

A Note on Frames in Banach Spaces

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Abstract

A method of constructing a new Schauder frame from a given Schauder frame is given. Also, we obtain conditions for a retro Banach frame to be a basis. Finally, we consider the block perturbation of Schauder frames and prove that it is again a Schauder frame.

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1. Introduction

Frames are generalizations of orthonormal bases in Hilbert spaces. The main property of frames which makes them useful is their redundancy. Representation of signals using frames is advantageous over basis expansions in a variety of practical applications. Many properties of frames make them useful in various applications in mathematics, science

and engineering. In particular, frames are widely used in sampling theory, wavelet theory, wireless communication, signal processing, image processing, differential equations, filter banks, geophysics, quantum computing, wireless sensor network, multiple-antenna code design and many more. The reason for such wide applications is that frames provide both great liberties in design of vector space decompositions, as well as quantitative measure on computability and robustness of the corresponding reconstructions. For a nice and comprehensive survey on various types of frames, one may refer to [1, 3] and the references therein.

The notion of frames has been extended to Banach spaces by Feichtinger and Grochenig [4]. They introduced the notion of atomic decomposition for Banach spaces. A more general concept called Banach frame for Banach spaces was introduced by Grochenig. Banach frames have been used in localization of frames which has decay properties. Also, Banach frames are used for defining cones associated with Banach frames. Casazza, Han and Larson [2] carried out a study on atomic decompositions and Banach frames. Schauder frames for Banach spaces were introduced by Han and Larson [6] and were further studied in [8, 9, 10, 11].

In this paper, A method of constructing a new Schauder frame from a given Schauder frame is given. Also, we obtain conditions for a retro Banach frame to be a basis. Finally, we consider the block perturbation of Schauder frames and prove that it is again a Schauder frame.

2. Preliminaries

Throughout this paper \mathcal{X} will denote an infinite dimensional Banach space over the scalar field $\mathbb{K}(\mathbb{R}, \mathbb{C})$, \mathcal{X}^* denotes the conjugate space of \mathcal{X} and $L(\mathcal{X}, \mathcal{X})$ denote the Banach space of all continuous linear mappings of \mathcal{X} into \mathcal{X} . For a sequence $\{x_n\} \in \mathcal{X}$ and $\{f_n\} \in \mathcal{X}^*$, $[x_n]$ denotes the closure of linear span of $\{x_n\}$ in the norm topology of \mathcal{X} and $[\tilde{f}_n]$ the closure of $\{f_n\}$. A sequence space S is called a *BK-space* if it is a Banach space and the co-ordinate functionals are continuous on S . That is the relations $x_n = \{\alpha_j^{(n)}\}, x = \{\alpha_j\} \in S, \lim_{n \rightarrow \infty} x_n = x$ imply $\lim_{n \rightarrow \infty} \alpha_j^{(n)} = \alpha_j$ ($j = 1, 2, 3, \dots$).

The concept of Atomic Decomposition was introduced and studied by Feichtinger and Grochenig [4] in 1988. They gave the following definition:

Definition 2.1. [4] Let \mathcal{X} be a Banach space and let \mathcal{X}_d be an associated Banach space of scalar valued sequences indexed by \mathbb{N} . Let $\{y_i\}$ be a sequence of elements from \mathcal{X}^* and $\{x_i\}$ a sequence of elements of \mathcal{X} . If:

1. $(\langle x, y_i \rangle) \in \mathcal{X}_d$, for each $x \in \mathcal{X}$,
2. The norms $\|x\|_{\mathcal{X}}$, and $\|(\langle x, y_i \rangle)\|_{\mathcal{X}_d}$ are equivalent,

3. $x = \sum_{i=1}^{\infty} (\langle x, y_i \rangle x_i)$, for all $x \in \mathcal{X}$,

then $(\{y_i\}, \{x_i\})$ is an **atomic decomposition** of \mathcal{X} with respect to \mathcal{X}_d . If the norm equivalence is given by $A\|x\| \leq \|(x, \{y_i\})\|_{\mathcal{X}_d} \leq B\|x\|$, then A and B are choice of atomic bounds for $(\{x_i\}, \{y_i\})$.

