrium point $A\left(\frac{r}{d_1}, 0, 0, 0\right)$, which always exist.

(b) A predator free equilibrium point $B\left(\hat{x}, \hat{y}, 0, 0\right)$, where $\hat{x} = \frac{d_2}{\beta}$, $\hat{y} = \frac{\beta r - d_1 d_2}{d_2 \beta}$ which exist

if $\beta r - d_1 d_2 > 0$

(c) Equilibrium point $C\left(x', y', 0, w'\right)$, where

$$x' = \frac{rc_2}{\beta d_4 + d_1 c_2}, \; y' = \frac{d_4}{c_2}, \; w' = \frac{1}{\alpha_2}\left(\frac{\beta r c_2 - \beta d_2 d_4 - c_2 d_1 d_2}{\beta d_4 + c_2 d_1}\right),$$

which exist if $(\beta r - d_1 d_2) c_2 > \beta d_2 d_4$

(d) An endemic semi positive equilibrium $D\left(\bar{x}, \bar{y}, \bar{z}, 0\right)$, where

$$\bar{x} = \frac{rc_1}{\beta d_3 + c_1 d_1}, \; \bar{y} = \frac{d_3}{c_1}, \; \bar{z} = \frac{1}{\alpha_1}\left(\frac{\beta r c_1 - \beta d_2 d_3 - c_1 d_1 d_2}{\beta d_3 + c_1 d_1}\right),$$

which exist if $(\beta r - d_1 d_2) c_1 > \beta d_2 d_3$

(e) An endemic positive equilibrium $E\left(x^*, y^*, z^*, w^*\right)$, where

$$x^* = \frac{r}{\beta y^* + d_1},$$
$$z^* = \frac{d_4 - c_2 y^*}{\lambda_1},$$
$$w^* = \frac{c_1 y^* - d_3}{\lambda_1}$$

And y^* is the positive root of the following equation

$Ay^{*2} + By^* + C = 0$ where

$A = \beta(\alpha_1 c_2 - \alpha_2 c_1),\ B = \alpha_1 d_1 c_2 - \beta d_2 \lambda_1 + \alpha_2 d_3 \beta - \alpha_2 d_1 c_1 - \alpha_1 d_4 \beta,$

$C = \beta r \lambda_1 - \alpha_1 d_1 d_4 - \lambda_1 d_1 d_2 + \alpha_2 d_1 d_3$

The above equation will have positive real solution if

$B^2 - 4AC > 0$ and one of the following conditions holds

(i) $-\frac{B}{A} > 0,\ \frac{C}{A} > 0$, then two positive roots for y^*

(ii) $\frac{C}{A} < 0$, then one positive roots for y^*

4. STABILITY ANALYSIS

The variational matrix for the system (1) is given by

$$\begin{bmatrix} -\beta y - d_1 & -\beta x & 0 & 0 \\ \beta y & \beta x - \alpha_1 z - d_2 - \alpha_2 w & -\alpha_1 y & -\alpha_2 y \\ 0 & c_1 z & -d_3 + c_1 y - \lambda_1 w & -\lambda_1 z \\ 0 & c_2 w & \lambda_1 w & -d_4 + c_2 y + \lambda_1 z \end{bmatrix}$$

The following theorems are direct consequences of linear stability analysis of the system (1)

Theorem-1:- The equilibrium point $A\left(\frac{r}{d_1}, 0, 0, 0\right)$ is locally asymptotically stable if $\beta r - d_1 d_2 < 0$

Proof: - The Eigen values for the equilibrium $A\left(\frac{r}{d_1}, 0, 0, 0\right)$ are given by

$$\xi_1 = -d_1,\quad \xi_2 = \frac{\beta r}{d_1} - d_2,\quad \xi_3 = -d_3,\quad \xi_4 = -d_4$$

Clearly $\xi_1, \xi_3, \xi_4 < 0$,

Now $\xi_2 < 0$

Iff $\beta r - d_1 d_2 < 0$

This means that stability of equilibrium point $A\left(\frac{r}{d_1}, 0, 0, 0\right)$ will result in non existence of equilibrium point $B\left(\hat{x}, \hat{y}, 0, 0\right)$.

Theorem -2:-The predator free equilibrium $B\left(\hat{x}, \hat{y}, 0, 0\right)$ if exist, is locally asymptotically stable for $c_1 < \frac{d_2 d_3}{r}$.

Proof: - The characteristic equation for the equilibrium $B\left(\hat{x}, \hat{y}, 0, 0\right)$ is given by

$$(\xi + d_4 - c_2\hat{y})(\xi + d_3 - c_1\hat{y})\left(\xi^2 + (\beta\hat{y} + d_1)\xi + \beta d_2\hat{y}\right) = 0$$

Here $\xi_1 = c_2\hat{y} - d_4, \xi_2 = c_1\hat{y} - d_3$ and ξ_3, ξ_4 are roots of third factor having the entire coefficients positive. So ξ_3, ξ_4 are clearly negative.

