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Equivalent Statements and Conjectures Associated with Pythagorean Triangles

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Abstract

In this paper, we try to logically establish equivalence of seven tantamounting statements that to some extent concern right triangles and some generalized statements of number theory. The prime purpose, in addition, is to show that how dominantly our designed set-up format of sides for a right triangle simplifies the pre-known routines known to establish each statements individually.

Key words: Area of right triangle, Congruent number**,** Co-primes, Congruum, Elliptic Curve

Abbreviations:

P.P.T. (Primitive Pythagorean triplet)

1. INTRODUCTION:

In this note we introduce Pythagorean triplets, say (a, b, h) for the odd or even integer '$\boldsymbol{a}$' in our format that assures to generate a primitive triplet. We have, over and above some known mathematical properties, established some more salient features which we claim to be the aftermath of our terminologies. The important feature is, what we primarily require is the first integer '$\boldsymbol{a}$' and rest of all principal feature pertaining to that right triangle with 'a' as the smaller leg, are on the tip of formula.

The next important role encircles the inter-connectivity/ cyclic equivalence of some statements. These are also established, shaking off the centuries old Eulerian time routine terminology, in our format and that dominantly claims brevity. In addition to this working along with, it helps develop mathematical insight.

1.1 Primitive Triplets and Congruent Number:

(a) P.P.T (Primitive Pythagorean Triplets): For any right triangle with integer sides, a, b, and h, considered in ascending order, we have the following propositions for (a, b, h) to be a P.P.T.

(1) $a, b, and\ h \in N$

(2) $g.c.d\ of\ a\ and\ b = (a, b) = 1$

(3) $a^2 + b^2 = h^2$

In this section we introduce our pattern of dealing with sides of any Pythagorean triplet of a right triangle. our pattern mainly focuses on the (first) smaller leg of a right triangle and finds the next one and the hypotenuses in its own terms. It focuses on the nature of the first leg being even or an odd integer. The importance of this procedure lies in the fact that it enables one to identify all other primitive triplets, if exists, by changing the integer value of 'i' (even or odd) as the case may be for the smaller leg. It cuts off all known and predefined possible routines that search for triplets, as the case is with Euler formula, and algorithmically comes to an end finding possible triplets. We first introduce P.P.T. with necessary condition on it.

For positive integer $\boldsymbol{a}, we\ have\ integers\ b = \frac{a^2 - i^2}{2i},\ and\ h = \frac{a^2 + i^2}{2i}$ that satisfy Pythagorean condition, $a^2 + b^2 = h^2$ [1],[2],[3] for some integer i.

- For '$\boldsymbol{a}$'**,** an odd integer, the integer '$\boldsymbol{i}$ ' is also an odd one that will undoubtedly satisfy

  $a^2 - i^2 \geq 0$ and $\boldsymbol{a} < \boldsymbol{b} < \boldsymbol{h}$**.** Feasibility of '$\boldsymbol{b}$' enjoins feasibility of primitive triplet for the integer '$\boldsymbol{a}$'**.**

- For '$\boldsymbol{a}$', an even integer, the integer '$\boldsymbol{i}$ ' is also an even one that will undoubtedly satisfy

  $a^2 - i^2 \geq 0$ and $\boldsymbol{a} < \boldsymbol{b} < \boldsymbol{h}$**.** Feasibility of '$\boldsymbol{b}$' enjoins feasibility of primitive triplet for the integer '$\boldsymbol{a}$'**.**

The above set-up with $\boldsymbol{a}$ and $\boldsymbol{b}$ **can claim for** $\boldsymbol{h} = \frac{\boldsymbol{a^2 + i^2}}{\boldsymbol{2i}}$ necessitating $a^2 + b^2 = h^2$

A right Pythagorean triangle with positive integral sides, can be, on the basis of the shorter leg, divided into two mutually exclusive infinite sets. We shall call them P_1 and P_2 on the basis of oddness and evenness of shorter legs respectively.

