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Second Hankel determinant for certain classes of analytic functions associated with Al-Oboudi Differential Operator

S. M. Patil
Department of Applied Sciences,
SSVPS B.S. Deore College of Engineering,
Deopur, Dhule, Maharashtra, India.

S. M. Khairnar
Professor & Head, Department of Engineering Sciences,
MIT Academy of Engineering,
Alandi, Pune-412105, Maharashtra, INDIA.

Abstract

This paper focuses on attaining the upper bound for the functional belonging to $\mathcal{S}(\alpha, \beta, \lambda)$ defined in the unit disk $\mathbb{D} = \{z : |z| < 1\}$ which is defined by Al-oboudi Operator.

AMS subject classification:
Keywords: Starlike Function, Hankel Determinant, Coefficient Bound.

1. Introduction

Let $\mathcal{A}$ denote the class of normalized analytic univalent function for the term,

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad where \quad z \in \mathbb{D} = \{z : |z| < 1\} \tag{1.1}$$

Let $\mathcal{S}(\alpha, \beta, \lambda)$ denote the starlike subclass of $\mathcal{S}$. It is well known that f $\in \mathcal{S}(\alpha, \beta, \lambda)$ if only if,

$$\Re\left\{\frac{zf'(z)}{f(z)} + \frac{\alpha z^2 f''(z)}{f(z)}\right\} > 0 \qquad , z \in \mathbb{D} \tag{1.2}$$

The choice of $\alpha = 0$ yields $\Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0, z \in \mathbb{D}$ the class of starlike function $\mathcal{S}^*$.

In 1966, Pommerenke [5] stated the qth Hankel determinant for $q \leq 1$, & $n \leq 0$ as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q+1} \\ a_{n+1} & \vdots & \cdots & \vdots \\ a_{n+q-1} & \vdots & \cdots & a_{n+2q-2} \end{vmatrix} \tag{1.3}$$

where a_n's are the coefficients of various power of z in f(z) defined by (1.1).

This determinant has also been considered by several authors. For example Noor [14] determined the rate of growth of H$_q(n)$ an $n \to \infty$ for function f given by (1.1) with bounded boundary.

One can easily observe that Fekete and Szegö [3] functional H$_2$(1). Fekete and Szegö then further generalized and estimate $|a_3 - \mu a_2^2|$ where μ is real & $f \in \mathcal{S}$.

We consider the Hankel determinant for the case $q = 2$ and $n = 2$,

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = |a_2a_4 - a_3^2| \tag{1.4}$$

Let D^n be the Salagean differential operator $D^n : A \to A, n \in \mathbb{N}$ defined by

$$\begin{aligned} D^0 f(z) &= f(z) \\ D' f(z) &= Df(z) = zf'(z) \\ &\vdots \\ D^n f(z) &= D(D^{n-1} f(z)) \end{aligned} \tag{1.5}$$

Let $n \in \mathbb{N}$ and $\lambda \geq 0$. Let $f \in \mathcal{S}$ denote with $D_\lambda^n : A \to A$ the Al-Oboudi operator [17] defined by

$$\begin{aligned} D_\lambda^0 f(z) &= f(z) \\ D_\lambda' f(z) &= (1-\lambda)f(z) + \lambda zf'(z) = D_\lambda f(z) \\ D_\lambda^n f(z) &= D_\lambda(D_\lambda^{n-1} f(z)) \end{aligned} \tag{1.6}$$

consider the operator D_λ^β which is defined as follows,

Definition 1.1. Let $\beta, \lambda \in R, \beta \geq 0, \lambda \geq 0$ and $f(z)$ defined as (1.1), we denote by D_λ^β the linear operator defined by $D_\lambda^\beta f(z) = z + \sum_{n=2}^{\infty}[1 + (n-1)\lambda]^\beta a_n z^n$. We now define the following class $\mathcal{S}(\alpha, \beta, \lambda)$.

Definition 1.2. Let $f(z)$ be given by (1.1) then $f(z) \in \mathcal{S}(\alpha, \beta, \lambda)$ if and only if

$$\Re\left\{\frac{zD_\lambda^{\beta+1} f(z)}{D_\lambda^\beta f(z)} + \frac{\alpha z^2 D_\lambda^{\beta+2} f(z)}{D_\lambda^\beta}\right\} > 0, \qquad z \in \mathbb{D} \tag{1.7}$$

When $\lambda = 1, \beta = 0, \alpha = 0$ subclass $\mathcal{S}(\alpha, \beta, \lambda) = \mathcal{S}^*$ we obtain an upper bound for functional $|a_2a_3 - a_3^2|$ in the class $\mathcal{S}(\alpha, \beta, \lambda)$.