For $\xi_1 < 0$, we have

$\frac{c_2 r}{d_2} < d_4$ (A)

For $\xi_2 < 0$, we have

$\frac{c_1 r}{d_2} < d_3$ (B)

Conditions (A) and condition (B) can be combined as $\frac{r}{d_2} < \text{Min}\left\{\frac{d_4}{c_2}, \frac{d_3}{c_1}\right\} = \frac{d_3}{c_1}$

i.e equilibrium $B\left(\hat{x}, \hat{y}, 0, 0\right)$ is locally asymptotically stable if $c_1 < \frac{d_2 d_3}{r}$

Theorem-3:- The equilibrium point $C\left(x', y', 0, w'\right)$, if exist is locally asymptotically stable if $c_1 d_4 - c_2 d_3 < 0$ and $a_1 a_2 - a_3 > 0$,where $a_1=\beta y'+d_1$, $a_2=c_2\alpha_2 y'w'+\beta^2 x'y'$, $a_3=(\beta y'+d_1)c_2\alpha_2 y'w'$.

Proof:- The Characteristic roots corresponding to the equilibrium $C\left(x', y', 0, w'\right)$, are given by the equation

$$(\xi + d_3 - c_1 y' + \lambda_1 w')\left(\xi^3 + a_1\xi^2 + a_2\xi + a_3\right) = 0$$

Here $\xi_1 = c_1 y' - \lambda_1 w' - d_3 < 0$ if $c_1 d_4 - c_2 d_3 < 0$.

Also ξ_2, ξ_3, ξ_4 are roots of second factor. Again by using Routh- Hurwitz criterion, the second bracket will have negative eigen values if $a_1 a_2 - a_3 > 0$

Theorem-4:- The endemic semi positive equilibrium $D(\bar{x},\bar{y},\bar{z},0)$ if exist is locally asymptotically stable if $\lambda_1 \beta r c_1^2 < \alpha_1 L(c_1 d_4 - c_2 d_3)$, where $L = \beta d_3 + c_1 d_1$ and $a_1 a_2 - a_3 > 0$,where $a_1 = \beta\bar{y} + d_1$, $a_2 = c_1\alpha_1\bar{y}\,\bar{z} + \beta^2\bar{x}\,\bar{y}$, $a_3 = (\beta\bar{y} + d_1)c_1\alpha_1\bar{y}\,\bar{z}$.

Proof:-The Characteristic roots corresponding to the equilibrium $D(\bar{x},\bar{y},\bar{z},0)$ are given by the equation

$$(\xi + d_4 - c_2\bar{y} - \lambda_1\bar{z})\left(\xi^3 + a_1\xi^2 + a_2\xi + a_3\right) = 0 = 0$$

Here $\xi_1 = c_2\bar{y} + \lambda_1\bar{z} - d_4 < 0$ if $\lambda_1 \beta r c_1^2 < \alpha_1 L(c_1 d_4 - c_2 d_3)$

Also ξ_2, ξ_3, ξ_4 are roots of second factor. Again by using Routh- Hurwitz criterion, the second bracket will have negative eigen values if $a_1 a_2 - a_3 > 0$

Theorem-5:- The positive endemic equilibrium $E(x^*, y^*, z^*, w^*)$ is locally asymptotically stable if $a_i > 0\,(i = 1, 2, 3, 4)$, $a_1 a_2 - a_3 > 0$, $a_1 a_2 a_3 - a_1^2 a_4 - a_3^2 > 0$.

where $a_1 = \beta y^* + d_1$, $a_2 = \left(\lambda_1^2 z^* w^* + c_1\alpha_1 y^* z^* + c_2\alpha_2 y^* w^*\right) + \beta^2 y^* x^*$

$a_3 = (\beta y^* + d_1)\left(\lambda_1^2 z^* w^* + c_1\alpha_1 y^* z^* + c_2\alpha_2 y^* w^*\right)$

$a_4 = (\beta y^* + d_1)\left(\alpha_2 c_1 \lambda_1 y^* z^* w^* - \alpha_1 c_2 \lambda_1 y^* z^* w^* + \beta^2 \lambda_1^2 y^* x^* z^* w^*\right)$

Proof:-The Characteristic equation corresponding to the equilibrium $E(x^*, y^*, z^*, w^*)$ is given by the equation

$$\xi^4 + a_1\xi^3 + a_2\xi^2 + a_3\xi + a_4 = 0$$

Here $\xi_1, \xi_2, \xi_3, \xi_4$ are roots of above equation which are negative by Routh Hurwitz criterion if $a_i > 0\,(i = 1, 2, 3, 4)$, $a_1 a_2 - a_3 > 0$, $a_1 a_2 a_3 - a_1^2 a_4 - a_3^2 > 0$

5. CONCLUSION

In this paper, a four species Prey- Predator model is proposed and analyzed. Consequently the population is compartmentalized into four classes i.e susceptible predator, infected predator, susceptible prey and infected prey. Further it is found that if E_2 exist then E_1 is not locally stable. Local stability analysis of all possible equilibrium points has been done. The conditions for the existence of equilibrium points and their stability have been established. Parametric conditions for the removal of infection from prey and predator species have been obtained. Existence and stability conditions of non-zero equilibrium point have also been worked out.

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