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New Results on Equienergetic Graphs of Small Order

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Abstract

The energy of a graph G is the sum of the absolute values of its eigenvalues. Two non-isomorphic graphs of same order are said to be equienergetic if their energies are equal. In this paper energy of trees of order upto 10 have been analysed in detail and equi-energetic trees were identified and its properties have been examined. We give a MATLAB approach to find the equienergetic graphs satisfying the condition $\mathscr{E}(G) = \mathscr{E}(G - e)$ where e is an edge of the graph G.

AMS subject classification: 05C50.
Keywords: Adjacency Matrix, Graph Energy.

1. Introduction

The concept of energy was introduced by Ivan Gutman in 1978 [8]. The chemical aspects of this concept are outlined in the book [9] The energy of a graph is calculated from the eigenvalues of the corresponding adjacency matrix of the graph.

All graphs considered in this paper are simple, finite and undirected. Let G be simple graph with the vertex set $V(G)$ consisting of n vertices labeled by $v_1, v_2, \dots v_n$ and an edge set $E(G)$. The **adjacency matrix** $A(G)$ of the graph G is the square matrix of order n, whose $(i, j)^{th}$ entry is equal to 1 if the vertices v_i and v_j are adjacent and is equal to zero, otherwise. That is,

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent.} \\ 0, & \text{otherwise.} \end{cases}$$

where a_{ij} is any element of $A(G)$.

Let G be a graph on n vertices. The eigen values of the adjacency matrix of G denoted by λ_i where $i = 1, 2, \ldots, n$ are said to be the eigenvalues of the graph G and form the spectrum of G [1]. The **energy** of a graph G is the sum of the absolute values of the eigen values of its adjacency matrix $A(G)$. That is,

$$\mathscr{E}(G) = \sum_{i=1}^{n} | \lambda_i | \tag{1.1}$$

When two graphs are structurally different one would naturally assume that energy of these graphs are also different. However, it was observed that this is not true always. That is, two graphs that are structurally different can have same energy. For example, consider the cycles C_3 and C_4. The eigenvalues of the adjacency matrix of these graphs are 2, -1, 1 and 2, 0, 0, 2 respectively so that $\mathscr{E}(C_3) = \mathscr{E}(C_4)$. It was this observation that led to the formulation of the concept of equi-energetic graphs.

The study on equi-energetic graphs were first considered in the year of 2004, independently by Stevanovic [2], Ramane and Walikar [5]. As seen earlier, two graphs G_1 and G_2 are said to be **equi-energetic**, if $\mathscr{E}(G_1) = \mathscr{E}(G_2)$. It is obvious that if the two graphs are isomorphic, both the graphs will have the same eigenvalues and hence they are equi-energetic. Therefore, researchers were interested in finding non-isomorphic graphs that are equi-energetic. R. Balakrishnan [10] proved that for any positive integer $n \leq 3$, there exists non-cospectral, equienergetic graphs of order $4n$.

Ramane, Walikar and Gutman [5] constructed equi-energetic line graphs in the year 2004. They proved that if G_1 and G_2 are any two regular graphs on n vertices with the degree $r \geq 3$, then $L^2(G_1)$ and $L^2(G_2)$ are equi-energetic and they generalised this result for any $k \geq 2$. The same authors [6] in the same year showed that $\mathscr{E}(L^2(G_1))$ and $\mathscr{E}(L^2(G_2)) = 2nr(r-2)$ for $r \geq 3$. They also presented a result on energy of complement of line graphs. That is, if G_1 and G_2 are any two regular graphs on n vertices with the degree $r \geq 3$, and if $L^2\overline{(G_1)}$ and $L^2\overline{(G_2)}$ are the complements of the graphs $L^2(G_1)$ and $L^2(G_2)$ then $\mathscr{E}(L^2\overline{(G_1)}) = \mathscr{E}(L^2\overline{(G_2)}) = 2nr(r-2)$. This led to the construction of infinite family of equi-energetic graphs of same order which are non-cospectral. Ramane, Walikar, Halkarni [7] proved yet another result that if G is a regular graph on n vertices of degree $r \geq 3$, then $\mathscr{E}(L^2(G)) = \mathscr{E}(L^2\overline{(G)})$ if and only if $G = K_6$. For other results on equienergetic graphs see [3, 4].