Later Grochenig [5] in 1991 introduced a more general concept for Banach spaces called Banach frame. He gave the following definition:

Definition 2.2. [5] Let \mathcal{X} be a Banach space and \mathcal{X}_d be an associated Banach space of Scalar-valued sequences, indexed by \mathbb{N} . Let $\{f_n\} \subset \mathcal{X}^*$ and $S : \mathcal{X}_d \rightarrow \mathcal{X}$ be given. Then $\Phi = (\{f_n\}, S)$ is called a **Banach frame** for \mathcal{X} with respect to \mathcal{X}_d if

1. $\{f_n(x)\} \in \mathcal{X}_d$ for each $x \in \mathcal{X}$,
2. there exist positive constants A and B with $0 < A \leq B < \infty$ such that

$$A\|x\|_{\mathcal{X}} \leq \|\{f_n(x)\}\|_{\mathcal{X}_d} \leq B\|x\|_{\mathcal{X}}, \quad x \in \mathcal{X} \tag{1}$$

3. S is a bounded linear operator such that

$$S(\{f_n(x)\}) = x, \quad x \in \mathcal{X}.$$

The positive constants A and B , respectively, are called lower and upper frame bounds of the Banach frame $\Phi = (\{f_n\}, S)$. The operator $S : \mathcal{X}_d \rightarrow \mathcal{X}$ is called the reconstruction operator (or, the pre-frame operator). The inequality (1) is called frame inequality.

In 2004 Jain, Kaushik and Vashist [7] define the notion of retro Banach frames in conjugate Banach spaces.

Definition 2.3. [7] Let \mathcal{X} be a Banach space and \mathcal{X}_d be a BK-space. Let $\{x_n\} \subset \mathcal{X}$ and $J : \mathcal{X}_d^* \rightarrow \mathcal{X}^*$ be given. The pair $(\{x_n\}, J)$ is called a **retro Banach frame** for \mathcal{X}^* with respect to \mathcal{X}_d^* if

1. $\{f(x_n)\} \in \mathcal{X}_d^*$, for all $f \in \mathcal{X}^*$.
2. There exist positive constants A and B with $0 < A \leq B < \infty$ such that

$$A\|f\|_{\mathcal{X}^*} \leq \|\{f(x_n)\}\|_{\mathcal{X}_d^*} \leq B\|f\|_{\mathcal{X}^*}, \text{ for all } f \in \mathcal{X}^*. \tag{2}$$

3. $J(\{f(x_n)\}) = f$, for all $f \in \mathcal{X}^*$,

where A and B are lower and upper bounds of retro Banach frames $(\{x_n\}, J)$. The operator $J : \mathcal{X}_d^* \rightarrow \mathcal{X}^*$ is called the reconstruction operator (or the preframe operator). The inequality (2) is called retro Banach frame inequality. A retro Banach frames $(\{x_n\}, J)$ is said to be exact if on removal of one x_j renders the collection $\{x_n\}_{n \neq j}$ no longer a retro Banach frame for \mathcal{X}^* .

One may note that if $(\{x_n\}, J)$ is an exact retro Banach frame, then there exists a sequence Φ_n in X^* such that $\Phi_i(x_j) = \delta_{ij}$, for all $i, j \in \mathbb{N}$.

Han and Larson [6], in 1997, introduced the concept of Schauder frame for a Banach Space. They defined a **Schauder frame** for a Banach space \mathcal{X} to be an inner direct summand (i.e. a compression) of a Schauder basis for \mathcal{X} . Similarly, an **unconditional Schauder** frame is a compression of an unconditional Schauder basis for \mathcal{X} .

Definition 2.4. Let \mathcal{X} be a Banach space. A sequence $(\{x_n\}, \{f_n\})$ ($\{x_n\} \subset \mathcal{X}, \{f_n\} \subset \mathcal{X}^*$) is called a **Schauder frame** for \mathcal{X} if

$$x = \sum_{n=1}^{\infty} f_n(x)x_n, \quad x \in \mathcal{X}. \quad (3)$$

A Schauder frame $(\{x_n\}, \{f_n\})$ is called unconditional if the series $x = \sum_{n=1}^{\infty} f_n(x)x_n$ converges unconditionally for all $x \in \mathcal{X}$.

Definition 2.5. Let \mathcal{X} be a Banach space. A sequence $(\{x_n\}, \{f_n\})$ ($\{x_n\} \subset \mathcal{X}, \{f_n\} \subset \mathcal{X}^*$) is called a **Schauder frame sequence** for \mathcal{X} if $(\{x_n\}, \{f_n\})$ is a Schauder frame for $[x_n]$.

