International Journal of Computational and Applied Mathematics.
ISSN 1819-4966 Volume 12, Number 2 (2017), pp. 603–613

http://www.ripublication.com/ijcam.htm

Certain Combinatorial Results on Two variable Hybrid Fibonacci Polynomials

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Abstract

Recently many researchers in combinatorics have contributed to the development of one variable polynomials related to hypergeometric family. Here a class of two variable hybrid Fibonacci polynomials is defined and proved certain combinatorial results.

AMS subject classification: 05A19, 11B39 and 33C45.
Keywords: Combinatorial Identities, Fibonacci Numbers and Hyper Geometric Functions in one or more Variables.

1. Introduction

The natural and the most beautiful three term recurrence relation is $F_{n+1} = F_n + F_{n-1}$, $F_1 = 1$, $F_2 = 1$, where F_n is the nth Fibonacci number [1, 2, 3, 13]. There are many one variable generalizations of F_n related to hypergeometric family [3, 4]. Two important such generalizations are Catalan polynomials and Jacobsthal polynomials denoted by $f_n^{(C)}(x)$ and $f_n^{(J)}(x)$ which exhibits many interesting combinatorial properties [4, 6, 7, 8, 9, 10, 12].

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Two hypergeometric polynomials, namely, $U_n(x)$ and $V_n(x)$ called Tchebyshev polynomials of second and third kind are given by

$$\begin{aligned} U_n(x) &= \frac{1}{2\sqrt{x^2-1}}\left[\left(x+\sqrt{x^2-1}\right)^{n+1}-\left(x-\sqrt{x^2-1}\right)^{n+1}\right] \\ &= (n+1)\,{}_2F_1\left(-n,\ n+2;\ \frac{3}{2};\ \frac{1-x}{2}\right) \end{aligned}$$

and

$$\begin{aligned} V_n(x) &= U_{n+1}(x)-U_n(x) \\ &= {}_2F_1\left(-n,\ n+1;\ \frac{1}{2};\ \frac{1-x}{2}\right) \end{aligned}$$

which have many applications in both pure and applied mathematics [5, 6, 7, 8, 9, 10, 11, 12]. It is interesting to note that $U_n\left(\frac{3}{2}\right) = F_{2n+2}$ and $V_n\left(\frac{3}{2}\right) = F_{2n+1}$ [12].

In the present paper, a two variable hybrid Fibonacci polynomials are defined which naturally generalizes Fibonacci-Catalan polynomials $f_n^{(C)}(x)$ [3] and Fibonacci-Jacobsthal polynomials $f_n^{(J)}(y)$ [3] by a nontrivial hybridization. Further the defined hybrid Fibonacci polynomials are shown to be directly connected to Tchebyshev polynomials of second and third kinds. Certain combinatorial results such as generating function, matrix identities, determinant formula and the direct formula using Pascal like table are proved for the hybrid Fibonacci polynomials.

2. Two Variable Hybrid Fibonacci Polynomials

Definition 2.1. The generalized hybrid Fibonacci polynomials in two variables x and y of degree n, denoted by $f_n^{(H)}(x,y)$ is

$$f_n^{(H)}(x,y) = \frac{1}{\sqrt{x^2+4y}}\left[\left(\frac{x+\sqrt{x^2+4y}}{2}\right)^n-\left(\frac{x-\sqrt{x^2+4y}}{2}\right)^n\right] \quad (2.1)$$

When $y=1$ in (2.1), the hybrid Fibonacci polynomials become Fibonacci-Catalan polynomials in terms of x, that is $f_n^{(H)}(x,1) = f_n^{(C)}(x)$. When $x=1$ in (2.1), they become Fibonacci-Jacobsthal polynomials in terms of y, that is $f_n^{(H)}(1,y) = f_n^{(J)}(y)$. When $x=1$ and $y=1$ in (2.1), they become Fibonacci numbers.