From the review of above results on equi-energetic graphs, it was observed that characterization of equi-energetic graphs is still an open problem. For a general graph this problem is too large to attempt. Therefore, we try to explore graphs of order less than 10 and find out equi-energetic graphs in this family. In particular, we attempt the following questions:

What are the non-isomorphic equi-energetic graphs of order less than 10? What are the characteristic features of such graphs?

2. Equi-energetic trees of order less than 10

If the two graphs are isomorphic then the eigenvalues of two graphs will be same and hence the two graphs will have the same energy. We now present the non-isomorphic co-spectral trees of order less than 10 having the same energy.

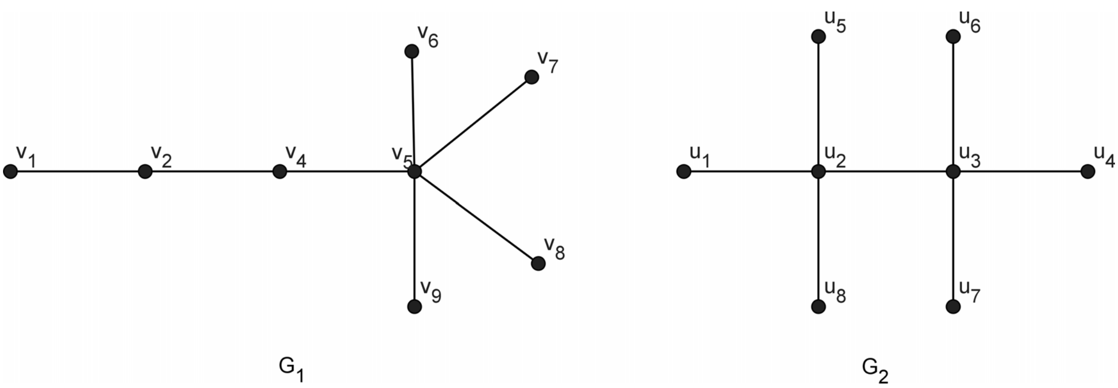


Figure 1: Non-isomorphic co-spectral trees on 8 vertices

The eigen values of the tree graphs in Figure 1 are -2.3028, -1.3028, $0^{(4)}$, 1.3028, 2.3028 and $\mathscr{E}(G_1) = \mathscr{E}(G_2) = 7.211$.

We have also found that there exists a pair of non-isomorphic cospectral trees on 9 vertices whose eigen values are -2.2361, -1.4142, -1, $0^{(3)}$, 1, 1.4142, 2.2361 and $\mathscr{E}(G_3) = \mathscr{E}(G_4) = 9.3006$ which is shown in Figure 2.

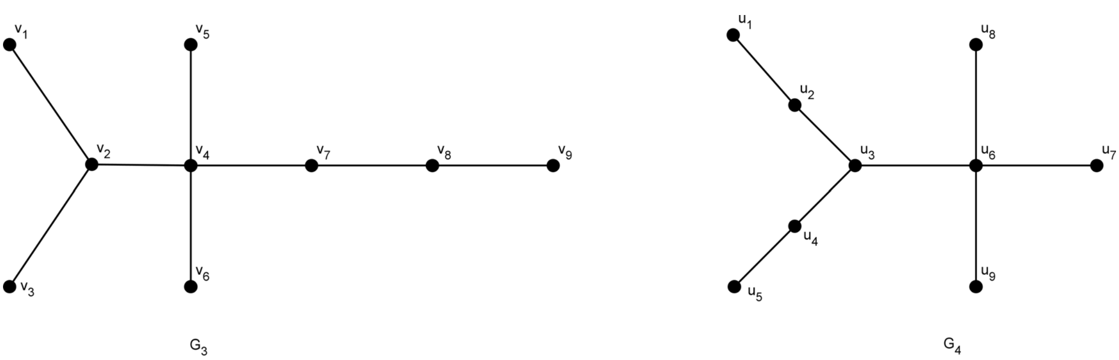
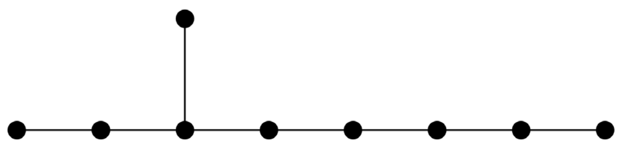


