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Modified weighted (0, 2)-interpolation

Shrawan Kumar

Department of Mathematics and Astronomy,
University of Lucknow, Lucknow.

Pankaj Mathur[1]

Department of Mathematics and Astronomy,
University of Lucknow, Lucknow.

Abstract

The aim of this paper is to give the existence, uniqueness, and explicit representation of the weighted (0, 2)-interpolation polynomials on the roots of all classical orthogonal polynomials.

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1. Introduction

In 1955, P. Turán [9] initiated the study of weighted (0, 2) interpolation, which means the determination of the polynomial $S_n(x)$ of minimum possible degree satisfying the conditions:

$$S_n(x_{j,n}) = \alpha_{j,n}\ , \quad (w(x)S_n(x))''(x_{j,n}) = \beta_{j,n}, \quad j = 1(1)n \tag{1}$$

where $\alpha_{j,n}, \beta_{j,n}$ are given arbitrary real numbers, $w(x) \in C^2(a, b)$ is a weight function and $\{x_{j,n}\}_{j=1}^{n}$ is the set of nodal points. Balázs [1] was the first to consider this problem by taking $x_j's$ as the zeros of the nth Ultraspherical polynomial $P_n^{\alpha}(x)$, $(\alpha > -1)$ and the weight function as $(1-x^2)^{\frac{\alpha+1}{2}}$. He proved that generally there does not exist any

[1]Corresponding author.

polynomial of degree $\leq 2n-1$ satisfying the conditions (1). However, by taking an additional condition:

$$S_n(0) = \sum_{j=1}^{n} \alpha_{j,n} \ell_{j,n}^2(0) \tag{2}$$

where 0 is not a nodal point and $\ell'_{j,n}s$ are the fundamental polynomials of Lagrange interpolation, he proved that there exists a unique interpolatory polynomial $S_n(x)$ of degree $\leq 2n$ (n even) satisfying the conditions (1) and (2). If n is odd, the uniqueness fails to exist. In another paper, Joó and Szili [4] considered the above problem on the zeros of Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, $(\alpha, \beta > -1)$ taking the weight function as $(1-x)^{\frac{\alpha+1}{2}}(1+x)^{\frac{\beta+1}{2}}$ and n such that $P_n^{(\alpha,\beta)}(0) \neq 0$. They proved that there exists a uniquely determined polynomial of degree $\leq 2n$ satisfying the conditions (1) and (2). If n is such that $P_n^{(\alpha,\beta)}(0) = 0$, the uniqueness fails to exist. Szili [8] also considered the weighted (0, 2) interpolation on the roots of classical orthogonal polynomials in the case when 0 is not a nodal point. He [7] has also given a survey of (0, 2) interpolation.

In weighted lacunary interpolation and Pál type interpolation processes, results have been obtained under the special condition of the type (2), which appears to be artificial. Also, in almost every lacunary interpolation it has been proved that, when 0 is a nodal point then either the interpolatory polynomial of minimum possible degree does not exist or if it exists, they are infinitely many.

In this paper we have considered the modified weighted (0, 2) - interpolation by taking the nodal points as the roots of the classical orthogonal polynomials $p_n(x)$, in both the cases when $p_n(0) \neq 0$ or $p_n(0) = 0$. Explicit representation of the interpolatory polynomial has been obtained by replacing the artificial looking condition (2) in each of the cases. Precisely we have shown that there exists a unique interpolatory polynomial $R_n(x)$ of degree $\leq 2n$ satisfying the conditions, for $i = 1, 2, \ldots, n$

$$R_n(x_{i,n}) = y_{i,n}, \quad (wR_n)''(x_{i,n}) = y''_{i,n} \tag{3}$$

$$\begin{cases} R_n(0) = y_{0,n}, & if \quad p_n(0) \neq 0 \\ or & \\ R'_n(0) = y'_{0,n}, & if \quad p_n(0) = 0 \end{cases} \tag{4}$$

where $\{x_{k,n}, k = 1, 2, \ldots, n, n \in N\}$ is a given system of the nodal points in finite or infinite interval (a, b), $w \in C^2(a, b)$ is a weight function and $\{y_{0,n}, y'_{0,n}, y_{k,n}, y''_{k,n}, k = 1, \ldots, n, n \in N\}$ are arbitrarily given real numbers. Other authors [2, 3, 5, 6, 10] have also considered similar modifications in Lacunary and Pál type interpolations.

