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Fractional Order Nonlinear Prey Predator Interactions

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Abstract

This paper analyses the dynamical behavior of a fractional order Prey – Predator Model. A discretization process is applied to obtain its discrete version. The fixed points are obtained and the stability properties are discussed. Time series and phase portraits are presented and the oscillation in the prey predator population is established via limit cycles.

Keywords: Fractional Order, discretization, Lotka - Volterra predator prey system, limit cycles

2010Subject classification: 34A08, 34D20, 92D25

1. INTRODUCTION

Mathematical modeling of interactions between species has drawn the attention of researchers [1, 3, 6]. Population models in ecology have been studied using ordinary differential equations, difference equations, partial differential equations, fractional order differential equations and stochastic models. Fractional order differential equations (FODE) are suitable to study the systems with memory which exists in most biological systems. FODE can be used to model many phenomena which cannot be modeled by integer order differential equations [2, 5]. There are several definitions for fractional derivatives, see [4].

Stability and dynamical analysis of fractional order Lotka – Volterra models can be found in [5, 7, 8, 9]. In this paper, we consider fractional order Lotka – Volterra predator prey system for the study of its dynamical behaviors.

2. DISCRETIZATION OF FRACTIONAL ORDER MODEL

In 1926, Volterra came up with a model to describe the evolution of predator and prey fish populations in the Adriatic Sea. They were proposed independently by Alfred J. Lotka in 1925 [6, 7]. The equations are

$$x' = ax - bxy; y' = -cy + dxy$$

The dynamics is periodic and system has the tendency to oscillate. Also the model generates neutral stability. In [7], authors considered

$${}^{c}D_{0+}^{\alpha} x = x(1-y); {}^{c}D_{0+}^{\beta} y = y(-1+x)$$

and compared the dynamics of classical model with fractional order. Paper [9] discusses the stability of the equilibrium points of the fractional order system

$$D^{\alpha} y_1 = y_1(a - by_2); D^{\beta} y_2 = y_2(-c + by_2)$$

Introducing the fractional order in the classical model, we obtain

$$D^{\alpha} x(t) = x(t)(a - by(t)); D^{\alpha} y(t) = y(t)(-c + dx(t)) \quad (1)$$

where $t > 0$ and $\alpha \in (0,1]$. Only the positive and finite values of the parameters have meaningful biological interpretation. Now, applying the discretization process for a fractional-order system described in [2, 10], we obtain the discrete fractional order predator prey system as follows:

$$x_{n+1} = x_n + \frac{s^{\alpha}}{\alpha\Gamma(\alpha)}(x_n(a - by_n)); y_{n+1} = y_n + \frac{s^{\alpha}}{\alpha\Gamma(\alpha)}(y_n(-c + dx_n)). \quad (2)$$

3. FIXED POINTS AND STABILITY

Fixed points are obtained by solving $D^{\alpha} x(t) = 0$; $D^{\alpha} y(t) = 0$. The fixed points of system (1) are

(a) Trivial fixed point $E_0 = (0,0)$ (Origin);

(b) Interior fixed point of coexistence $E_1 = \left(\frac{c}{d}, \frac{a}{b}\right)$ (Interior).

We next study the local stability of the fixed points. The Jacobian matrix J of system (1) evaluated at the fixed point (x^*, y^*) is given by

$$J(x^*, y^*) = \begin{bmatrix} a - by^* & -bx^* \\ dy^* & -c + dx^* \end{bmatrix}$$

The determinant of the Jacobian $J(x^*, y^*)$ is $Det = adx^* - c(a - by^*)$

Theorem 1. *The fixed point* E_0 *is locally asymptotically stable if* $|a| < 1, |c| < 1$, *otherwise unstable fixed point.*

Proof: The Jacobian matrix at E_0 is given by $J(E_0) = \begin{bmatrix} a & 0 \\ 0 & -c \end{bmatrix}$.

Hence the eigenvalues of $J(E_0)$ are $\lambda_1 = a$ and $\lambda_2 = -c$. Thus E_0 is stable when $|a| < 1$, $|c| < 1$. Otherwise E_0 is unstable fixed point.

Theorem 2. *The fixed point* E_1 *is locally asymptotically stable if* $|A| < 1$, *where* $A = \pm i\sqrt{ac}$. *Otherwise unstable fixed point.*

Proof: The Jacobian matrix at E_1 is given by

$$J(E_1) = \begin{bmatrix} 0 & \dfrac{-bc}{d} \\ \dfrac{ad}{b} & 0 \end{bmatrix}.$$

The eigenvalues of the matrix $J(E_1)$ are $\lambda_{1,2} = \pm\sqrt{-ac}$. Hence E_1 is locally asymptotically stable when $|A| < 1$, and unstable $|A| > 1$.

We will now discuss the dynamics of the discretized fractional – order Lotka – Volterra predator prey model (2). The dynamical behaviors of model (2) is determined by the parameters a, b, c, d, s and α.

