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On Partially Ordered Disemigraphs

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Abstract

Partially ordered sets serve as the starting point of several algebraic structures. In this paper we introduce the concept of partially ordered disemigraphs. Here we are introducing the possibility of partial ordering on the subdisemigraphs of a Disemigraph.

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1. Introduction

Semigraphs is the concept introduced by E. Sampathkumar [5] it can be said to be a natural generalization of concept of graphs. A semigraph resembles a graph when drawn on a plane. Semigraphs can be used to model concepts such as road networks, family relations, etc. In [1], J. Blessed Singh defined the po-digraphs which is in other way equivalent to the concept of poset in set theory. The concept of po-disemigraph can be extended to

2. Preliminaries

In this section some basic definitions [1, 2, 3, 4] and examples needed later are given.

Definition 2.1. A semigraph G is a pair (V, X) where V is a nonempty set whose elements are called vertices of G and X is a set of n-tuples called edges of G of distinct vertices for $n \geq 2$ satisfying the following conditions.

(1) Any two edges have atmost one vertex in common.

(2) Two edges $(u_1, u_2, \dots, u_n)$ and $(v_1, v_2, \dots, v_m)$ are considered to be equal, if and only if,

  (a) $m = n$ and

  (b) either $u_i = v_i$ for $1 \le i \le n$ or $u_i = v_{n-i+1}$ for $1 \le i \le n$.

Thus the edge $(u_1, u_2, \dots, u_n)$ is the same as the edge $(u_n, u_{n-1}, \dots, u_1)$.

u_1 and u_n are the end vertices of edge E and u_i, $2 \le i \le n-1$ are the middle vertices of E.

A semigraph G may be drawn as a set of points representing the vertices. An edge $E = (v_1, v_2, \dots, v_n)$ is represented by a Jordan curve joining the points corresponding to the vertices $(v_1, v_2, \dots, v_n)$ in the same order as they appear in E. The end points of the curve are denoted by thick dots. The vertices in between the end points are the middle vertices denoted by small circles. If an end vertex of some edge E, is the middle vertex of some other edge E', a small tangent is drawn to the circle at the end of E.

All the vertices on an edge of a semigraph are considered to be adjacent to one another for obvious reasons.The vertices are divided into four types namely end vertices, middle vertices, middle-end vertices and isolated vertices.

Example 2.2. Semigraph $G = (V, X)$ where $V = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ and

$$X = \{(u_1, u_2, u_3), (u_1, u_6), (u_3, u_6, u_5), (u_4, u_5), (u_3, u_4), (u_4, u_6)\}.$$

is given in Fig. 1.

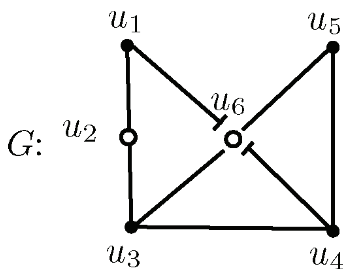


Fig. 1

Definition 2.3. A *subedge* of an edge $E = (v_1, v_2, \dots, v_n)$ is a k-tuple $E' = (v_{i_1}, v_{i_2}, \dots, v_{i_k})$ where $1 \le i_1 < i_2 < \dots < i_k \le n$ or $1 \le i_k < i_{k-1} < i_{k-2} \dots < i_1 \le n$. E' is said to be induced by the set of vertices $\{v_{i_1}, v_{i_2}, \dots, v_{i_k}\}$.

Definition 2.4. A partial edge E is a $(j - i + 1)$-tuple. $E(v_i, v_j) = (v_i, v_{i+1}, \dots, v_j)$ where $1 \le i \le n$. Thus a subedge E' of an edge E is a partial edge if and only if any two consecutive vertices in E' are also consecutive vertices of E.

Definition 2.5. A semigraph $G' = (V', X')$ is a subsemigraph of a semigraph $G = (V, X)$ if $V' \subseteq V$ and the edges in G' are subedges of G.

Definition 2.6. A *directed semigraph* or *disemigraph* D is a finite set of objects called vertices together with a (possibly empty) set of ordered n-tuples of distinct vertices of D for various $n \geq 2$ called *directed edges* or *arcs*, satisfying the condition C given below.

Suppose $a = (u_1, u_2, \ldots, u_n)$ is an arc. Then for $1 \leq i < j \leq n$, u_i is adjacent to u_j and u_j is adjacent from u_i. Thus each u_i is adjacent to u_j, $1 \leq i < j \leq n$ and each u_j is adjacent from u_i, $1 \leq i < j \leq n$.