Three Term Recurrence Relations

By direct verification using the definition, one can show that the following recurrence relation is satisfied by the generalized hybrid fibonacci polynomials in two variables:

$$\begin{aligned} f_{n+1}^{(H)}(x,y) &= xf_n^{(H)}(x,y)+yf_{n-1}^{(H)}(x,y), \quad (2.2) \\ f_1^{(H)}(x,y) &= 1,\ \ f_2^{(H)}(x,y)=x,\ f_3^{(H)}(x,y)=x^2+y,\ \ n=1,2,3,\dots\ . \end{aligned}$$

Illustration

The following table with first five initial polynomials of hybrid Fibonacci polynomials in two variables, Fibonacci-Catalan polynomial and Fibonacci-Jacobsthal polynomials illustrates the nontrivial hybridization.

n	$f_n^{(H)}(x, y)$	$f_n^{(C)}(x, 1)$	$f_n^{(J)}(1, y)$
1	1	1	1
2	x	x	1
3	$x^2 + y$	$x^2 + 1$	$1 + y$
4	$x^3 + 2xy$	$x^3 + 2x$	$1 + 2y$
5	$x^4 + 3x^2y + y^2$	$x^4 + 3x^2 + 1$	$1 + 3y + y^2$

Theorem 2.2. The hybrid polynomial can be connected to Tchebychev polynomials of second kind $U_n(x)$ and third kind $V_n(x)$ as follows:

1. $f_{2n+2}^{(H)}(x, y) = xy^n U_n\Big(1 + \dfrac{x^2}{2y}\Big)$

2. $f_{2n+1}^{(H)}(x, y) = y^n V_n\Big(1 + \dfrac{x^2}{2y}\Big).$

Proof.

(1) Consider

$$\begin{aligned} f_{2n+2}^{(H)}(x, y) &= \frac{1}{\sqrt{x^2+4y}}\left[\Big(\frac{x+\sqrt{x^2+4y}}{2}\Big)^{2n+2} - \Big(\frac{x-\sqrt{x^2+4y}}{2}\Big)^{2n+2}\right]. \\ &= \frac{x}{2y\sqrt{\Big(1+\frac{x^2}{2y}\Big)^2 - 1}}\left[\left[\Big(1+\frac{x^2}{2y}\Big)y + y\sqrt{\Big(1+\frac{x^2}{2y}\Big)^2 - 1}\right]^{n+1}\right. \\ &\quad \left. - \left[\Big(1+\frac{x^2}{2y}\Big)y - y\sqrt{\Big(1+\frac{x^2}{2y}\Big)^2 - 1}\right]^{n+1}\right]. \end{aligned}$$

$$= \frac{x.y^{n+1}}{2y\sqrt{\left(1+\frac{x^2}{2y}\right)^2-1}}\left[\left[\left(1+\frac{x^2}{2y}\right)+\sqrt{\left(1+\frac{x^2}{2y}\right)^2-1}\right]^{n+1} - \left[\left(1+\frac{x^2}{2y}\right)-\sqrt{\left(1+\frac{x^2}{2y}\right)^2-1}\right]^{n+1}\right]$$