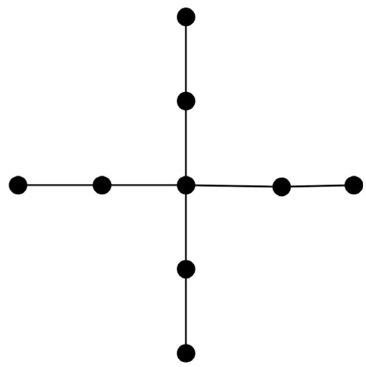
Figure 2: Non-isomorphic co-spectral trees on 9 vertices

Observation 2.1. Although, the graphs G_1, G_2 on 8 vertices and G_3, G_4 on 9 vertices have the same number of vertices, edges and equi-energetic, they both differ in number of pendent vertices, the diameter, and in the degree sequence.

Now we explore further to identify non-cospectral tree graphs that are equi-energetic. The trees given in Figure 3 is one such pair.



G5



G6

Figure 3: Non-cospectral equi-energetic trees on 9 vertices.

2.1. Equi-energetic graphs satisfying the condition $\mathscr{E}(G) = \mathscr{E}(G - e)$

Let G be the graph of order less than 10, we present the list of graphs which satisfies the relation

$$\mathscr{E}(G) = \mathscr{E}(G - e)$$

where e is an edge of the graph G. We observe that there is exactly one graph on 6 vertices and 2 graphs on 9 vertices satisfying this condition which are shown in the fgured given below respectively:

In the Figure 4, M_1 and M_2 are two graphs on 6 vertices, where M_2 is obtained from M_1 by deleting an edge v_5v_6. The eigenvalues of the graph M_1 are 2.732, 1.414, 0, -0.732, -1.414, -2 whose energy is $\mathscr{E}(M_1) = 8.292$ and the eigenvalues of the graph M_2 are 2.414, 1.732, -0.414, -1, -1, -1.732 whose energy is $\mathscr{E}(M_2) = 8.292$. Hence the graph M_1 and the subgraph M_2 obtained from M_1 by deleting a single edge are non-co-spectral equi-energetic graphs.

In Figure 5, M_3 and M_4 are the two graphs on 9 vertices, where M_4 is obtained from M_3 by deleting an edge v_6v_9. The eigenvalues of the graph M_3 are -2, $-1^{(5)}$, 0, 0.2679, 3, 3.7321 whose energy is $\mathscr{E}(M_3) = 14$ and the eigenvalues of the graph M_4 are -2, $-1^{(5)}$, 0.5858, 3, 3.4142 whose energy is $\mathscr{E}(M_4) = 14$. Hence the graph M_3 and the subgraph M_4 obtained from M_3 by deleting a single edge are non-co-spectral equi-energetic graphs.