We will now discuss the stability of fixed points of model (2). The Jacobian matrix J of model (2) evaluated at the fixed point (x^*, y^*) is given by

$$J(x^*, y^*) = \begin{bmatrix} 1 + \dfrac{s^\alpha}{\alpha\Gamma(\alpha)}(a - by^*) & -\dfrac{s^\alpha}{\alpha\Gamma(\alpha)} bx^* \\ \dfrac{s^\alpha}{\alpha\Gamma(\alpha)} dy^* & 1 + \dfrac{s^\alpha}{\alpha\Gamma(\alpha)}(-c + dx^*) \end{bmatrix}. \quad (3)$$

The characteristic equation of the Jacobian matrix is

$$\lambda^2 - Tr\lambda + Det = 0. \quad (4)$$

where Tr is the trace and Det is the determinant of the Jacobian matrix $J(x^*, y^*)$ and they are

$$Tr = 2 + \frac{s^\alpha}{\alpha\Gamma(\alpha)}(a - c + dx^* - by^*)$$

$$Det = 1 + \frac{s^\alpha}{\alpha\Gamma(\alpha)}(a - c + dx^* - by^*) + \left(\frac{s^\alpha}{\alpha\Gamma(\alpha)}\right)^2 (adx^* + bcy^* - ac) \quad (5)$$

To study the stability properties of the model (2), we present the following results that can be easily proved by using the relation between roots and coefficients of the characteristic equation (4). Let λ_1 and λ_2 be the two roots of Eq. (4), are the eigen values of the Jacobian matrix

$$J = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}$$

evaluated at the fixed point of system (1). Then we have

Lemma 3. (i) *A fixed point* (x^*, y^*) is called a sink if $|\lambda_1| < 1$ and $|\lambda_2| < 1$, so the sink is locally asymptotically stable.

(ii) A fixed point (x^*, y^*) is called source if $|\lambda_1| > 1$ and $|\lambda_2| > 1$, so the source is locally unstable.

(iii) A fixed point (x^*, y^*) is called a saddle if $|\lambda_1| > 1$ and $|\lambda_2| < 1$ (or $|\lambda_1| < 1$ and $|\lambda_2| > 1$).

(iv) A fixed point (x^*, y^*) is called non-hyperbolic if either $|\lambda_1| = 1$ or $|\lambda_2| = 1$.

Theorem 4. *If* $0 < s < \sqrt[\alpha]{\frac{2\alpha\Gamma(\alpha)}{c}}$ *then the fixed point* E_0 *is a saddle point. If* $s > \sqrt[\alpha]{\frac{2\alpha\Gamma(\alpha)}{c}}$ *, then* E_0 *is a source and if* $s = \sqrt[\alpha]{\frac{2\alpha\Gamma(\alpha)}{c}}$ *, then* E_0 *is non-hyperbolic.*

Proof: The Jacobian matrix J at E_0 is given by $J(E_0)=\begin{bmatrix} 1+\frac{s^\alpha}{\alpha\Gamma(\alpha)}a & 0 \\ 0 & 1-\frac{s^\alpha}{\alpha\Gamma(\alpha)}c \end{bmatrix}$.

Hence, the eigenvalues are $\lambda_1 = 1+\frac{s^\alpha}{\alpha\Gamma(\alpha)}a$ and $\lambda_2 = 1-\frac{s^\alpha}{\alpha\Gamma(\alpha)}c$. Since $a>0$, then $|\lambda_1|>1$. Thus the fixed point E_0 is a saddle point if $0<s<\sqrt[\alpha]{\frac{2\alpha\Gamma(\alpha)}{c}}$, source if $s>\sqrt[\alpha]{\frac{2\alpha\Gamma(\alpha)}{c}}$ and non-hyperbolic if $s=\sqrt[\alpha]{\frac{2\alpha\Gamma(\alpha)}{c}}$.

Theorem 5. *The positive fixed point* $E_1=\left(\frac{c}{d},\frac{a}{b}\right)$ *of the system (2) is locally asymptotically stable if* $\beta<0$, *where* $\beta=\frac{s^\alpha}{\alpha\Gamma(\alpha)}A$ *such that* $A=\sqrt{-ac}$.

Proof: The Jacobian matrix evaluated at the fixed point E_1 has the form

$$J(E_1)=\begin{bmatrix} 1 & -\frac{s^\alpha}{\alpha\Gamma(\alpha)}\frac{bc}{d} \\ \frac{s^\alpha}{\alpha\Gamma(\alpha)}\frac{ad}{b} & 1 \end{bmatrix}.$$

The trace and determinant of the Jacobian matrix $J(E_1)$ are given by

$$Tr(J(E_1))=2, \qquad Det=1+\left(\frac{s^\alpha}{\alpha\Gamma(\alpha)}\right)^2(ac).$$