3. Main Results

We begin this section with the following result and gives a method of constructing a new Schauder frame from a given Schauder frame.

Theorem 3.1. Let $(\{x_n\}, \{f_n\})$ ($\{x_n\} \subset \mathcal{X}, \{f_n\} \subset \mathcal{X}^*$) be a Schauder frame for \mathcal{X} with $\inf_{1 \leq n < \infty} \|x_n\| > 0$. Let $\{\alpha_n\}$ be a sequence of scalars such that $\alpha_n \neq 0$, for all $n \in \mathbb{N}$. Let $\{y_n\}$ be a sequence defined by

$$y_n = \sum_{i=1}^n \alpha_i x_i, \quad n \in \mathbb{N}.$$

Then there exists a sequence $\{g_n\} \subset \mathcal{X}^*$ such that (y_n, g_n) is a Schauder frame for \mathcal{X} provided that the sequence $\left\{ \frac{\|y_n\|}{|\alpha_{n+1}|} \right\}$ is bounded.

Proof. Define $\{g_n\} \subset \mathcal{X}^*$ by

$$g_n = \frac{f_n}{\alpha_n} - \frac{f_{n+1}}{\alpha_{n+1}}, \quad n \in \mathbb{N}.$$

Then

$$\begin{aligned} \sum_{i=1}^n g_i(x)y_i &= \sum_{i=1}^n \left(\frac{f_n(x)}{\alpha_n} - \frac{f_{n+1}(x)}{\alpha_{n+1}} \right) \left(\sum_{j=1}^i \alpha_j x_j \right) \\ &= \sum_{i=1}^n f_i(x)x_i - \frac{f_{n+1}(x)}{\alpha_{n+1}}y_n, \quad x \in \mathcal{X} \quad n \in \mathbb{N}. \end{aligned} \tag{4}$$

Since (x_n, f_n) is a Schauder frame,

$$x = \sum_{n=1}^{\infty} f_n(x)x_n.$$

Therefore

$$|f_n(x)| \leq \frac{1}{\inf_{1 \leq j < \infty} \|x_j\|} \|f_n(x)x_n\| \longrightarrow 0 \text{ as } n \rightarrow \infty. \tag{5}$$

Since $\left\{ \frac{\|y_n\|}{|\alpha_{n+1}|} \right\}$ is bounded, it follows from (4) that

$$\lim_n \sum_{i=n}^{\infty} g_i(x)y_i = x.$$

Hence $(\{y_n\}, \{g_n\})$ is a Schauder frame for \mathcal{X} . ■

Note that a basis for \mathcal{X} is always a retro Banach frame for \mathcal{X} but a retro Banach frame for \mathcal{X} need not be a basis of \mathcal{X} . One may easily verify that even an exact retro Banach frame for \mathcal{X} need not be a basis of \mathcal{X} .

In the following result we obtain conditions under which an exact retro Banach frame for \mathcal{X} is a basis of \mathcal{X} .

Theorem 3.2. Let $(\{x_n\}, T)$ be an exact retro Banach frame for \mathcal{X} with admissible sequence $\{\Phi_n\} \in \mathcal{X}^*$. If there exists a constant $C \geq 1$ such that

$$\left\| \sum_{i=1}^n \Phi_i(x)x_i \right\| \leq C\|x\|, \quad x \in \mathcal{X}, \quad n \in \mathbb{N}, \tag{6}$$

then $\{x_n\}$ is a basis of \mathcal{X} with associated sequence of functionals $\{\Phi_n\} \in \mathcal{X}^*$.

Proof. Consider $z = \sum_{i=1}^m a_i x_i$. Then for all $n \geq m$, we have

$$\begin{aligned} \sum_{i=1}^n \Phi_i(z)x_i &= \sum_{i=1}^n \Phi_i\left(\sum_{j=1}^m a_j x_j\right)x_i \\ &= \sum_{i=1}^n \sum_{j=1}^m a_j \delta_{ij} x_i \\ &= \sum_{j=1}^m a_j x_j \\ &= z. \end{aligned}$$

For each $n \in \mathbb{N}$, let

$$u_n(z) = \sum_{i=1}^n \Phi_i(z)x_i, \quad z \in \text{span}[x_1, \dots, x_n].$$

Then $\{u_n\}$ is a sequence of continuous linear operators on \mathcal{X} such that by hypothesis

$$\|u_n(z)\| \leq C\|z\|, \quad z \in \text{span}[x_1, \dots, x_n].$$

Also $\lim_{n \rightarrow \infty} u_n(z) = z$, for all z in a dense subset of \mathcal{X} . Therefore

$$\lim_{n \rightarrow \infty} u_n(x) = x, \quad \text{for all } x \in \mathcal{X}.$$

Hence $\{x_n\}$ is a basis of \mathcal{X} . ■

Next we consider block perturbation of Schauder frames and prove that block perturbation of a Schauder frame is again a Schauder frame.