We define, Set $P_1 = \{(a, b, h) | a, b, and\ h \in N, a < b, (a, b) = 1, h - b = 1\}$.Triplets like,

(3, 4, 5), (7, 24, 25) $\in P_1$.These triplets are called odd Pythagorean triplets.

Set $P_2 = \{(a, b, h) | a, b, and\ h \in N, a < b, (a, b) = 1, h - b = 2\}$. Triplets like, (8, 15, 17),

(12, 35, 37) $\in P_2$. These triplets are called even Pythagorean triplets.

We like to pass a final remark that there are many odd integers, e.g. 33. 39,… etc., and many even integers, e.g. 20, 28, …., which possess more than one primitive triplets that corresponds to different odd or even values of '$\boldsymbol{i}$' as the case may be for the integer '$\boldsymbol{a}$'.

(b) Congruent number: A positive integer n is called a congruent number if it is an area of a right triangle with positive rational sides. E.g. 6 is congruent number for in terms of measure in square units it shows an area right triangle with sides $\boldsymbol{a = 3, b = 4}$ **and** $\boldsymbol{h = 5}$. it can be established that '1' is not a congruent number" that is, we cannot find a right angle with non-zero positive rational sides whose area, in square units, equals '1' [5],[6].(Proof of this statement is shown in annexure).

We, being reliant on our sound experience, put forward the following conjectures without proof (Some of them, are dealt with an annexure).

2. SOME CONJECTURES:

In this section we make some conjectures and try to prove in our way that broadly refers the sides to a right triangle. [Pl. refer annexure]

2.1 Conjecture regarding primitive Pythagorean triplets.

1. Every odd integer other than '1' has **at least*** one primitive Pythagorean triplets.

   (* Some odd integers of the form $3(2n + 9)$ $, n \in N$ have more than one primitive triplets).

   [e.g. the integers 33,39, 45,… possess more than one primitive triplets.]

2. Every even integers of the form $2(2n + 1), n \in N$, has **no** primitive triplets for $n \in N \cup \{0\}$

   [e.g., the integers 2,6,10, 14,… has no primitive triplets.]

3. Every even integers other than those which fall in $\{4(2n+3)/n \in N \cup \{0\}\}$ are likely to possess more than one primitive triplets.

   [e.g. the integers 20, 28, 36,….. possess more than one primitive triplets.]

4. The number of primitive triplets associated with odd integers and even integers mentioned above keep on increasing indefinitely and its distribution, without loss of generality, is not known.

2.2 Equivalent Statements: In this section we shall make efforts to establish some statements pertaining to some basic properties of integers and basics of right trianglesSet of following seven Tantamounting statements related to important characteristics of right angle triangle. These are quoted as under.

S1: $a^4 - b^4 \neq c^2$ for $a, b\ and\ c$ non-zero integers.

S2: There are no two Pythagorean triangles in which the two legs of one triangle are the leg and hypotenuse of the other right triangle.

S3: Area of any right angle triangle with positive rational sides cannot be a perfect square of rational number.

S4: If three square numbers from an arithmetic progression, then the difference between consecutive numbers [known as **congrumm**] in the progression cannot be a square of any non-zero rational.

S5: The only rational point on the Elliptic curves are $0, -1$, and 1.

S6: The two legs of a right angle triangle cannot be a perfect square of integers.

S7: $a^4 + b^4 \neq c^2$ for $a, b\ and\ c$ non-zero integers.

2.2.1 Proof of the statements: Now we prove the statements using known properties

S1: $a^4 - b^4 \neq c^2$ for $a, b\ and\ c$ non-zero integers.

Proof: If $a^4 - b^4 = c^2$ then $c^2 + b^4 = a^4$,Therefore, (c, b^2, a^2) is Pythagorean primitive triplet.

