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Fuzzy Optimization of Multi Item Inventory Model with Imprecise Production and Demand

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Abstract

A Fuzzy Inventory model is developed for production of Multi-items with shortages being allowed and fully backlogged. The preparation time which is the time gap between decision and actual production of items is considered in a fuzzy environment. Then it is reduced to crisp preparation time using nearest interval approximation and then reduced to multi- objective optimization problem. The computation is carried out by the method of defuzzification using Global Criteria Method.

Keywords: Interval arithmetic, fuzzy inventory model, multi-item, production level, defuzzification.

1. INTRODUCTION

In the conventional inventory models different types of vagueness in different parameters and functions are experienced which are very difficult to be solved by the probability inventory modelling approaches. To define inventory optimization tasks in such environment and to interpret optimal solutions, fuzzy set theory rather than probability theory is more convenient. Considering the fuzzy set theory in inventory modelling renders an authenticity to the model formulated since fuzziness is the closest possible approach to reality.

In 1970, Zadeh et al had proposed some strategies for decision making in fuzzy environment. Wide applications of fuzzy set theory can be found in zimmermann [19]. In the crisp inventory models, all the parameters in the total cost are known and have definite values, but in the practical situations it is not at all possible. Hence fuzzy set theory comes to the rescue but different inventory models occur not only due to the various fuzzy cost parameters but also due to the fuzziness in other variables and constraints.

Lead- time is the time gap between placement of order and the actual receipt of the orders. It is a crucial factor to be considered in the field of business. So far, most of the researches have dealt with either constant or stochastic lead-time. In practice, it is difficult to forecast the lead-time in a definite and precise manner and at times, the past records are also not available to form a probability distribution for the lead-time. Hence the only choice left to the decision maker is to define the lead-time parameter imprecisely by a fuzzy number. Generally, lead time is associated with EOQ model i.e. instantaneous procurement or purchase of the lot. But, in a production system, the scenario is different. Here the time gap between the decision production and the actual commencement of production known as preparation time plays a key role in the time analysis of inventory control models. This preparation time means the time to collect the raw-materials, to arrange skilled/unskilled labourers to get machine ready for production, etc., which in turn influences the set-up cost of the system. For the first time, Mahapatra and Maiti [10] formulated and solved production inventory models for a deteriorating/breakable item with varying preparation time.

In this paper we develop a fuzzy production inventory model for multi items in a warehouse or factory. The formulated EPQ model in crisp sense is solved by generalized reduced gradient method which is a single objective optimization problem. The problem of minimizing the average total cost in the fuzzy sense is solved by Global Criteria method.

2. PRELIMINARIES

Basic definitions and theories

Definition 2. 1: Let X denote the universe of discourse. Then the fuzzy subset A of X is defined by its membership function $\mu_A(x): X \to [0,1]$ which assigns a real number $\mu_A(x)$ in the interval $[0,1]$, to each element $x \in X$, where the value of $\mu_A(x)$ at x shows the grade of membership of $x \in A$.

Definition 2.2: A fuzzy set A on $\mathbb{R}$ is convex if $A\left[\lambda x_1+(1-\lambda)x_2\right]\ge min\left[A(x_1),A(x_2)\right]$ for all $x_1,x_2\in\mathbb{R}$ and $\lambda\in[0,1]$, where min denotes minimum operator.

Definition2. 3: A fuzzy set A in the universe of discourse X is called a normal fuzzy set if there exists at least one $x\in X$, such that $\mu_A(x)=1$.

Definition 2.4: A fuzzy set which is both convex and normal is called as a fuzzy number in the universe discourse X. Fuzzy numbers are the fuzzy sets that are normalized and convex.

Definition 2.5: A fuzzy set A defined on X and any number $\alpha\in[0,1]$, then $\alpha-$cut is defined as $\{x\in X/\mu_A(x)\ge\alpha\}$. Figure 1. shows a fuzzy number A of universe of discourse X with $\alpha-$cuts.

Definition 2.6: The fuzzy number A is said to be Triangular fuzzy number if it is fully determined by (a_1,a_2,a_3) of crisp numbers such that $a_1<a_2<a_3$ whose membership function, representing triangle, can be denoted by

$$\mu_A(x)=\begin{cases}\dfrac{x-a_1}{a_2-a_1} & a_1\le x\le a_2\\ \dfrac{a_3-x}{a_3-a_2} & a_2\le x\le a_3\\ 0 & otherwise\end{cases}.$$

The $\alpha-$cuts of A where $0\le\alpha\le1$ is $A(\alpha)=\left[A_L(\alpha),A_R(\alpha)\right]$, where $A_L(\alpha),A_R(\alpha)$ are the left and right end points of $A(\alpha)$. If $A=(a_1,a_2,a_3)$, then $A_L(\alpha)=a_1+\alpha(a_2-a_1)$, $A_R(\alpha)=a_3-\alpha(a_3-a_2)$.