C: for any two distinct vertices u and v in a disemigraph D there is at most one arc containing u and v such that u is adjacent to v and at most one arc containing u and v such that v is adjacent to u.

Definition 2.7. A disemigraph D_1 is a *subdisemigraph* of a disemigraph D if $V(D_1) \subseteq V(D)$ and $E(D_1) \subseteq E(D)$ This is denoted by $D_1 \subseteq D$, where $V(D)$ is the vertex set of D and $E(D)$ is the edge set of D. A subdisemigraph D_1 is a spanning subdisemigraph of D if D_1 has the same vertex set as D.

Definition 2.8. A disemigraph D is *simple* if any two arcs either contain at most one vertex or all vertices in common.

Two vertices u and v are adjacent if u is either adjacent to or adjacent from v. It may happen that u is adjacent to v in one arc and v is adjacent to u in another arc.

Example 2.9. The disemigraph D with $V(D) = \{v_1, v_2, v_3, v_4, v_5\}$ and the arc set $E(D) = \{(v_5, v_4), (v_4, v_3, v_2), (v_5, v_3), (v_4, v_1), (v_1, v_2), (v_3, v_1, v_5)\}$ is given below

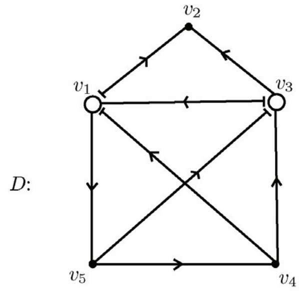


Fig. 2

Definition 2.10. A collection $\{E_1, E_2, E_3, \ldots, E_n\}$ of edges of a semigraph $G = (V, X)$ is said to be a *chain* if (i) $|E_i \cap E_{i+1}| = 1$ and (ii) $|E_i \cap E_j| = 0$, for every $i \neq j$, $j \neq i + 1$, i.e., no edges are adjacent except the consecutive ones.

Definition 2.11. [6] Given a non empty set L, a binary relation on L is called a partial ordering if the following relations hold:

(i) $x \leq x$(reflexivity)

(ii) $x \leq y$ and $y \leq x$ implies $x = y$(anti-symmetry)

(iii) $x \leq y$ and $y \leq z$ implies $x \leq z$ (transitivity) for every $x, y, z \in L$. A set L with a partial order relation $\leq$ is called a partially ordered set, or briefly poset, and is denoted by $(L, \leq)$.

Definition 2.12. A poset L is called a lattice if every finite subset of L has a supremum ($\bigvee$) and an infimum ($\bigwedge$). A lattice is called complete if every subset of L has a supremum and an infimum.

3. The Partially Ordered Disemigraph

In this section we are looking into the partial ordering on the set of subdisemigraphs of a disemigraph.

Definition 3.1. Let D be a disemigraph and $\mathfrak{D}$ be a collection of subdigraphs of D. Then $\mathfrak{D}$ together with a relation R is said to form a *partially ordered disemigraph* if the following properties hold.

(i) $H_i \; R \; H_i \forall H_i \in \mathfrak{D}$

(ii) If $H_i \; R \; H_j \text{and} H_j \; R \; H_i \text{then} H_i = H_j \forall H_i, H_j \in \mathfrak{D}$

(iii) If $H_i \; R \; H_j$ and $H_j \; R \; H_k$ then $H_i R H_k \; \forall H_i, H_j, H_k \in \mathfrak{D}$.

Analogus to the term po-set for partially ordered set and po-digraph for partially ordered digraph we can use the term po-disemigraph for partially ordered disemigraph.

3.1. The Graphical Representation of Po-disemigraph

The graphical representation of $(\mathfrak{D}, \subseteq)$ of a disemigraph. Let D be a disemigraph and $\{H_i\}$ be a collection of sub disemigraphs of D, which is denoted by $\mathfrak{D} = \{H_i\}$ then $(\mathfrak{D}, \subseteq)$ the po-disemigraph of D has a graphical representation where H_i are the vertices of the disemigraph of $\mathfrak{D}$. For that graph the edges are the sequence of vertices following the '$\subseteq$' relation. By reflexive property every subdisemigraph is related to itself. But by the definitions, digraph [4, 5] and disemigraph [1] have no loops hence we ignore the arcs corresponding to reflexive property. A vertex H_i is an end vertex of the podisemigraph $\mathfrak{D}$ only if either there exists no H_j such that $H_j \subseteq H_i$ in $\mathfrak{D}$ or there exists no H_j such that $H_i \subseteq H_j$ in $\mathfrak{D}$. Also an H_i is a middle vertex whenever for an H_i there exists H_j and H_k in $\mathfrak{D}$ such that $H_j \subseteq H_i \subseteq H_k$. If $H_i \subseteq H_j$ then (H_i, H_j) is an arc.