Case-1 if c is an odd number, greater than 1,then for a primitive triplet [for $i = 1$]

We have $b^2 = \frac{c^2-1}{2} \Rightarrow c^2 = 2b^2 + 1$, and $a^2 = \frac{c^2+1}{2} \Rightarrow c^2 = 2a^2 - 1$

From the above results, we have

$c^2 = 2b^2 + 1 = 2a^2 - 1$

$\Rightarrow 2b^2+2=2a^2 \Rightarrow b^2+1=a^2$ It means that $(b,1,a)$ is P.P.T. It is a contradiction as 1 cannot be any member of any P.P.T.

[For (a,b,c) a P.P.T, we have positive integers $a, b,$ and c so that $\boldsymbol{a} < \boldsymbol{b} < \boldsymbol{c}$ and in this case integer $\boldsymbol{b} < \mathbf{1}$ **is** not possible.]

Case-2 Let $c > 2$ an even integer, then following our pattern of getting remaining members of even triplet,

For i =2, we have $b^2 = \frac{c^2-4}{4}, \Rightarrow 4b^2+4=c^2$

and $a^2 = \dfrac{c^2+4}{4} \Rightarrow 4a^2 = c^2+4 \Rightarrow c^2 = 4a^2-4$

Combining the above results, we get

$c^2 = 4b^2+4 = 4a^2-4 \Rightarrow 4b^2+8=4a^2 \Rightarrow b^2+2=a^2$ this means that $(b,\sqrt{2},a)$ is P.P.T; but it is a contradiction, as one side $\sqrt{2}$ is an irrational number.

Thus, in both the cases we get contradiction; hence we can conclude that (c,b^2,a^2) is not a Pythagorean Primitive Triplet.

This proves that $a^4 - b^4 \neq c^2$. This proves S1.

S2: There are no two Pythagorean triangles in which the two legs of one triangle are the leg and hypotenuse of the other right triangle.

Proof: Let us consider two right triangles, say ABC and PQR. Let the two sides, a leg and c hypotenuse, of triangle PQR share two legs of triangle ABC.

If possible, then we have two Pythagorean triplets such that $c^2 + a^2 = d^2$ from below (fig.**1.1**) and $b^2 + a^2 = c^2$ from below (fig.**1.2)**. so, $c^2 - a^2 = b^2$.

From the above results we get $c^4 - a^4 = (bd)^2 = p^2$ [b and d are natural numbers and $bd = p$]

which is not possible following **S1 and hence S2** is proved.

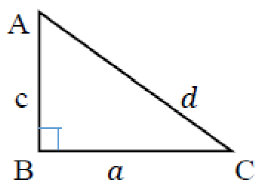


fig (1.1)

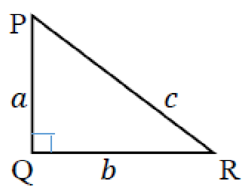


fig (1.2)

S3: Area of any right angle triangle with positive rational sides cannot be a perfect square of any rational number.

Proof: for $a \in N$ with $a > 1$ and $a \notin \{(4n+2)/n \in N \cup \{0\}\}$.

We have $\left(a, \frac{a^2 - i^2}{2i}, \frac{a^2 + i^2}{2i}\right)$ a primitive Pythagorean triplets. (1)

If a is an odd integer then i is also an odd integer with $i \in N$

[a is an even integer then i is also an even integer with $i \in N$]

Now area of a right triangle $= \frac{1}{2}\left(a \times \frac{a^2 - i^2}{2i}\right)$ (2)

To prove that area of triangle cannot be a perfect square of a rational number.

Case-1 For a - odd then by assuming $i = 1$ we have area $= \frac{1}{2}\left(a \times \frac{a^2 - 1}{2}\right) = \frac{1}{4}a(a^2 - 1)$

where a and $(a^2 - 1)$ both are co-primes.

For area of a triangle to be a perfect squares, we must have product of both co-primes a and $(a^2 - 1)$ to be perfect square.

This is a contradiction as for any $a \in N - \{1\}$, $(a^2 - 1)$ cannot be a perfect square.

We conclude that area of right angle triangle cannot be a perfect square.