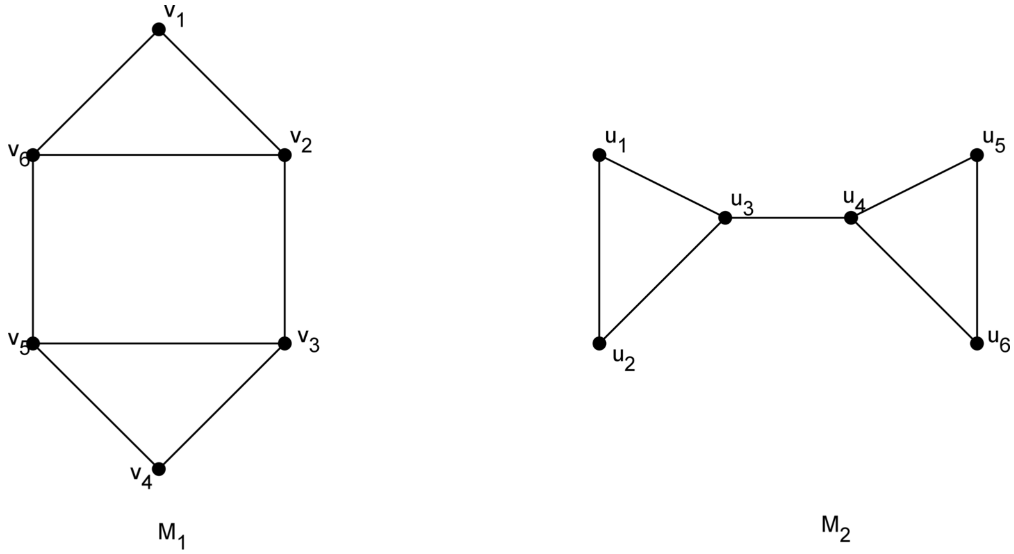


Figure 4: Equi-energetic graph on 6 vertices with $\mathscr{E}(M_1) = \mathscr{E}(M_2) = 8.292$

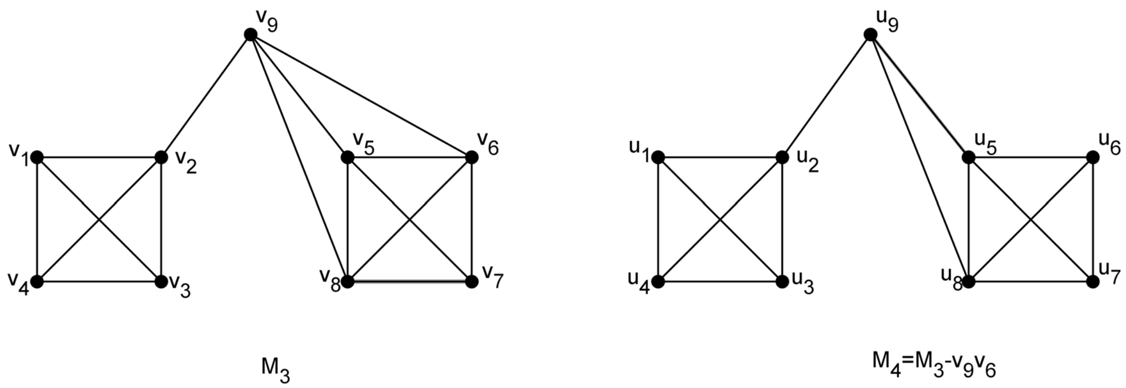


Figure 5: Equi-energetic graphs on 9 vertices.

In Figure 6, M_5 and M_6 are the two graphs on 9 vertices, where M_6 is obtained from M_5 by deleting an edge v_6v_9. The eigenvalues of the graph M_5 are -2, -1.5616, $-1^{(3)}$, 0, 0.2679,2.5616 3.7321 whose energy is $\mathscr{E}(M_5) = 13.1231$ and the eigenvalues of the graph M_6 are -2, -1.5616, $-1^{(3)}$, 0, 0.5858, 2.5616, 3.4142 whose energy is $\mathscr{E}(M_6) = 13.1231$. Hence the graph M_5 and the sub graph M_6 obtained from M_5 by deleting a single edge are non-cospectral equi-energetic graphs.