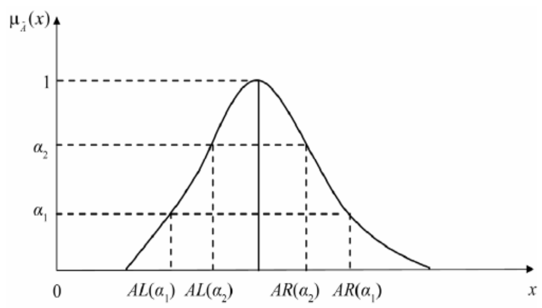

Figure 1. Fuzzy number Ã with α-cuts .

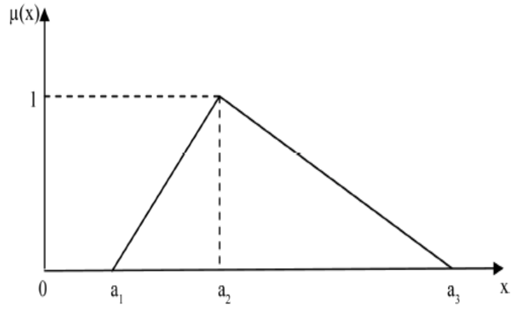

Figure 2.Triangular Fuzzy Number.

Definition 2.7: Operations on Fuzzy Numbers

Let $X = [x_L, x_R] = \{x : x_L \le x \le x_R, x \in \mathbb{R}^+\}$ where $[x_L, x_R]$ left and right limits of X respectively. Let $* \in \{+, -, \bullet, /\}$ be a binary operation on the set of positive real numbers. If X and Y are closed intervals then $X * Y = \{x * y : x \in X, y \in Y\}$ defines a binary operation on the set of closed intervals.In the case of division,it is assumed that $0 \not\in Y$ and the formula is given by

$$\frac{X}{Y} = \left[\frac{x_L}{y_L}, \frac{x_R}{y_R}\right] \text{ where } 0 \not\in Y, 0 \le x_L \le x_R \text{ and } 0 < y_L \le y_R$$

$$X + Y = [x_L, x_R] + [y_L, y_R] = [x_L + y_L, x_R + y_R]$$

$$kX = \begin{cases} (kx_L, kx_R), & for\, k \ge 0 \\ (kx_R, kx_L), & for\, k < 0 \quad k\, is\, a\, real\, number \end{cases}$$

Definition 2.8: Preference Relations of Intervals:

The decision maker's preference between interval costs are defined for minimization problems and let the uncertain costs be represented by intervals X and Y respectively. Then the preference relation connecting in, by the left and right limits of intervals is defined as below:

The preference relation $\le_{LR}$ between $X = [x_L, x_R]$ and $Y = [y_L, y_R]$ is

$X \le_{LR} Y$ if $x_L \le y_L$ and $x_R \le y_R$, $X <_{LR} Y$ if $X \le_{LR} Y$ and $x_R \ne y_R$.

The order relation $\le_{LR}$ represents the decision makers performance for the alternative with the least minimum cost, that is if, $X \le_{LR} Y$, then X is preferred to Y .

Definition 2.9: The Nearest Interval Approximation

Suppose X and Y are two fuzzy numbers with α-cuts $\left[X_L(\alpha), X_R(\alpha)\right]$ and $\left[Y_L(\alpha), Y_R(\alpha)\right]$ respectively. Then the distance between X and Y is

$$d\left(X,Y\right)=\sqrt{\int_0^1\left(X_L(\alpha)-Y_L(\alpha)\right)^2 d\alpha+\int_0^1\left(X_R(\alpha)-Y_R(\alpha)\right)^2 d\alpha}$$

Given X is a fuzzy number. We have to find a closed interval $C_d\left(X\right)$ which is nearest to X with respect to the metric d. We can do it since each interval is also a fuzzy number with constant α-cut for all $\alpha\in[0,1]$. Hence $\left(C_d\left(X\right)\right)_\alpha=[C_L, C_R]$. Now we have to minimize

$$d\left(X,C_d\left(X\right)\right)=\sqrt{\int_0^1\left(X_L(\alpha)-C_L(\alpha)\right)^2 d\alpha+\int_0^1\left(X_R(\alpha)-C_R(\alpha)\right)^2 d\alpha}$$

with respect to C_L and C_R. In order to minimize $d\left(X,C_d\left(X\right)\right)$, it is sufficient to minimize the function $D(C_L, C_R)=d^2\left(X,C_d\left(X\right)\right)$. The first partial derivatives are

$$\frac{\partial D(C_L,C_R)}{\partial C_L}=-2\int_0^1 X_L(\alpha)d\alpha+2C_L \text{ and } \frac{\partial D(C_L,C_R)}{\partial C_R}=-2\int_0^1 X_R(\alpha)d\alpha+2C_R$$

when we solve $\frac{\partial D(C_L,C_R)}{\partial C_L}=0$ and $\frac{\partial D(C_L,C_R)}{\partial C_R}=0$.