2. Preliminary Results

The following lemmas are required to prove our main results. Let $\mathbb{P}$ be the family of all function p analytic in the unit disc $\mathbb{D}$ for which $\Re p(z) > 0$ and

$$p(z) = 1 + c_1z + c_2z^2 + \ldots \tag{2.1}$$

Lemma 2.1. Let $p \in \mathbb{P}$ then $|c_k| \le 2, k = 1, 2, \ldots$ and the inequality is sharp.

Lemma 2.2. Let $p \in \mathbb{P}$ then

$$\begin{aligned} 2c_2 &= c_1^2 + x(4 - c_1^2) \\ 4c_3 &= c_1^3 + 2xc_1(4 - c_1^2) - x^2c_1(4 - c_1^2) + 2y(1 - |x|^2)(4 - c_1^2) \end{aligned} \tag{2.2}$$

for some x and y such that $|x| \le 1, |y| \le 1$.

Theorem 2.3. Let $f(z) \in \mathcal{S}^*(\alpha, \beta, \lambda)$ then

$$|a_2a_4 - a_3^2| \le \frac{1}{(1 + 3\alpha)(1 + 2\lambda)^{2\beta}} \tag{2.3}$$

Proof. Let $f(z) \in \mathcal{S}^*(\alpha, \beta, \lambda)$ then there exist $p(z) \in \mathbb{P}$ such that

$$zD_\lambda^{\beta+1} f(z) + 2z^2 D_\lambda^{\beta+2} f(z) = (D_\lambda^\beta f(z))p(z) \qquad for \quad some \quad z \in E \tag{2.4}$$

Equating the coefficients,

$$a_2 = \frac{c_1}{(1 + 2\alpha)(1 + \lambda)^\beta} \tag{2.5}$$

$$a_3 = \frac{c_2}{(1 + 2\lambda)^\beta(2 + 6\alpha)} + \frac{c_1^2}{(1 + 2\alpha)(2 + 6\alpha)(1 + 2\lambda)^\beta} \tag{2.6}$$

$$\begin{aligned} a_4 = {} & \frac{c_1^3}{(1 + 2\alpha)(2 + 6\alpha)(3 + 12\alpha)(1 + 3\lambda)^\beta} + \frac{c_1c_2(3 + 8\alpha)}{(1 + 2\alpha)(2 + 6\alpha)(3 + 12\lambda)(1 + 3\lambda)^\beta} \\ & + \frac{c_3}{(1 + 3\lambda)^\beta(3 + 12\lambda)} \end{aligned} \tag{2.7}$$

From (2.5) & (2.6) it is easily established that

$$|a_2a_4 - a_3^2| = \Bigg| \frac{c_1^4}{(1+2\alpha)^2(1+6\alpha)(3+12\alpha)(1+\lambda)^\beta(1+3\lambda)^\beta} + \frac{c_2c_1^2(3+8\alpha)}{(1+2\alpha)(2+6\alpha)(3+12\alpha)(1+\lambda)^\beta(1+3\lambda)^\beta} + \frac{c_1c_3}{(1+2\alpha)(3+12\alpha)(1+\lambda)^\beta(1+3\lambda)^\beta} - \left\{\left[\frac{c_2}{(1+2\lambda)^\beta(2+6\lambda)} + \frac{c_1^2}{(1+2\alpha)(2+6\alpha)(1+2\lambda)^\beta}\right]^2\right\}\Bigg| \tag{2.8}$$

By using lemma,

$$\begin{aligned} 2c_2 &= c_1^2 + x(4-c_1^2) \\ 4c_3 &= c_1^3 + 2xc_1(4-c_1^2) - x^2c_1(4-c_1^2) + 2y(1-|x|^2)(4-c_1^2) \end{aligned} \tag{2.9}$$

for some x and y such that $|x| \le 1$, $|y| \le 1$.