$$= xy^n U_n\left(1+\frac{x^2}{2y}\right).$$

(2) Consider

$$\begin{aligned}
f^{(H)}_{2n+1}(x,y) &= \frac{1}{\sqrt{x^2+4y}}\left[\left(\frac{x+\sqrt{x^2+4y}}{2}\right)^{2n+1} - \left(\frac{x+\sqrt{x^2+4y}}{2}\right)^{2n+1}\right] \\
&= \frac{1}{2\sqrt{x^2+4y}}\left[\left(\frac{x+\sqrt{x^2+4y}}{2}\right)^{2n} - \left(\frac{x+\sqrt{x^2+4y}}{2}\right)^{2n}\right] \\
&\quad + \frac{1}{2}\left[\left(\frac{x+\sqrt{x^2+4y}}{2}\right)^{2n} - \left(\frac{x+\sqrt{x^2+4y}}{2}\right)^{2n}\right] \\
&= \frac{x^2.y^n}{4y\sqrt{\left(1+\frac{x^2}{2y}\right)^2-1}}\left[\left[\left(1+\frac{x^2}{2y}\right)+\sqrt{\left(1+\frac{x^2}{2y}\right)^2-1}\right]^{n} - \left[\left(1+\frac{x^2}{2y}\right)-\sqrt{\left(1+\frac{x^2}{2y}\right)^2-1}\right]^{n}\right] \\
&+ \frac{y^n}{2}\left[\left[\left(1+\frac{x^2}{2y}\right)+\sqrt{\left(1+\frac{x^2}{2y}\right)^2-1}\right]^{n} - \left[\left(1+\frac{x^2}{2y}\right)-\sqrt{\left(1+\frac{x^2}{2y}\right)^2-1}\right]^{n}\right] \\
&= \frac{x^2}{2y}y^n U_{n-1}\left(1+\frac{x^2}{2y}\right) + y^n T_n\left(1+\frac{x^2}{2y}\right).
\end{aligned}$$

But $T_n(x)$ satisfies

$$T_n\left(1+\frac{x^2}{2y}\right) = \left(1+\frac{x^2}{2y}\right)U_{n-1}\left(1+\frac{x^2}{2y}\right) - U_{n-2}\left(1+\frac{x^2}{2y}\right).$$

$$\begin{aligned}
f_{2n+1}^{(H)}(x,y) &= y^n\Big[\Big[2\Big(1+\frac{x^2}{2y}\Big)U_{n-1}\Big(1+\frac{x^2}{2y}\Big) - U_{n-2}\Big(1+\frac{x^2}{2y}\Big)\Big] - U_{n-1}\Big(1+\frac{x^2}{2y}\Big)\Big]\\
&= y^n\Big[U_n\Big(1+\frac{x^2}{2y}\Big) - U_{n-1}\Big(1+\frac{x^2}{2y}\Big)\Big]\\
&= y^n V_n\Big(1+\frac{x^2}{2y}\Big).
\end{aligned}$$

■

3. Combinatorial Properties of Hybrid Polynomials

In this section, the combinatorial properties such as generating function, matrix identities and determinant formula and are direct formula using Pascal like table for the two variable hybrid Fibonacci polynomials are stated with proof.

Theorem 3.1. The generating function for generalized hybrid polynomials in two variable is

$$\sum_{n=0}^{\infty} f_n^{(H)}(x,y)t^n = \frac{t}{1-tx-yt^2}$$

Proof. Keeping in the mind the three term recurrence relation for $f_n^{(H)}(x,y)$. We proceed with the derivation. Put $f(x,y,t) = \sum_{n=0}^{\infty} f_n^{(H)}(x,y)t^n$. We write

$$\begin{aligned}
f(x,y,t) &= f_0^{(H)}(x,y) + f_1^{(H)}(x,y)t + \cdots + f_{n+1}^{(H)}(x,y)t^{n+1} + \cdots\\
-xtf(x,y,t) &= -xf_0^H(x,y)t - xf_1^H(x,y)t^2 - \cdots - xf_n^H(x,y)t^{n+1} - \cdots\\
-yt^2f(x,y,t) &= -f_0^{(H)}(x,y)yt^2 - f_1^{(H)}(x,y)yt^3 - \cdots + f_{n-1}^{(H)}(x,y)yt^{n+1} - \cdots .
\end{aligned}$$

Summing all the three expressions on both sides, we get

$$\begin{aligned}
(1-tx+yt^2)f(x,y,t) &= 0 + 1t - tx.0\\
f(x,y,t) &= \frac{t}{1-tx-yt^2}.
\end{aligned}$$

■

Theorem 3.2. The generalized polynomials in two variable can be expressed in matrix form in odd and even functions are as follows:

(1) $$\begin{bmatrix} f_{2n+2}^{(H)}(x,y) & y f_{2n}^{(H)}(x,y) \\ -y f_{2n}^{(H)}(x,y) & -y^2 f_{2n-2}^{(H)}(x,y) \end{bmatrix} = x \begin{bmatrix} x^2+2y & y \\ -y & 0 \end{bmatrix}^n .$$

(2) $$\begin{bmatrix} f_{2n+3}^{(H)}(x,y) & y f_{2n+1}^{(H)}(x,y) \\ f_{2n+1}^{(H)}(x,y) & y f_{2n-1}^{(H)}(x,y) \end{bmatrix} = \begin{bmatrix} x^2+y & y \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x^2+2y & y \\ -y & 0 \end{bmatrix}^n .$$

Proof.