We get $C_L^*=\int_0^1 X_L(\alpha)d\alpha$ and $C_R^*=\int_0^1 X_R(\alpha)d\alpha.$

Also $\frac{\partial D^2(C_L^*,C_R^*)}{\partial C_L^2}=2>0,$ $\frac{\partial D^2(C_L^*,C_R^*)}{\partial C_R^2}=2>0$ and the corresponding value of the Hessian Matrix is

$$H\left(C_L^*,C_R^*\right)=\frac{\partial D^2\left(C_L^*,C_R^*\right)}{\partial C_L^2}.\frac{\partial D^2\left(C_L^*,C_R^*\right)}{\partial C_R^2}-\left[\frac{\partial D^2\left(C_L^*,C_R^*\right)}{\partial C_L.\partial C_R}\right]^2=4>0$$

So $D\left(C_L^*,C_R^*\right)$ i.e $d\left(X,\mathrm{C}_d\left(X\right)\right)$ is the global minimum. Therefore the interval

$C_d\left(X\right)=\left\lfloor\int_0^1 X_L(\alpha)d\alpha, \int_0^1 X_R(\alpha)d\alpha\right\rfloor$ is the nearest interval approximation of fuzzy number X with respect to metric d. Let $X=\left(x_1,x_2,x_3\right)$ be a fuzzy number. The αlevel interval of X is defined as $\left(X\right)_{\alpha} = \left[X_L\left(\alpha\right), X_R\left(\alpha\right)\right]$ when X is a triangular fuzzy number then $X_L\left(\alpha\right) = x_1 + \alpha\left(x_2 - x_1\right)$ and $X_R\left(\alpha\right) = x_3 - \alpha\left(x_3 - x_2\right)$. By nearest interval approximation method lower and upper limits of the interval are respectively given by

$$C_L = \int_0^1 X_L\left(\alpha\right) d\alpha = \int_0^1 \left[x_1 + \alpha(x_2 - x_1)\right]\left(\alpha\right) d\alpha = \frac{1}{2}\left(x_2 + x_1\right)$$

$$C_R = \int_0^1 X_R\left(\alpha\right) d\alpha = \int_0^1 \left[x_3 - \alpha(x_3 - x_2)\right]\left(\alpha\right) d\alpha = \frac{1}{2}\left(x_2 + x_3\right).$$

Therefore the interval number of X as a triangular fuzzy number is given by $\left[\left(\frac{x_1 + x_2}{2}\right), \left(\frac{x_2 + x_3}{2}\right)\right]$.

Definition 2.10: Multi- Objective Non- Linear Problem:

Multi Objective Non-Linear Problem (MONLP) with interval valued parameters can be stated as below:

$$\text{Minimize } Z\left(x\right) = \sum_{i=1}^{k} C_i \prod_{j=1}^{n} x_j^{a_{ij}} \qquad (1)$$

$$\text{Subject to } \sum_{i=1}^{k} A_i x_j \le B_j \text{ , } x_j > 0, \left(j = 1, 2, \ldots, n\right), \; x = \left(x_1, x_2, ..., x_n\right).$$

where $C_i = \left[c_{Li}, c_{Ri}\right]$, $A_i = \left[a_{Li}, a_{Ri}\right]$ and $B_j = \left[b_{Lj}, b_{Rj}\right]$.

Let us formulate the original problem (1) as a Multi-Objective Non –Linear Problem. Since the objective function $Z\left(x\right)$ and the constraints contain some parameters represented by intervals, it is natural that the solution set of (1) should be defined by preference relations between intervals.

So the interval valued objective function $Z\left(x\right)$ in terms of right and left limits $Z_R\left(x\right)$ and $Z_L\left(x\right)$, its centre $Z_C\left(x\right)$ respectively becomes:

$$Z_R\left(x\right) = \sum_{i=1}^{k} C_{Ri} \prod_{j=1}^{n} x_j^{a_{ij}}, \; Z_L\left(x\right) = \sum_{i=1}^{k} C_{Li} \prod_{j=1}^{n} x_j^{a_{ij}} \text{ and } Z_C\left(x\right) = \frac{1}{2}\left[Z_R\left(x\right) + Z_L\left(x\right)\right].$$

Thus the problem (1) is transformed into

Minimize $\{Z_R, Z_C\}$

Subject to $\sum_{i=1}^{k} a_{Li}\, x_j \le b_{Rj}, \quad \sum_{i=1}^{k} a_{Ri}\, x_j \le b_{Lj},\ x_j > 0,\ (j = 1, 2, \ldots, n),\ x = (x_1, x_2, ..., x_n).$

Definition 2.11: A multi – objective Non- linear optimization problem (MONLP) is convex if all the objective functions and the feasible region are convex.