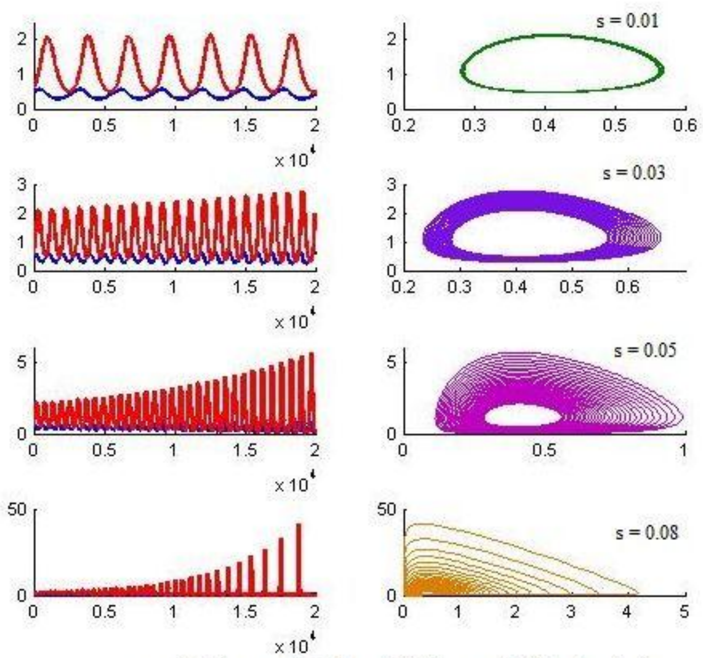
Hence the eigenvalues are $\lambda_{1,2}=1\pm\frac{s^\alpha}{\alpha\Gamma(\alpha)}\sqrt{-(ac)}$. Thus E_1 is locally asymptotically stable when $\beta<0$. The next lemma is an immediate consequence from Theorem (5).

Lemma 6. *The positive fixed point* $E_1=\left(\frac{c}{d},\frac{a}{b}\right)$ *of the system (2) is unstable if* $\beta>0$, *where* $\beta=\frac{s^\alpha}{\alpha\Gamma(\alpha)}A$ *such that* $A=\sqrt{-ac}$ *holds.*

4. PERIODIC SOLUTIONS AND LIMIT CYCLES

In this section, examples are presented to illustrate the periodic oscillations in the prey predator populations. All of the trajectories form closed orbits. Mainly, we present the orbits of the solutions x and y with phase plane diagrams for the fractional order predator-prey system (2). Ecologists applied the classical model to the data on the Canadian lynx snow shoe hare interaction which exhibited periodic oscillation resulting in limit cycles. A closed trajectory implies cycle periodic oscillation.

Example 1. *For* $\alpha = 0.99, a = 0.1; b = 0.09; c = 0.45; d = 1.1$ *and* $s = 0.01, 0.03, 0.05$ *and* $s = 0.08$ *, the time plots and phase portraits are presented. These values results in solutions with different amplitudes. Oscillatory behavior of the population is established by the existence of limit cycles, See Figure 1.*



α = 0.99; a = 0.1; b = 0.09; c = 0.45; d = 1.1

Example 2. *For the initial values* $x = 0.5, y = 0.6$ *and* $\alpha = 0.99;$ $s = 0.05,$ $a = 0.0025;$ $b = 0.0089;$ $c = 0.495$ *and* $d = 0.999$ *, the Eigen values are* $\lambda_{1,2} = 1.0000 \pm i0.0018,$ *so that* $|\lambda_{1,2}| = 1.00000032 > 1$ *Hence the interior fixed point* E_1 *is unstable. The orbit and the phase diagram illustrate the result, See Figure 2.*

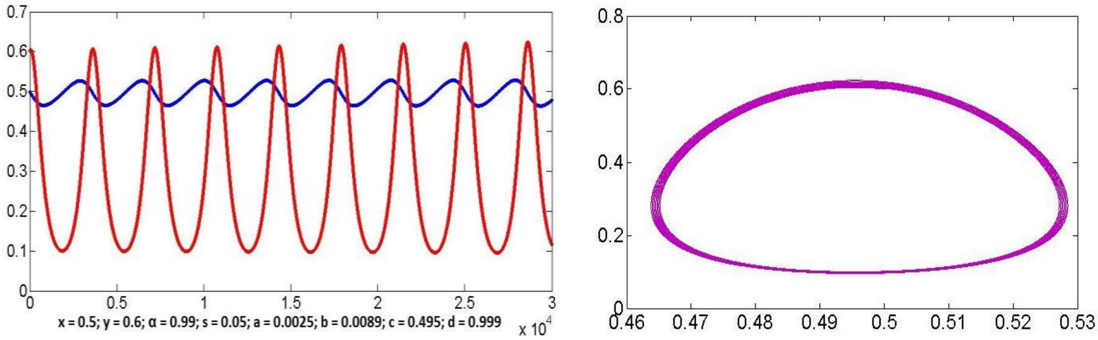


Figure 2. Time Series and Phase Portrait for interior fixed point E_1.

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