(1) The theorem is proved by using the principle of mathematical induction: For $n = 1$, the result is obvious.
For $n = 2$,

$$\begin{bmatrix} f_{6}^{(H)}(x,y) & y f_{4}^{(H)}(x,y) \\ -y f_{4}^{(H)}(x,y) & -y^2 f_{2}^{(H)}(x,y) \end{bmatrix} = x \begin{bmatrix} x^2+2y & y \\ -y & 0 \end{bmatrix}^2 .$$

$$\begin{bmatrix} f_{6}^{(H)}(x,y) & y f_{4}^{(H)}(x,y) \\ -y f_{4}^{(H)}(x,y) & -y^2 f_{2}^{(H)}(x,y) \end{bmatrix} = \begin{bmatrix} x^5+4x^3y+3xy^2 & x^3y+2xy^2 \\ -(x^3y+2xy^2) & -xy^2 \end{bmatrix}$$

is true. We assume the result is true for $n = k$,

$$\begin{bmatrix} f_{2k+2}^{(H)}(x,y) & y f_{2k}^{(H)}(x,y) \\ -y f_{2k}^{(H)}(x,y) & -y^2 f_{2k-2}^{(H)}(x,y) \end{bmatrix} = x \begin{bmatrix} x^2+2y & y \\ -y & 0 \end{bmatrix}^k .$$

Now we prove the result is true for $n = k+1$. Consider

$$\begin{bmatrix} f_{2k+2}^{(H)}(x,y) & y f_{2k}^{(H)}(x,y) \\ -y f_{2k}^{(H)}(x,y) & -y^2 f_{2k-2}^{(H)}(x,y) \end{bmatrix} \begin{bmatrix} x^2+2y & y \\ -y & 0 \end{bmatrix}$$
$$= x \begin{bmatrix} x^2+2y & y \\ -y & 0 \end{bmatrix}^k \begin{bmatrix} x^2+2y & y \\ -y & 0 \end{bmatrix}$$

On simplification by using the three term recurrence relation for hybrid polynomials. We get

$$\begin{bmatrix} f_{2k+4}^{(H)}(x,y) & y f_{2k+2}^{(H)}(x,y) \\ -y f_{2k+2}^{(H)}(x,y) & -y^2 f_{2k}^{(H)}(x,y) \end{bmatrix} = x \begin{bmatrix} x^2+2y & y \\ -y & 0 \end{bmatrix}^{k+1} .$$

The proof of matrix identity 2 is similar to that of the identity 1. ■

For $y = 1$, in the Theorem 3.1, we obtain matrix identities for Fibonacci–Catalan polynomials in the following form:

$$(3)\quad \begin{bmatrix} f^{(C)}_{2n+2}(x) & y f^{(C)}_{2n}(x) \\ -f^{(C)}_{2n}(x) & f^{(C)}_{2n-2}(x) \end{bmatrix} = x \begin{bmatrix} x^2+2 & 1 \\ -1 & 0 \end{bmatrix}^n .$$

$$(4)\quad \begin{bmatrix} f^{(C)}_{2n+3}(x) & f^{(C)}_{2n+1}(x) \\ f^{(C)}_{2n+1}(x) & y f^{(C)}_{2n-1}(x) \end{bmatrix} = \begin{bmatrix} x^2+1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x^2+2 & 1 \\ -1 & 0 \end{bmatrix}^n .$$