2. Explicit Representation (When $p_n(0) = 0$)

The interpolatory polynomial $R_n(x)$ of degree $\leq 2n$ satisfying the conditions (3) and (4), when $p_n(0) = 0$, can be uniquely represented as

$$R_n(x) = \sum_{k=1}^{n} y_{k,n} A_{k,n}(x) + \sum_{k=1}^{n} y''_{k,n} B_{k,n}(x) + y'_{0,n} C_{0,n}(x) \tag{5}$$

where $C_{0,n}(x)$, $\{B_{k,n}(x)\}_{k=1}^{n}$ and $\{A_{k,n}(x)\}_{k=1}^{n}$ are the fundamental polynomials of (0, 2) interpolation each of degree $\leq 2n$ satisfying the conditions, for $j, k = 1, 2, \ldots, n$

$$C_{0,n}(x_{j,n}) = 0, \quad (wC_{0,n})''(x_{j,n}) = 0, C'_{0,n}(0) = 1, \tag{6}$$

$$B_{k,n}(x_{j,n}) = 0, (wB_{k,n})''(x_{j,n}) = \delta_{kj}, B'_{k,n}(0) = 0 \tag{7}$$

and

$$A_{k,n}(x_{j,n}) = \delta_{kj}, (wA_{k,n})''(x_{j,n}) = 0, A'_{k,n}(0) = 0 \tag{8}$$

where $w(x) \in C^2$ is a weight function such that

$$\{w(x)p_n(x)\}''_{x_{j,n}} = 0, \quad j = 1, 2, \ldots, n. \tag{9}$$

The explicit form of these polynomials is given in the following

Lemma 2.1. For $p_n(0) = 0$, the interpolatory polynomials $C_{0,n}(x)$, $\{B_{k,n}(x)\}_{k=1}^{n}$ and $\{A_{k,n}(x)\}_{k=1}^{n}$ each of degree $\leq 2n$ satisfying the conditions (6), (7) and (8) respectively are given by

$$C_{0,n}(x) = \frac{p_n(x)}{p'_n(0)}, \tag{10}$$

for $k = 1, 2, \ldots, n$

$$B_{k,n}(x) = \frac{p_n(x)}{2w(x_{k,n})p'_n(x_{k,n})} \int_0^x \ell_{k,n}(x)dx \tag{11}$$

where

$$\ell_{k,n}(x) = \frac{p_n(x)}{(x - x_{k,n})p'_n(x_{k,n})} \tag{12}$$

and

$$A_{k,n}(x) = \ell^2_{k,n}(x) + \frac{p_n(x)}{p'_n(x_{k,n})} \int_0^x \frac{\ell'_{k,n}(x) + \left[a_{k,n}(x - x_{k,n}) - \ell'_{k,n}(x_{k,n})\right] \ell_{k,n}(x)}{(x - x_{k,n})} dx \tag{13}$$

where $\ell_{k,n}(x)$ is given by (12) and

$$a_{k,n} = \frac{w''(x_{k,n})}{2w(x_{k,n})} + \frac{2w'(x_{k,n})}{w(x_{k,n})}\ell'_{k,n}(x_{k,n}) + 2(\ell'_{k,n}(x_{k,n}))^2.$$

Proof. Obviously, $C_{0,n}(x)$ given by (10) is a polynomial of degree $\leq 2n$ with $C_{0,n}(x_{j,n}) = 0$, $j = 1, 2, \ldots, n$, $[w(x)C_{0,n}(x)]''_{x_{j,n}} = 0$, due to (9) and $C'_{0,n}(0) = 1$. Thus $C_{0,n}(x)$ given by (10) satisfy all the conditions (6).