Observation 2.2. Consider the two graphs M_5 and M_6 in Figure 6. We observe that the graphM_5 is obtained by adding blocks $K_4 - e$ and K_2 to the graph on 5 vertices $K_5 - e$. Before adding the blocks, the energy of the graph on 5 vertices, that is the energy of $K_5 - e$ graph is 7.2916. Similarly, the graph M_6 is obtained by deleting an edge v_9v_6 of

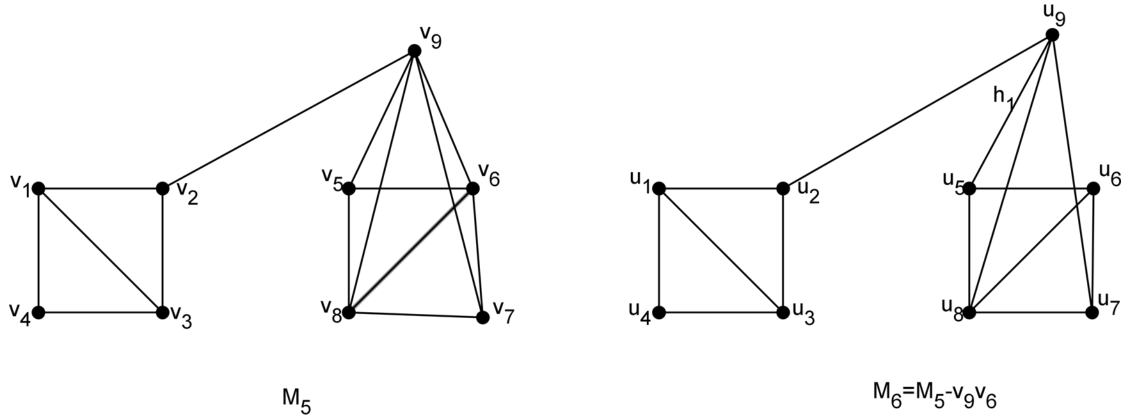


Figure 6: Another pair of equi-energetic graphs on 9 vertices.

the graph M_5. Before adding the blocks, the energy of the graph on 5 vertices is 6.4722. But after adding the blocks to both the graphs on 5 vertices we observe that the energy of two graphs remain same.

2.2. MATLAB code to find the Equi-energetic graphs satisfying the condition $\mathscr{E}(G) = \mathscr{E}(G - e)$

```
function []= energyedgedelete(A)
% Enter the Adjacency matrix
m=length(A);
f=eig(A);
E1=sum(abs(f));
for i=1:m
    if (i~=m)
        B=zeros(m);
        B(i,m)=1;
        B(m,i)=1;
        C=A−B;
        h=eig(C);
        E2=sum(abs(h));
        epsilon=0.00000001;
    if abs(E1−E2)≤ epsilon
        fprintf('\n matrix with energy E1= %1.12f.\n',E1);
        fprintf('\n has equienergetic submatrices given by : \n');
        fprintf('after removing edge e=(%1.0f,%1.0f):\n', i, m);
        display(C)
        fprintf('\n with energy E2= %1.12f\n', E2);
  end
 end
end
```

We establish the relation between the energy of the cartesian product and the tensor product of two graphs K_n and K_2.

Gutman [5] stated in his paper that if G_1 and G_2 are the two graphs with the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $\mu_1, \mu_2, \ldots, \mu_n$ respectively, then

1. The eigen values of $G_1 \Box G_2$ are the product $\lambda_i \mu_j$, for all $i = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, s$.

2. The eigen values of $G_1 \times G_2$ are the $\lambda_i + \mu_j$, for all $i = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, s$.

Proposition 2.3. If K_n is a complete graph on n vertices, then $\mathscr{E}(K_n \Box K_2) = \mathscr{E}(K_n \times K_2)$.

Proof. Let $G = K_n \Box K_2$ and $H = K_n \times K_2$. It is obvious that the number of vertices in G and H are equal. We know that $\mathscr{E}(K_n) = 2(n-1)$ with the eigen values n-1(with with multiplicity 1) and -1(with multiplicity n-1).

$$\mathscr{E}(K_n \Box K_2) = 4(n-1)$$
$$\mathscr{E}(K_n \times K_2) = 4(n-1)$$

Hence, both G and$H$ are of same energy. Using MATLAB to find the eigenvalues of both G and H we observe that the eigenvalues are different. Hence the graphs G and$H$ are non-cospectral equi-energetic graphs. ∎

Remark 2.4.