Case-2 For a - even then by assuming $i = 2$ we have area $= \frac{1}{2}\left(a \times \frac{a^2 - 4}{4}\right) = \frac{1}{2}a\left(\left(\frac{a}{2}\right)^2 - 1\right)$.

In this case, also following the same argument as given in above paragraph $\left(\left(\frac{a}{2}\right)^2 - 1\right)$ cannot be a perfect square.

Hence in the both cases we have established that area of right angle triangle with integral sides cannot be a perfect square.

S4: If three square numbers from an arithmetic progression, then the difference between consecutive numbers [known as congrumm] in the progression cannot be a square of any non-zero rational.

Proof – Let, for some integers a, b, and c, three squares as shown, be in the arithmetic progression be $(b-a)^2$, c^2, $(b+a)^2$.

If possible, difference of these consecutive square numbers is,

$c^2-(b-a)^2=(b+a)^2-c^2=p^2$ (say)

For a right angle triangle triplet (a,b,c), we have $a^2+b^2=c^2$

From above congrumm relation,

$c^2-(b-a)^2=p^2$

$\Rightarrow 2ab=p^2$

So, $p^2=2ab=4\left(\frac{1}{2}ab\right)$ = 4 × area of right angle triangle,

Following S3, area of right angle triangle cannot be a perfect square and hence S4 is established.

Thus, $c^2-(b-a)^2=(b+a)^2-c^2\neq p^2$.

So congrumm cannot be a perfect square.

S5: The only rational point on the Elliptic curves are 0, -1, and 1.

Proof: According to the above case -1 in S3,

Area $A=\frac{1}{2}\left(a\times\frac{a^2-1}{2}\right)=\frac{1}{4}a(a^2-1)$

Letting $A=y^2$, we have $y^2=\frac{1}{4}a(a^2-1)=\frac{1}{4}a(a-1)(a+1)$

So $(2y)^2=a(a-1)(a+1)\Rightarrow (\mathrm{Y})^2$ = a (a-1) (a+1) this means that Y can vanish only for

$a=0,1$, and -1.[for any other value of a , $y\notin Q$.]

This proves S5.

S6: The two legs of a right angle triangle cannot be a perfect square of integers.

Proof: To prove that both the legs of a right triangle cannot be perfect square of integers.

If so, let for some a, b and $c \in N - \{1,2\}$, (a^2, b^2, c) be a P.P.T.

For some m, and n [$m, n \in N$ with . One of m or n is even and the other one be odd] $a^2 = 2mn, b^2 = m^2 - n^2$ and $c = m^2 + n^2$.

Now, Area of right angle triangle, $A = \frac{1}{2}a^2b^2 = \frac{1}{2}(2mn)(m^2 - n^2) = mn(m^2 - n^2)$.

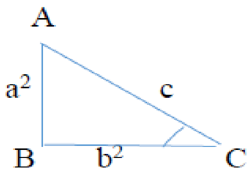
Letting $mn = \alpha^2, m^2 - n^2 = \beta^2$ then $A = \alpha^2\beta^2 = (\alpha\beta)^2$, which is contradiction as **S3** the area of right angle triangle cannot be perfect square.

Hence, we conclude that (a^2, b^2, c) cannot be a P.P.T. this means that first two legs of right angle triangle cannot be a perfect square and hence S6.

S7: $a^4 + b^4 \neq c^2$ for a, b and c non-zero integers.

Proof: If $a^4 + b^4 = c^2$ then (a^2, b^2, c) is Primitive Pythagorean Triplets as they are integers.