Since $\ell_{k,n}(x)$, given by (12), is a polynomial of degree $\leq n-1$ hence $B_{k,n}(x)$, given by (11), is a polynomial of degree $\leq 2n$ such that $B_{k,n}(x_{j,n}) = 0$, $j = 1, 2, \ldots, n$ and $B'_{k,n}(0) = 0$. Lastly, multiplying (7) by $w(x)$ and differentiating twice, then due to (9), we get $[w(x)B_{k,n}(x)]''_{x_{j,n}} = \delta_{jk}$. Thus $B_{k,n}(x)$ given by (11) satisfy all the conditions (7).

Since,

$$\lim_{x \to x_{k,n}} \left[\ell'_{k,n}(x) + \left\{a_{k,n}(x - x_{k,n}) - \ell'_{k,n}(x_{k,n})\right\} \ell_{k,n}(x)\right] = 0$$

hence $\{A_{k,n}(x)\}_{k=1}^n$ given by (13) is a polynomial of degree $\leq 2n$ with $A_{k,n}(x_{j,n}) = \delta_{jk}$, $j, k = 1, 2, \ldots, n$ and $A'_{k,n}(0) = 0$. On multiplying (13) by $w(x)$ and differentiating twice, we have $[w(x)A_{k,n}(x)]''_{x=x_{j,n}} = 0$, $j = 1, 2, \ldots, n$. Thus $A_{jk}(x)$, given by (13) satisfies all the conditions (8) which proves the Lemma. ■

3. Explicit Representation (When $p_n(0) \neq 0$)

The interpolatory polynomial $R_n(x)$ of degree $\leq 2n$ satisfying the conditions (3) and (4), when $p_n(0) \neq 0$, can be uniquely represented as

$$R_n(x) = \sum_{k=0}^{n} y_{k,n} A^*_{k,n}(x) + \sum_{k=1}^{n} y''_{k,n} B^*_{k,n}(x) \tag{14}$$

where $\{A^*_{k,n}(x)\}_{k=0}^n$, $\{B^*_{k,n}(x)\}_{k=1}^n$ are the fundamental polynomials of (0, 2) interpolation each of degree $\leq 2n$. On taking, $x_{0,n} = 0$, these fundamental polynomials satisfy the conditions, for $k = 1, 2, \ldots, n$

$$\begin{cases} B^*_{k,n}(x_{j,n}) = 0, \quad j = 0, 1, \cdots, n \\ (wB^*_{k,n})''(x_{j,n}) = \delta_{kj}, \quad j = 1, \cdots, n \end{cases} \tag{15}$$

and for $k = 0, 1, \ldots, n$

$$\begin{cases} A^*_{k,n}(x_{j,n}) = \delta_{kj}, \quad j = 0, 1, \ldots, n \\ (wA^*_{k,n})''(x_{j,n}) = 0 \quad j = 1, \ldots, n. \end{cases} \tag{16}$$

The explicit form of these polynomials is given in the following

Lemma 3.1. For $p_n(0) \neq 0$, the interpolatory polynomials $\{A^*_{k,n}(x)\}_{k=0}^n$, $\{B^*_{k,n}(x)\}_{k=1}^n$ each of degree $\leq 2n$ satisfying the conditions (15) and (16) respectively are given by for $k = 1, 2, \ldots, n$

$$B^*_{k,n}(x) = B_{k,n}(x)$$

where $\{B_{k,n}(x)\}_{k=1}^n$ are given by (11),

$$A^*_{0,n}(x) = \frac{p_n(x)}{p_n(0)},$$

and for $k = 1, 2, \ldots, n$

$$A^*_{k,n}(x) = A_{k,n}(x) - \ell^2_{k,n}(0)A_{0,n}(x)$$

where $\{A_{k,n}(x)\}_{k=1}^n$ are given by (13).

The proof of this lemma is similar to that of Lemma 2.1, we omit details.

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