Definition 2.12: The Best choice of the objective vector $Z_A(t', t_0)$ which are functions of the variable t', t_0 are said to be pareto-optimal solution to the MONLP if and only if there exists unique solution (t'^*, t_0^*) in the feasible region such that $Z_{A_i}(t', t_0) \le Z_{A_i}(t'^*, t_0^*)$ for all i. If the multi-objective optimization problem is convex, then every locally pareto-optimal solution is also globally pareto-optimum[13].

3. NOTATIONS AND ASSUMPTIONS

To develop our inventory model we need the following notations and assumptions:-

1. Demand is dependent on unit production cost and the current stock level.
2. Shortages are allowed & backlogged fully.
3. Preparation time in production of new items is allowed and fuzzy in nature.
4. Set up cost is dependent on preparation time
5. Production cost, set up cost and shortage cost are all known constants. Shortages are allowed and fully backlogged.
6. Time of plan is infinite
7. $i =$ index of items where our inventory system involves multi items (non–deteriorating),say 'i'where i ranges from 1 to n.
8. $I_i(t) =$ Inventory level at any time 't' for i^{th} term
9. $I_{\max} =$ Maximum inventory level in a time cycle 't'
10. $I_s =$ Maximum shortage allowed in a time cycle 't'
11. $T =$ Fuzzy preparation time for the next production cycle.

12. t_0 = Cycle length of a cycle.
13. t' = Re – production time i.e. when the next production starts which depends on decision maker.
14. u = Set up cost which depends on the fuzzy preparation time and hence it has the form

    $u = u_1 - u_2 T^{\gamma}$ where $0 \le \gamma \le 1$, u_1 and u_2 are constants so chosen to best fit the set up cost.
15. h = Holding cost per unit per unit item
16. s = Shortage cost per unit per unit item
17. p_i = Production cost per unit per unit item
18. Γ = Total production cost for all 'i' items $\Gamma = \sum_{i=1}^{n} p_i$
19. D_i = Demand Rate for item 'i' which depends on production price and stock and so it

    has the form $D_i\left(I_i(t)\right) = \begin{cases} \Gamma^{-\varepsilon}\left(\alpha + \beta\left[I_i(\mathrm{t})\right]^{1-\delta}\right) & \text{if } I_i(t) > 0 \\ \alpha\Gamma & \text{if } I_i(t) \le 0 \end{cases}$

    where α and β are positive real constants.
20. K_i = Rate of production for i^{th} *item* which has the form $K_i = \mu D_i$ where $\mu \ge 1$.

4. MODEL FORMULATION

4.1 CRISP MODEL : The Inventory production cycle starts with shortages at time $t = 0$ and at time t_1, it reaches the maximum shortage level I_s and these shortages are backlogged fully and after time t_2 the shortages go to the level zero and then the inventory again shoots up to the level I_{max} in time t_3. After this the production is halted, the inventory on hand again declines due to demand and reaches the level zero at time t_0. This is represented in the following Figure(3)

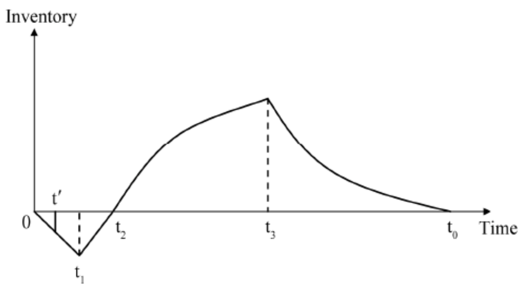

Figure (3): Inventory Begins with backlogged shortages and end with no inventory

The change of inventory level can be described by the following differential equations:

$$\frac{dI_i(\mathrm{t})}{dt}=\begin{cases}-D_i & if\ 0\le t\le t_1\\ K_i-D_i & if\ t_1\le t\le t_2\\ K_i-D_i & if\ t_2\le t\le t_3\\ -D_i & if\ \mathrm{t}_3\le t\le t_0\end{cases} \qquad (2)$$

With the boundary conditions $I_i(0)=I_i(t_2)=I_i(t_0)=0$. These differential equations can be rewritten as follows:

$$\frac{dI_i(\mathrm{t})}{dt}=\begin{cases}-\alpha\,\Gamma^{-\varepsilon} & if\ 0\le t\le t_1\\ (\mu-1)\alpha\,\Gamma^{-\varepsilon} & if\ t_1\le t\le t_2\\ (\mu-1)\Gamma^{-\varepsilon}\left(\alpha+\beta\,I_i(\mathrm{t})^{1-\beta}\right) & if\ t_2\le t\le t_3\\ -\Gamma^{-\varepsilon}\left(\alpha+\beta I_i(\mathrm{t})^{1-\beta}\right) & if\ \mathrm{t}_3\le t\le t_0\end{cases} \qquad (3)$$

Solving 3(i) and 3(ii) with boundary conditions in the time intervals $[0,t_1]$ and $[t_1,t_2]$ and also putting $\delta=0$ we have the following equations:

$$I_i(\mathrm{t})=\begin{cases}-\alpha\,\Gamma^{-\varepsilon}\,t\\ -(\mu-1)\alpha\,\Gamma^{-\varepsilon}\,(t_2-t)\end{cases} \qquad (4)$$