Case-1 if a^2 is an odd integer greater than 1, then by known formula[1], $b^2 = \frac{a^4-1}{2}$ and $c = \frac{a^4+1}{2}$ or $c = b^2 + 1$. We consider the triangle ABC



Let $\angle$ ACB $= \theta$

Now,[2] $\sin\theta = \frac{a^2}{c} = \frac{a^2}{b^2+1}$,

$$\cos^2\theta = 1 - \sin^2\theta = 1 - \left(\frac{a^2}{b^2+1}\right)^2 = \frac{(b^2+1)^2 - a^4}{(b^2+1)^2} = \frac{b^4 + 2b^2 + 1 - a^4}{(b^2+1)^2} \quad (3)$$

$$\text{But } \cos^2\theta = \left(\frac{b^2}{c}\right)^2 = \frac{b^4}{(b^2+1)^2} \quad (4)$$

From (3) and (4) we must have $2b^2 + 1 - a^4 = 0$

$\Rightarrow 2b^2 + 1 = a^4 \Rightarrow (\sqrt{2}b)^2 + 1 = (a^2)^2$

this means that $(\sqrt{2}b, 1, a^2)$ is P.P.T. but it is a contradiction, [For 1 cannot participate in any primitive triplet or $\sqrt{2}b$ is not rational for b a rational]

Case-2 if a^2 is even and square of an integer, greater than 2, then $b^2 = \frac{a^4-4}{4}$,

$c = \frac{a^4+4}{4}$ or $c = b^2 + 2$ again, this is not possible on the basis of the same argument as above case-1.

Thus, in both the cases we get contradiction. So we can conclude that, (a^2, b^2, c) is not a Pythagorean Primitive Triplets.

Hence, $a^4 + b^4 = c^2$ is not true and hence S7.

3. CONCLUSION

Since the inception of Pythagorean era and the time that different mathematicians contributed in this area, innumerable details and facts have been creeped in. We have also substantially added to enhance the accumulated records. Some of the conjectures are added, e.g. S6, and established also while some known and established earlier have been painted in the new format that we have designed amenable to concise the existing proceedings. This pattern does not limit the structure but on the contrary opens wide the gates searching for new avenues to approach.

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ANNEXURES:

A-1 We prove that '1' is not a congruent number.

Proof: Suppose that 1 is congruent number then there exists a right triangle with positive rational sides whose area is 1.

We know that from Pythagorean triplet $\left(a, \frac{a^2 - i^2}{2i}, \frac{a^2 + i^2}{2i}\right)$

Area $= \frac{1}{2}\left(a \times \frac{a^2 - 1}{2}\right)$.

[For 'a' an odd integer other than 1 and $i = 1$ assuring the first primitive triplet.]. Area of the said triangle = 1 implies that

$$1 = \frac{1}{4}a(a^2 - 1) \Rightarrow a(a^2 - 1) = 4$$

$\therefore (a-1)a(a+1) = 4$, which is not possible as the product of three consecutive integers cannot be 4. Hence, the result.

A-2 The following table strengthens conjecture no.-1 and 3.

Numbers of the Form $3(2n + 9) = \boldsymbol{a}$	Number of Triplets	Value of i
$**$ 45,63,81.99,117,153,171, 243,351,459,513,621,675 729,783, 837, 891, 945, 999, ...	1	1
33,39,51,57, ... ,135, ... .567, . ..	2	(1,9), ((1,25), (1,49) ...
105,165,195,231, ...	3	(1,9,25), (1,9,49) ... ..
315,429, ...	4	(1,25,49,81), (1,9,121,169)

It apparently looks that there is some symmetrical pattern in the value of 'i' but it is not easy to generalize as the difference between consecutive numbers keeps on increasing.

Numbers of the Form $4(2n + 3) = \boldsymbol{a}$	Numbers of Triplets	Value of i
20,28, 36,44, 52,68,76 92,100,108,116,124 … ….	2	2, 8
60,84,132,140,156 … ..	3	(2,8,18), ((2,8,50) …
204,228,276, … ….	4	(2,8,18,72), (2,8,50,200) … ..
780,924, … …	6	2,8,18,72,98,242

At this stage it is a note-worthy fact that the even integer $a = 660$ has 7 primitive triplets. The first 10 integers multiples of 660 exhibits different nature in generating triplets. The integer 2640 (= 660 x 4) and 3960 (=660x6) have 6 primitive triplets; 4620 (= 660 x 7) has **13** primitive triplets and the others have 7 primitive triplets.