$$= \Bigg| \frac{c_1^4}{(1+2\alpha)^2(2+6\alpha)(3+12\alpha)(1+\lambda)^\beta(1+3\lambda)^\beta} + \frac{c_1^2(3+8\alpha)\left[\frac{c_1^2+x(4-c_1)}{2}\right]}{(1+2\alpha)^2(2+6\alpha)(3+12\alpha)(1+\lambda)^\beta(1+3\lambda)^\beta} + \frac{c_1\left[\frac{c_1^3+2(4-c_1^2)c_1x-c_1(4-c_1^2)x^2+2(1-c_1^2(1+x^2)y}{4}\right]}{(1+2\alpha)(3+12\alpha)(1+\lambda)^\beta(1+3\lambda)^\beta} - \frac{\left[\frac{c_1^2+x(4-c_1^2)}{2}\right]}{(1+2\lambda)^{2\beta}(2+6\alpha)^2} - \frac{2c_1^2\left[\frac{c_1+x(4-c_1^2)}{2}\right]}{(1+2\alpha)(2+6\alpha)^2(1+2\lambda)^{2\beta}} + \frac{c_1^4}{(1+2\alpha)^2(2+6\alpha)^2(1+2\lambda)^\beta}\Bigg| \tag{2.10}$$

Substituting for c_2 and c_3 from (2.2) and since $|c_1| < 2$ by lemma. Let $c_1 = c$ and assume without restriction that $c \in \{02\}$ we obtain, by triangle inequality,

$$\begin{aligned} |a_2a_4 - a_3^2| \leq & \frac{c^4}{(1+2\alpha)^2(2+6\alpha)(3+12\alpha)(1+\lambda)^\beta(1+3\lambda)^\beta} \\ & + \frac{(3+8\alpha)c^4}{2(1+2\alpha)^2(2+6\alpha)(3+12\alpha)(1+\lambda)^\beta(1+3\lambda)^\beta} \\ & + \frac{\rho c(4-c^2)(3+8\alpha)}{2(1+2\alpha)^2(2+6\alpha)(3+12\alpha)(1+\lambda)^\beta(1+3\lambda)^\beta} \\ & + \frac{c^4+2c^2(4-c^2)-c^2(4-c^2)\rho^2+2(1-c^2)(1-\rho^2)}{4(1+2\alpha)(3+2\alpha)(1+\lambda)^\beta(1+3\lambda)^\beta} \\ & + \frac{(c^2+2\rho c^2(4-c^2)+\rho^2(4-c^2))^2}{4(1+2\lambda)^{2\beta}(2+6\alpha)^2} \\ & + \frac{c^2(c+\rho(4-c^2))}{(1+2\alpha)(2+6\alpha)^2(1+2\lambda)^{2\beta}} + \frac{c^4}{(1+2\alpha)^2(2+6\alpha)^2(1+2\lambda)^{2\beta}} \\ & \leq F(\rho) \end{aligned} \tag{2.11}$$

with $\rho = |x| \leq 1$ furthermore,

$$\begin{aligned} F'(\rho) \leq & \frac{c(4-c^2)(3+8\alpha)}{2(1+2\alpha)^2(2+6\alpha)(3+12\alpha)(1+\lambda)^\beta(1+3\lambda)^\beta} \\ & + \frac{2c^2(4-c^2)-2\rho c^2(4-c^2)-4\rho(1-c^2)}{4(1+2\alpha)(3+2\alpha)(1+\lambda)^\beta(1+3\lambda)^\beta} \\ & + \frac{2c^2(4-c^2)+2\rho(4-c^2)^2}{4(1+2\lambda)^{2\beta}(2+6\alpha)^2} + \frac{c^2(4-c^2)}{(1+2\alpha)(2+6\alpha)(1+2\lambda)^{2\beta}} \end{aligned} \tag{2.12}$$

and with elementary calculus we can show that $F'(\rho) > 0$ for $\rho > 0$. This implies that F is an increasing function and thus the upper bound for (2.8) corresponds to $\rho = 1$ & $c = 0$ gives,

$$|a_2a_3 - a_3^2| \leq \frac{1}{(1+3\alpha)^2(1+2\lambda)^{2\beta}} \tag{2.13}$$

Remark 2.4. When we replace λ and β by 0 we get corresponding results of T.V. Sudharshan & R. Vijaya. ■

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