So at time $t=t_1$, the above equations become

$$-\alpha\,\Gamma^{-\varepsilon}t_1=-(\mu-1)\alpha\,\Gamma^{-\varepsilon}\,(t_2-t_1)\Rightarrow \mu t_1=(\mu-1)t_2 \qquad (5)$$

In a Similar manner solving 3(iii) and 3(iv) with the boundary conditions $I_i(t_2)=0=I_i(t_0)$ in the time intervals $[t_2,t_3]$ and $[t_3,t_0]$ respectively we have the following equations:

$$I_i(t)=\begin{cases}\dfrac{\alpha}{\beta}\left\{e^{\theta(t-t_2)}-1\right\}\\ \dfrac{\alpha}{\beta}\left\{e^{\beta\Gamma^{-\varepsilon}(t_0-t)}-1\right\}\end{cases} \tag{6}$$

where $\theta=(\mu-1)\beta\Gamma^{-\varepsilon}$.

At time $t=t_3$, the above equations become

$$\frac{\alpha}{\beta}\left\{e^{\theta(t_3-t_2)}-1\right\}=\frac{\alpha}{\beta}\left\{e^{\beta\Gamma^{-\varepsilon}(t_0-t_3)}-1\right\}\Rightarrow t_3=\frac{t_0}{\mu}+t_1,\ \text{using (5).} \tag{7}$$

Now calculating the different total costs as follows:

Total shortage cost during time intervals $[0,t_1]$ and $[t_1,t_2]$ is calculated as below:

$$\begin{aligned}C_s&=-s\left[\int_0^{t_1}I_i(t)\,dt+\int_{t_1}^{t_2}I_i(t)\,dt\right]\\&=\frac{1}{2}s\,\alpha\Gamma^{-\varepsilon}t_1^2+\frac{1}{2}s(\mu-1)\alpha\Gamma^{-\varepsilon}(t_2-t_1)^2\end{aligned} \tag{8}$$

where $t_1=T+t'$, $t_2=\dfrac{\mu}{\mu-1}t_1$ and $t_3=\dfrac{t_0}{\mu}+t_1$.

Total holding cost during the time intervals $[t_2,t_3]$ and $[t_3,t_0]$ is calculated as follows:

$$\begin{aligned}C_h&=h\left[\int_{t_2}^{t_3}I_i(t)\,dt+\int_{t_3}^{t_0}I_i(t)\,dt\right]\\&=\frac{h\alpha}{\beta}\left[\frac{1}{\theta}\left(e^{\theta(t_3-t_2)}-1\right)+t_2\right]-\frac{h\alpha}{\beta}\left[\frac{1}{\beta\Gamma^{-\varepsilon}}\left(1-e^{\beta\Gamma^{-\varepsilon}(t_0-t_3)}\right)+t_0\right]\end{aligned} \tag{9}$$

Total production cost during time intervals $[t_1,t_2]$ and $[t_2,t_3]$ is calculated as follows:

$$\begin{aligned}C_p&=\Gamma\int_{t_1}^{t_2}\mu\alpha\Gamma^{-\varepsilon}\,dt+\Gamma\int_{t_2}^{t_3}\mu\Gamma^{-\varepsilon}\left(\alpha+\beta I_i(t)\right)dt\\&=\Gamma^{1-\varepsilon}\mu\alpha(t_2-t_1)+\frac{1}{\theta}\Gamma^{1-\varepsilon}\mu\alpha\left(e^{\theta(t_3-t_2)}-1\right)\end{aligned} \tag{10}$$

Hence thetotalaverage cost $Z_A(t',t_0)$ is given by the following equations:

$Z_A\left(t',t_0\right)=\dfrac{TC}{t_0}$, where $TC=C_s+C_h+C_p+u.$

Hence the proposed model in the crisp sense is formulated as follows:

$$\text{Minimize } Z_A\left(t',t_0\right)=\frac{1}{t_0}\left\{\frac{1}{2}s\,\alpha\Gamma^{-\varepsilon}\,t_1^{\,2}+\frac{1}{2}s\left(\mu-1\right)\alpha\Gamma^{-\varepsilon}\left(t_2-t_1\right)^2\right.$$
$$+\frac{h\alpha}{\beta}\left[\frac{1}{\theta}\left(e^{\theta(t_3-t_2)}-1\right)+t_2\right]-\frac{h\alpha}{\beta}\left[\frac{1}{\beta\Gamma^{-\varepsilon}}\left(1-e^{\beta\Gamma^{-\varepsilon}(t_0-t_3)}\right)+t_0\right] \quad (11)$$
$$\left.+\Gamma^{1-\varepsilon}\mu\alpha\left(t_2-t_1\right)+\frac{1}{\theta}\Gamma^{1-\varepsilon}\mu\alpha\left(e^{\theta\left(t_3-t_2\right)}-1\right)+u_1-u_2\,T^{\,\gamma}\right\}.$$