For $x = 1$ in the Theorem 3.1, we obtain matrix identities for Fibonacci-Jacobsthal polynomials in the following form:

$$(5)\quad \begin{bmatrix} f^{(J)}_{2n+2}(y) & y f^{(J)}_{2n}(y) \\ -y f^{(J)}_{2n}(y) & -y^2 f^{(J)}_{2n-2}(y) \end{bmatrix} = \begin{bmatrix} 1+2y & y \\ -y & 0 \end{bmatrix}^n .$$

$$(6)\quad \begin{bmatrix} f^{(J)}_{2n+3}(y) & y f^{(J)}_{2n+1}(y) \\ -y f^{(J)}_{2n+1}(y) & y f^{(J)}_{2n-1}(y) \end{bmatrix} = \begin{bmatrix} 1+y & y \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1+2y & y \\ -y & 0 \end{bmatrix}^n .$$

By applying determinants on both sides of matrix identities we obtain the following Corollary.

Corollary 3.3. We have

1. $[f^{(H)}_{2n}(x, y)]^2 - [f^{(H)}_{2n+2}(x, y)][f^{(H)}_{2n-2}(x, y)] = xy^{2n-2}$
2. $[f^{(H)}_{2n+3}(x, y)][f^{(H)}_{2n-1}(x, y)] - [f^{(H)}_{2n+1}(x, y)]^2 = x^2 y^{2n-1}$.

Determinants Formulas

We state the following theorem for generalized hybrid polynomials of odd even polynomials in two variable without proof because they follow directly from their three term recurrence relations.

Theorem 3.4. The determinants formulas for generalized hybrid polynomials for odd and even polynomial matrix are

$$f^{(H)}_{2n+2}(x, y) = x \begin{vmatrix} x^2+2y & -y & 0 & \cdots & 0 & 0 \\ -y & x^2+2y & -y & 0 & \cdots & 0 \\ 0 & -y & x^2+2y & \ddots & -y & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & -y & x^2+2y \end{vmatrix}_{nXn}$$

$$f_{2n+1}^{(H)}(x,y) = \begin{vmatrix} x^2+2y & -y & 0 & \cdots & 0 & 0 \\ -y & x^2+2y & -y & 0 & \cdots & 0 \\ 0 & -y & x^2+2y & \ddots & -y & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & -y & x^2+2y \end{vmatrix}_{nXn} - y \begin{vmatrix} x^2+2y & -y & 0 & \cdots & 0 & 0 \\ -y & x^2+2y & -y & 0 & \cdots & 0 \\ 0 & -y & x^2+2y & \ddots & -y & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & -y & x^2+2y \end{vmatrix}_{(n-1)X(n-1)}$$

Special cases
(1) For $y = 1$, we deduce

$$f_{2n+2}^{(C)}(x) = x \begin{vmatrix} x^2+2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & x^2+2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & x^2+2 & \ddots & -1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & -1 & x^2+2 \end{vmatrix}_{nXn}$$

$$f_{2n+1}^{(C)}(x) = \begin{vmatrix} x^2+2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & x^2+2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & x^2+2 & \ddots & -1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & -1 & x^2+2 \end{vmatrix}_{nXn} - \begin{vmatrix} x^2+2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & x^2+2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & x^2+2 & \ddots & -1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & -1 & x^2+2 \end{vmatrix}_{(n-1)X(n-1)}.$$

(2) For $x = 1$, we deduce

$$f_{2n+2}^{(J)}(y) = \begin{vmatrix} 1+2y & -y & 0 & \cdots & 0 & 0 \\ -y & 1+2y & -y & 0 & \cdots & 0 \\ 0 & -y & 1+2y & \ddots & -y & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & -y & 1+2y \end{vmatrix}_{nXn}$$

$$f_{2n+1}^{(J)}(y) = \begin{vmatrix} 1+2y & -y & 0 & \cdots & 0 & 0 \\ -y & 1+2y & -y & 0 & \cdots & 0 \\ 0 & -y & 1+2y & \ddots & -y & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & -y & 1+2y \end{vmatrix}_{nXn} - y \begin{vmatrix} 1+2y & -y & 0 & \cdots & 0 & 0 \\ -y & 1+2y & -y & 0 & \cdots & 0 \\ 0 & -y & 1+2y & \ddots & -y & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & -y & 1+2y \end{vmatrix}_{n-1Xn-1}.$$