Example 3.2. Consider the disemigraph D and $\mathfrak{D} = \{H_1, H_2, H_3, H_4\}$ given below $(\mathfrak{D}, \subseteq)$ is a po-disemigraph.

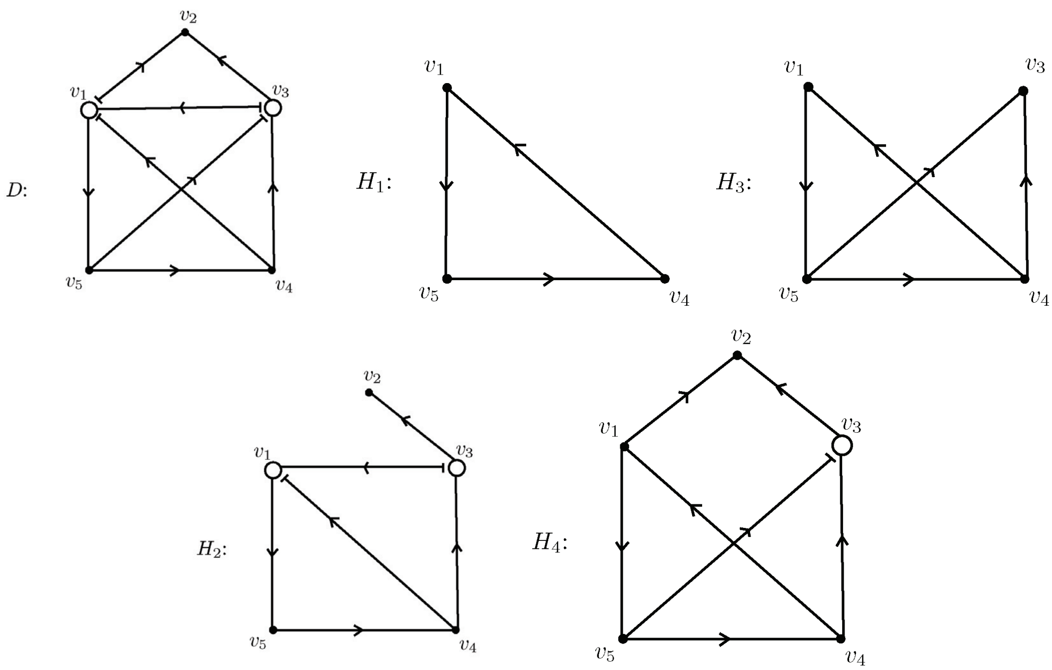


Fig. 4

The graphical representation of $\mathfrak{D}$ is as follows

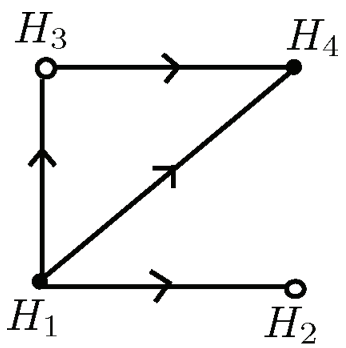
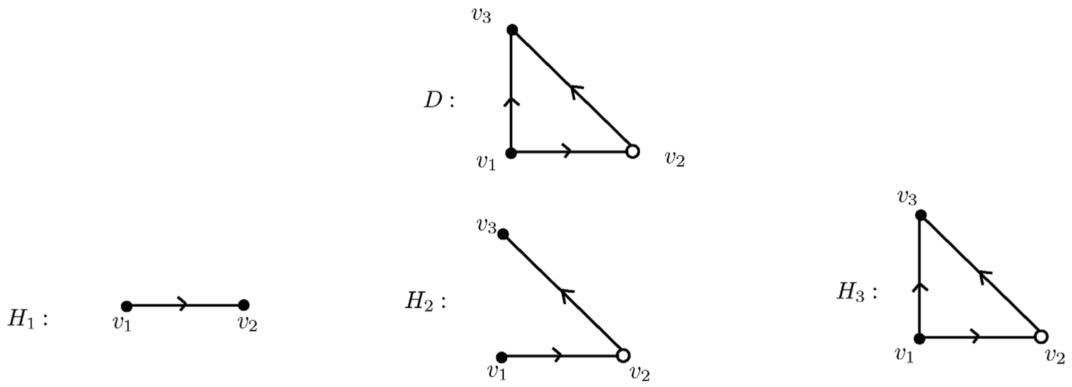


Fig. 5

Definition 3.3. A disemigraph D is a po-disemigraph if there exists a po-disemigraph $(\mathfrak{D}, \subseteq)$ of D such that the graphical representation of $\mathfrak{D}$ is isomorphic to disemigraph D.