4.2 Fuzzy Model:

Due to uncertainty in the environment preparation time T is a fuzzy number represented by an apt interval arithmetic number in the form $T=\left[T_L,T_R\right]$. Hence substituting $t_1=\left[T_L,T_R\right]+t'=\left[t_{1L},t_{1R}\right]$ (say) and also for the rest of the time factors by their corresponding fuzzy parameters in the interval arithmetic number we have the following equations:

$$t_2=\frac{\mu}{\mu-1}t_1=\left[t_{2L},t_{2R}\right]\quad(\text{say})$$
$$t_3=\frac{t_0}{\mu}+t_1=\left[t_{3L},t_{3R}\right]\quad(\text{say})$$

The corresponding different costs associated with the proposed inventory model with their left & right fuzzified α -cuts are deduced as follows :

Total shortage cost $C_S=\left[C_{S_L},S_{S_R}\right]$ say, where

$$C_{s_L}=\frac{1}{2}s\alpha\,\Gamma^{-\varepsilon}\,t_{1L}^{\;2}+\frac{1}{2}s\omega t_{2L}^{\;2}-s\omega t_{2R}t_{1R}+\frac{1}{2}s\omega t_{1L}^{\;2} \quad (12)$$

where $\omega=(\mu-1)\alpha$.

Total holding cost $C_h=\left[C_{h_L},C_{h_R}\right]$ say, where

$$C_{h_L}=\frac{h\alpha}{\beta\theta}\left[e^{\theta\left(t_{3L}-t_{2R}\right)}-1\right]+\frac{h\alpha}{\beta}t_{2L}-\frac{h\alpha}{\beta^2\,\Gamma^{-\varepsilon}}\left[1-e^{\beta\Gamma^{-\varepsilon}\left(t_0-t_{3R}\right)}\right]-\frac{h\alpha}{\beta}t_0 \quad (13)$$

Total production cost $C_p = \left[C_{P_L}, C_{P_R} \right]$ say, where

$$C_{P_L} = \Gamma^{(1-\varepsilon)} \mu\alpha\left(t_{2L} - t_{1R}\right) + \Gamma^{1-\varepsilon} \frac{\mu\alpha}{\theta}\left[e^{\theta\left(t_{3L} - t_{2R}\right)} - 1 \right] \tag{14}$$

The set up cost u can be written as

$$u = \left[u_1 - u_2 T_R^{\gamma}, u_1 - u_2 T_L^{\gamma} \right] = \left[u_L, u_R \right] \text{ say} \tag{15}$$

Similarly the right α-cuts of the fuzzified average total cost can be got by substituting R instead of L and L instead of R in the above formulae. Hence the total average cost now equals

$$Z_A = \frac{1}{t_0}\left[TC \right] = \frac{1}{t_0}\left\{ \left[C_{s_L}, C_{s_R} \right] + \left[\mathrm{C}_{h_L}, C_{h_R} \right] + \left[C_{p_L}, C_{p_R} \right] + \left[u_L, u_R \right] \right\} = \left[Z_{A_L}, Z_{A_R} \right] \tag{16}$$

where

$$Z_{A_L} = \frac{1}{t_0}\left[C_{s_L} + C_{h_L} + C_{p_L} + u_L \right],\ Z_{A_R} = \frac{1}{t_0}\left[C_{s_R} + C_{h_R} + C_{p_R} + u_R \right] \text{ and}$$

$$Z_{A_C} = \frac{1}{2}\left[Z_{A_L} + Z_{A_R} \right]$$

5. SOLUTION METHODOLOGY

5.1 Crisp model: To get the optimum value of average total cost we have to solve

$$\frac{\partial}{\partial t'}\left(Z_A\left(t', t_0\right)\right) = 0 \text{ and } \frac{\partial^2}{\partial t'^2}\left(Z_A\left(t', t_0\right)\right) > 0 \quad \frac{\partial\left(Z_A\left(t', t_0\right)\right)}{\partial t_0} = 0 \ \& \ \frac{\partial^2}{\partial t_0^{\,2}}\left(Z_A\left(t', t_0\right)\right) > 0$$

$$\frac{\partial}{\partial t'}\left(Z_A\left(t', t_0\right)\right) = \frac{1}{t_0}\left(\frac{\partial TC}{\partial t'}\right) = \frac{1}{t_0}\left\{ \frac{-h\alpha}{\beta(\mu-1)} e^{\theta\left(\frac{t_0}{\mu} - \frac{T}{\mu-1} - \frac{t'}{\mu-1}\right)} + \frac{h\alpha\mu}{\beta(\mu-1)} - \frac{h\alpha}{\beta} e^{\beta\Gamma^{-\varepsilon}\left(\frac{\mu-1}{\mu}t_0 - \left(T+t'\right)\right)} \right.$$