1. If $T(G)$ is the total graph of G, then for all $n \geq 4$, $K_2 \Box T(K_n)$ and $K_2 \times T(K_n)$ are equi-energetic.

2. If $K_{n,n}$ is the complete bipartite graph on $2n$ vertices, and if $L(K_{n,n})$ is the line graph of the complete bipartite graph then, $K_2 \Box L(K_{n,n})$ and $K_2 \times L(K_{n,n})$ are equi-energetic.

3. If K_n is the complete graph on n vertices and $K_{n,n}$ is the complete bipartite graph on n vertices then $\mathscr{E}(K_n) = \mathscr{E}(K_{n-1,n-1})$ for all $n \geq 3$.

4. $\mathscr{E}(K_n \boxtimes K_2) = \mathscr{E}(K_n \Box K_2) + \mathscr{E}(K_2)$.

Observation 2.5. There are several non-cospectral equi-energetic graphs including the following.

1. The complete graph K_4 and the sub graph of the complete graph K_5, obtained by deleting only the edges of K_5 which makes a cycle C_3 are equi-energetic.

2. The complete graph K_5 and the sub graph of the complete graph K_6, obtained by deleting only the edges of K_6 which makes a cycle C_6 are equi-energetic.

3. The complete graph K_6 and the sub graph of the complete graph K_8, obtained by deleting only the edges of K_8 which makes a a complete graph K_5 are equi-energetic.

4. The complete graph K_7 and the sub graph of the complete graph K_7, obtained by deleting only the edges of K_7 which makes a cycle C_4 are equi-energetic. Also the complete graph K_9 and the sub graph of the complete graph K_9, obtained by deleting only the edges of K_9 which makes a path P_3 are equi-energetic.

3. Conclusion

Most of the results we have obtained in this paper is based on the study of the exact values of the energy of the graphs under consideration. Further studies are to be done and appropriate proof techniques are to be employed to generalize these results. However the results obtained can act as a pointer for articulating and proving the general results.

References

[1] D. Cvetkovic, M. Petric, “Connected graphs on six vertices”, *Discrete Math.*, vol. 50, pp. 37–49, 1984.

[2] D. Stevanovic, I. Gutman, V. Brankov, “Equienergetic chemical trees”, *J. Serb. Chem. Soc.*, vol. 69, pp. 549–553, 2004.

[3] D. Stevanovic, “Energy and NEPS of graphs”, *Lin. Multilin. Algebra*, vol. 53, pp. 67–34, 2005.

[4] G. Indulal, A. Vijaykumar, “On a pair of equienergetic graphs”, *MATCH Commun. Math. Comput. Chem.*, vol. 55, pp. 91–94, 2006.

[5] H. S. Ramane, H. B. Walikar, I. Gutman, “Equienergetic graphs”, *Kragujevac J. Math.*, vol. 26, pp. 5–13, 2004.

[6] H. S. Ramane, H. B. Walikar, I. Gutman, “Another class of equienergetic graphs”, *Kragujevac J. Math.*, vol. 26, pp. 15–18, 2004.

[7] H. S. Ramane, H. B. Walikar, I. Gutman, S. B. Halkarni, “Equienergetic complement graphs”, *Kragujevac J. Math.*, vol. 27, pp. 67–74, 2005.

[8] I. Gutman, “The energy of a graph”, *Ber. Math. Stat. Sekt.*, pp. 1–22, 1978.

[9] I. Gutman, O. E. Polansky, *Mathematical Concepts in Organic Chemistry*, Springer-Verlag, Berlin, 1986.

[10] R. Balakrishnan, “The energy of a graph”, *Linear Algebra Appl.*, vol. 387, pp. 287–295, 2004.