Example 3.4. The disemigraph D with $\mathfrak{D} = \{H_1, H_2, H_3\}$ given below is isomorphic to D. Hence D is a po-disemigraph.



Then $\mathfrak{D}$:

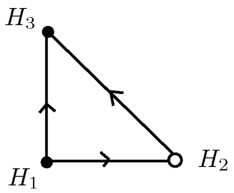


Fig. 6

Proposition 3.5. The graphical representation of a po-disemigraph is not symmetric.

Proof. Let $(\mathfrak{D}, R)$ be a po-disemigraphs. If $H_i, H_j \in \mathfrak{D}$; then only $H_i R H_j$ or $H_j R H_i$ since if $H_i R H_j$ and $H_j R H_i$ then $H_i = H_j$ hence po-disemigraph is not symmetric. ■

Proposition 3.6. The po-disemigraph $\mathfrak{D}$ has only two end vertices in it, all the other vertices are middle vertices if any two subdisemigraphs in it are compareable.

Proof. If R is a realation on $\mathfrak{D}$ and if any two subdisemigraphs in it are compareable then by definition of po-disemigraph there exists a minimal element $H_0 \in \mathfrak{D}$ such that H_0 is always less than or equal to H_i for all $H_1 \in \mathfrak{D}$. In the same way the maximal element $H_n \in \mathfrak{D}$ also exists. Then H_0 and H_n are the only end vertices and all the other vertices will be middle vertices. ■

Theorem 3.7. The subsubset of a po-disemigraph is itself a po-disemigraph under the same inclusion relation.

Proof. Let $\mathcal{H}$ be a subdisemigraph of a po-disemigraph $\mathfrak{D}$. Then

(i) $H_i R H_i \forall H_i \in \mathcal{H}$ as $H_i \in \mathfrak{D}$

(ii) If $H_i R H_j$ and $H_j R H_i$ then $H_i = H_j \forall H_i, H_j \in \mathcal{H}$ as $H_i, H_j \in \mathfrak{D}$

(iii) If $H_i R H_j$ and $H_j R H_k$ then $H_i R H_k \forall H_i, H_j, H_k \in \mathcal{H}$ as $H_i, H_j, H_k \in \mathfrak{D}$.

Hence the theorem. ■

Definition 3.8. A subdisemigraph H_i in po-disemigraph $(\mathfrak{D}, R)$ is said to be maximal (minimal) subdisemigraph if for any other $H_j \in \mathfrak{D}$, H_j is a subdisemigraph of H_i (H_i is a subdisemigraph of any other $H_j \in \mathfrak{D}$).

Definition 3.9. Given any subdisemigraphs H_i, H_j in a po-disemigraph $(\mathfrak{D}, \subseteq)$ if either $H_j \subseteq H_i$ or $H_i \subseteq H_j$ then the po-disemigraph is totally ordered.

Theorem 3.10. If the po-disemigraph $(\mathfrak{D}, \subseteq)$ contains a maximal subdisemigraph and a minimal subdisemigraph of D, then it contains only two end vertices.

Proof. First to prove the existence of a maximal subdisemigraph, consider any vertex in the disemigraph associated with $(\mathfrak{D}, \subseteq)$. If it is not a maximal subdisemigraph there will be an edge leaving that vertex, move to the neighbouring vertex through that edge. Continue this till we reach the end vertex from which no edge leaves. The subdisemigraph represented by that vertex will be the maximal subdisemigraph in $\mathfrak{D}$.

To find the minimal subdisemigraph in $(\mathfrak{D}, \subseteq)$ reverse the direction of the edges in the disemigraph of $(\mathfrak{D}, \subseteq)$ and find the end vertex in it as above starting from any vertex. The end vertex found will be the minimal vertex in $(\mathfrak{D}, \subseteq)$. ■

In example 2.1.1. H_1 is the minimal subdisemigraph but there is no maximal subdisemgraph hence it has more than two end vertices. In example 2.1.3. there is a minimal subdisemigraph and maximal subdisemgraph in $\mathfrak{D}$. Hence it has only two end vertices. The concept of partially ordered disemigraphs have a lattice structure also.

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