$$\left. + \Gamma^{1-\varepsilon}\frac{\mu\alpha}{\mu-1} - \Gamma^{1-\varepsilon}\frac{\mu\alpha}{\mu-1} e^{\theta\left\{\frac{t_0}{\mu} - \frac{T}{\mu-1} - \frac{t'}{\mu-1}\right\}} + s\alpha\Gamma^{-\varepsilon}\left(T + t'\right) + s\alpha\Gamma^{-\varepsilon}\left(\frac{T}{\mu-1} + \frac{t'}{\mu-1}\right) \right\} = 0$$

$$\frac{\partial}{\partial t_0}\left(Z_A\left(t',t_0\right)\right)=\frac{1}{t_0}\frac{\partial TC}{\partial t_0}-\frac{1}{t_0^{\,2}}TC=\frac{1}{t_0}\left\{\frac{h\alpha}{\beta\mu}e^{\theta\left(\frac{t_0}{\mu}-\frac{T}{\mu-1}-\frac{t'}{\mu-1}\right)}+\frac{h\alpha(\mu-1)}{\beta\mu}e^{\beta\Gamma^{-\varepsilon}\left(\frac{\mu-1}{\mu}t_0-\left(T+t'\right)\right)}\right.$$
$$\left.-\frac{h\alpha}{\beta\mu}+\Gamma^{1-\varepsilon}\frac{\mu\alpha}{\mu}e^{\theta\left\{\frac{t_0}{\mu}-\frac{T}{\mu-1}-\frac{t'}{\mu-1}\right\}}\right\}-\frac{1}{t_0^{\,2}}TC=0 \tag{17}$$

which implies $\frac{\partial TC}{\partial t_0}=\frac{1}{t_0}TC$.

Hence the average total cost is strictly convex since all the positive minors of its Hessian matrix are strictly positive.

Since we have

$$\frac{\partial^2 Z_A\left(t',t_0\right)}{\partial t'^2}>0,\ \frac{\partial^2 Z_A\left(t',t_0\right)}{\partial t_0^{\,2}}>0 \text{ and } \frac{\partial^2 Z_A\left(t',t_0\right)}{\partial t'^2}\times\frac{\partial^2 Z_A\left(t',t_0\right)}{\partial t_0^2}-\left\{\frac{\partial^2 Z_A\left(t',t_0\right)^2}{\partial t_0\partial t'}\right\}^2>0.$$

5.2 Fuzzy Model: The objective functions of the fuzzified optimization problem i.e. $Z_{A_L}, Z_{A_C}, Z_{A_R}$ are all functions of t' & t_0 as t_1, t_2 and t_3 are dependent variables which depend on t' and t_0. This total average cost minimization problem with interval objective function is converted to multi objective non-linear optimization problem, whose objectives are to minimize the centre Z_{A_C} and right limit Z_{A_R} of the interval objective function. As seen in model I, we can also show and hence Z_{A_C} are strictly convex and can also be seen that an objective vector $\left(Z^*_{A_C}, Z^*_{A_C}\right)$ is pareto-optimal if there does not exist another objective vector (Z_{A_C}, Z_{A_R}) such that $Z_{A_i}\left(t',t_0\right)\le Z_{A_i}\left(t'^*,t_0^*\right)$ for all $i=C,R$ and $Z_{A_j}\left(t',t_0\right)<Z_{A_j}\left(t'^*,t_0^*\right)$ for at least one index $j=C,R$. Hence $\left(Z^*_{A_C}, Z^*_{A_R}\right)$ is pareto-optimal if the decision vector corresponding to it is pareto-optimal and also since our multi objective optimization problem is convex, we know every locally pareto-optimal solution is also globally pareto-optimum [13]. Hence the multi objective problem is solved by Global Criteria Method whose solving steps are given below:

Step-1: The multi-objective programming problem is converted to single objective problem using only one objective at a time.

Step-2: A pay-off matrix is formed from the above last table as follows:

$$\begin{pmatrix} \tilde{Z}_{A_C}^{\min} & \tilde{Z}_{A_R}^{\max} \\ \tilde{Z}_{A_C}^{\max} & \tilde{Z}_{A_R}^{\min} \end{pmatrix}$$

Step-3: Find the ideal objective vector from the above pay-off matrix of step-2, for example

($\tilde{Z}_{A_C}{}^{min}\ \tilde{Z}_{A_R}{}^{\min}$) say and its corresponding value of ($\tilde{Z}_{A_C}{}^{\max}\ \tilde{Z}_{A_R}{}^{\max}$).Then its auxiliary problem is solved as below:

$$\text{Global Criteria } Z_{GC} = \text{Minimize}\left\{\left(\frac{\tilde{Z}_{A_C}(t',t_0)-\tilde{Z}_{A_C}^{\min}}{\tilde{Z}_{A_R}^{\max}-\tilde{Z}_{A_C}^{\min}}\right)^{\omega}+\left(\frac{\tilde{Z}_{A_C}(t',t_0)-\tilde{Z}_{A_R}^{\min}}{\tilde{Z}_{A_R}^{\max}-\tilde{Z}_{A_R}^{\min}}\right)^{\omega}\right\}^{\frac{1}{\omega}}$$

where $1 \le \omega < \infty$ and for calculation sake we take the value of ω as 2. Hence the solution set given above is referred to as Global Criteria.