(3) For $x = 1$ and $y = 1$, we deduce

$$F_{2n+2} = \begin{vmatrix} 3 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 3 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 3 & \ddots & -1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & -1 & 3 \end{vmatrix}_{nXn}$$

$$F_{2n+1} = \begin{vmatrix} 3 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 3 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 3 & \ddots & -1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & -1 & 3 \end{vmatrix}_{nXn} - \begin{vmatrix} 3 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 3 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 3 & \ddots & -1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & -1 & 3 \end{vmatrix}_{(n-1)X(n-1)}.$$

Theorem 3.5. The explicit formula for hybrid polynomial in two variable is

$$f_n^{(H)}(x, y) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-k}{k} x^{n-1-2k} y^k. \tag{3.3}$$

Proof. The result is proved by using mathematical induction method.
For $n = 1$,

$$f_1^{(H)}(x, y) = \binom{0-0}{0} x^{0-0} y^0 = 1.$$

For $n = 2$,

$$f_2^{(H)}(x, y) = \binom{1-0}{0} x^{1-0} y^0 = x.$$

We assume the result is true for $n = m$. i.e.,

$$f_m^{(H)}(x, y) \;=\; \sum_{k=0}^{\lfloor\frac{m-1}{2}\rfloor} \binom{m-1-k}{k} x^{m-1-2k} y^k.$$

We prove the result is true for $n = m + 1$.

By applying the three term recurrence relation (2.2) for the hybrid polynomials

$$\begin{aligned}
f_{m+1}^{(H)}(x, y) &= x f_m^{(H)}(x, y) + y f_{m-1}^{(H)}(x, y) \\
&= x \sum_{k=0}^{\lfloor\frac{m-1}{2}\rfloor} \binom{m-1-k}{k} x^{m-1-2k} y^k + y \sum_{k=0}^{\lfloor\frac{m-2}{2}\rfloor} \binom{m-2-k}{k} x^{m-2-2k} y^k \\
&= x\Big[\binom{m-1}{0} x^{m-1} y^0 + \binom{m-2}{1} x^{m-3} y^1 + \binom{m-3}{2} x^{m-5} y^2\Big] \\
&+ y\Big[\binom{m-2}{0} x^{m-2} y^0 + \binom{m-3}{1} x^{m-4} y^1 + \binom{m-4}{2} x^{m-6} y^2\Big] \\
&= \binom{m-1}{0} x^m y^0 + \binom{m-1}{1} x^{m-2} y + \binom{m-2}{2} x^{m-4} y^2 + \cdots \\
&= \sum_{k=0}^{\lfloor\frac{m}{2}\rfloor} \binom{m-k}{k} x^{m-2k} y^k.
\end{aligned}$$

When $y = 1$ in (3.3), it becomes the explicit formula for the Fibonacci-Catalan polynomial in terms of x.

$$f_n^{(H)}(x, y) = f_n^{(C)}(x) = \sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor} \binom{n-1-k}{k} x^{n-1-2k}.$$

When $x = 1$ in (3.3), it becomes the explicit formula for the Fibonacci-Jacobsthal polynomial in terms of y.

$$f_n^{(H)}(x, y) = f_n^{(J)}(y) = \sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor} \binom{n-1-k}{k} y^k.$$

When $x = 1$ and $y = 1$ in (3.3), the hybrid polynomials becomes the explicit formula for Fibonacci numbers

$$F_n = \sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor} \binom{n-1-k}{k}.$$

■

Acknowledgement

The second author would like to thank both UGC-SWRO, F. No. FIP/12th Plan/KADA018 TF02 and Govt. of Karnataka (DCE) and the third author would like to thank UGC, Govt. of India for encouraging this work under Post Doctoral Fellowship For SC/ST Candidates Order No. F./PDFSS - 2014 -15-ST-KAR-10116.

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