Theorem 3.3. Let $(\{x_n\}, \{f_n\})$ ($x_n \in E$, $f_n \in E^*$) be a Schauder frame with $x_n \neq 0$, for all $n \in \mathbb{N}$ and $\inf_{1 \leq n \leq \infty} \|x_n\| > 0$. Let $\{m_n\}$ and $\{p_n\}$ be increasing sequence in $\mathbb{N} \cup \{0\}$ such that $m_0 = 0$ and $m_{n-1} + 1 \leq p_n \leq m_n$, $n \in \mathbb{N}$. Let $\{z_n\}$ be a sequence in E of the form

$$z_k = \begin{cases} x_k, & \text{if } k \neq p_n \\ x_{p_n} + y_n, & \text{if } k = p_n, n \in \mathbb{N}, \end{cases}$$

where $y_n = \sum_{i=m_{n-1}+1}^{p_n-1} f_i(x) x_i + \sum_{i=p_n+1}^{m_n} f_i(x) x_i$ and $\|y_n\| \leq M < \infty$, $n \in \mathbb{N}$.

Then there exists a sequence $\{\Phi_n\} \subset E^*$ such that (z_n, Φ_n) is a Schauder frame for E .

Proof. Define $\{\Phi_n\} \subset E^*$ by

$$\Phi_l = \begin{cases} f_l - \alpha_l f_{p_n}, & \text{if } l \neq p_n, m_{n-1} + 1 \leq l \leq m_n, n \in \mathbb{N} \\ f_{p_n}, & \text{if } l = p_n, n \in \mathbb{N}. \end{cases}$$

Then

$$\sum_{l=1}^k \Phi_l(x) z_l = \begin{cases} \sum_{j=1}^k f_j(x) x_j + f_{p_n}(x) \sum_{i=m_{n-1}+1}^k f_i(x) x_i, & \text{if } m_{n-1} + 1 \leq k \leq p_n - 1 \\ \sum_{j=1}^k f_j(x) x_j + f_{p_n}(x) \sum_{i=k+1}^{m_n} f_i(x) x_i, & \text{if } p_n \leq k \leq m_n, n \in \mathbb{N}. \end{cases}$$

Note that

$$\left\| \sum_{i=1}^n f_i(x) x_i \right\| \leq \alpha \left\| \sum_{i=1}^{\infty} f_i(x) x_i \right\| = \alpha \|x\|, x \in E; n \in \mathbb{N} \text{ and } \alpha \geq 1.$$

Thus, we obtain

$$\left\| \sum_{m_{n-1}+1}^k f_i(x) x_i \right\| \leq \alpha \|y_n\| \leq \alpha M \quad (m_{n-1} \leq k \leq p_n - 1; n \in \mathbb{N})$$

and

$$\left\| \sum_{k+1}^{m_n} f_i(x) x_i \right\| \leq \alpha \|y_n\| \leq \alpha M \quad (p_n \leq k \leq m_n; n \in \mathbb{N}).$$

Also since $\inf_{1 \leq n \leq \infty} \|x_n\| > 0$ and $\sum_{i=1}^{\infty} f_i(x) x_i$ converges, we have

$$|f_n(x)| \leq \frac{\|f_n(x) x_n\|}{\inf_{1 \leq j \leq \infty} \|x_j\|} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

So

$$\lim_n f_n(x) = 0, x \in E, \text{ i.e., } \lim_{n \rightarrow \infty} f_{p_n}(x) = 0, x \in E.$$

Therefore, for every $\epsilon > 0$ and $x \in E$, there exists an $N \in \mathbb{N}$ such that

$$\left\| \sum_{i=1}^k \Phi_i(x) z_i - \sum_{j=1}^k f_j(x) x_j \right\| < \epsilon, \text{ for all } k > N.$$

Hence $x = \sum_{i=1}^{\infty} \Phi_i(x) z_i$, for all $x \in E$. ■

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