6. NUMERICAL EXAMPLE

Consider an inventory model with the following parametric values:

6.1 Crisp Model:

$\alpha = 300, \beta = 2,\ \varepsilon = 0.7, \gamma = 0.5, h = Rs1.5 / unit / year,\ u_1 = Rs\,2000 / unit / year,$
$u_2 = Rs300 / unit / year, s = Rs15 / unit / year, \mu = 1.8, T = 0.6\, year$

By Graded Based Non –linear optimization Method we get the solution of the Crisp Model as:
$Z_A(t',t_0) = 1356.35,\ t' = 0.6001609,\ t_0 = 6.939239,\ I_s = 37.82707,\ I_{\max} = 72.84892$

6.2 Fuzzy Model:

$\mu = 1.8\ , \alpha = 300,\ \beta = 2, \varepsilon = 0.7, \gamma = 0.5,\ h = Rs.1.5 / unit / year,\ \Gamma = Rs.25 / unit / year,$
$u_1 = Rs.2000 / unit / year,\ u_2 = Rs.300 / unit / year, s = Rs.15 / unit / year,\ T_L = 0.5, T_R = 0.8$

Using the above data, we get the following pay –off matrix $\begin{pmatrix} Z_{A_C} & Z_{A_R} \\ 1364.005 & 1753.147 \\ 1385.503 & 1722.588 \end{pmatrix}$

By Global Criteria Method we get the solution of the fuzzy model

$Z_{A_C} = 1369.40,\ Z_{A_R} = 1730.23,\ Z_{A_L} = 1008.61,\ t' = 0.6327567, t_0 = 7.941731, GC = 0.3543743$

SENSITIVITY ANALYSIS

A Sensitivity analysis of the variables $t' \& t^0$ for the crisp model has been made where the parameters α, μ, T, Γ are changed from -50% to +50%. The corresponding change of $Z_A^*\left(t', t^0\right),\ I_s^*, I_{\max}^*$ are enumerated in Table I.

Table: I. SENSITIVITY ANALYSIS

Parameter	% changes	% *of* $Z_A(t'_, t_0)$	t'^*	t_0^*	I_S	$I_{\max}$
α	-50	-42.09005	1.051922	9.5283	26.03292	54.05133
	-20	-16.43247	0.7275575	7.693093	33.47436	66.22929
	$+20$	$+16.07881$	0.506768	6.373380	41.86019	78.68090
	$+50$	$+39.71182$	0.4040906	5.737264	47.47090	86.34708
	-50	*No solution*	–	–	–	–
	-20	-4.761578	0.4635499	8.362338	33.52132	55.21444
μ	$+20$	$+2.550066$	0.6733307	6.306766	40.13326	83.86810
	$+50$	$+4.740787$	0.7361900	5.817491	42.11448	94.57485
	-50	$+0.7166992$	0.9207250	7.061990	38.47522	74.44632
	-20	$+0.2596661$	0.7276148	6.983796	38.06201	73.42793
T	$+20$	-0.2364422	0.4733806	6.898645	37.61337	72.32223
	$+50$	-0.5588499	0.2841310	6.843172	37.32184	71.60373
	-50	-2.621729	0.2851214	5.590982	45.31972	108.9680
	-20	-1.285355	0.4873798	6.480728	40.06651	83.00596
Γ	$+20$	$+1.360556$	0.7014259	7.332111	36.10404	65.42835
	$+50$	$+3.408616$	0.8376034	7.836238	34.11446	57.30315

7. CONCLUTIONS

The generalized reduced gradient method has been applied to get the solution of crisp model which is a single objective optimization problem, whereas fuzzy model is a multi-objective optimization problem whose objectives are to minimize the centre Z_{A_C} and right limit Z_{A_R} of the interval objective function. These two objective functions take care of both average case and the worst case. In this fuzzy model the pareto-optimal solutions obtained using Global Criteria Method give the best possible solution in terms of best choice to be made by decision maker as the preparation time is fuzzy in nature. The preparation time for the next production cycle is highly significant as it impacts several costs like set up cost, production costs, shortage costs, etc. It is also seen that the demand of a commodity decreases with the increase in production costs but increases with the increase of stock put up on display and vice versa is true. The proposed model is generally a production model with demand dependent production rate and the unit production cost which is assumed to be constant for each of the items under production which is not so in real terms as it varies with the preparation time and quantity produced. This model can also be improvised as time dependent production rate, partially lost sales, inflation, quantity discounts on the displayed stock are the different factors which can be considered to make this model more pragmatic and dynamic in the future scope of research.

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