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Approximate analytical solution for the fingero-imbibition phenomenon by optimal homotopy analysis method

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Abstract

In the present paper, we apply an optimal homotopy analysis method to obtain approximate analytical solution of the nonlinear partial differential equation governing fingero-imbibition phenomenon with appropriate boundary conditions. The discrete squared residual is used to obtain the optimal value of convergence-control parameter to guarantee the convergence of the obtained approximate series solution. The solution represents the saturation of injected water.

AMS subject classification:
Keywords: Fluid flow, Homogeneous porous medium, Fingero-imbibition phenomenon, Optimal Homotopy analysis method, Discrete Squared Residual.

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1. Introduction

This work presents an important phenomenon of fingero-imbibition in double phase flow through homogeneous porous medium which is due to the simultaneous occurrence of fingering and imbibition in the porous medium.

It is assumed that injecting phase (water) is less viscous than the resident phase (oil), initiates the process of imbibition and as a consequence, the resident phase is pushed by the injected phase giving rise to the fingering phenomenon.

When a porous medium filled with one phase (oil) is brought into the contact of another phase (water) which is preferentially wetting, then the spontaneous flow of the wetting phase into the medium and counter flow of the resident phase from the medium without any external force. This is known as imbibition phenomenon. Besides this if a phase (oil) contained in a porous medium is displaced by another phase of lesser viscosity, then instead of regular displacement of the whole front, protuberances may occur which shoot through the porous medium at relatively very great speed giving rise to the fingering phenomenon. This simultaneous occurrence of both phenomena was termed as fingero-imbibition by Verma.

Many researchers have discussed this problem from different viewpoints and solved by different methods: Verma [14], Mehta [15], Patel and Mehta [1], Yadav and Mehta [4], Mishra and Pradhan [6], Patel and Rabari [3], Patel and Desai [21] etc. In this work, we have obtained the approximate analytical solution of this problem by optimal homotopy analysis method.

In this work, the underlying assumptions are that the two phases are immiscible and the injected phase is less viscous as well as preferentially wetting.

This phenomenon has gained considerable current interest in petroleum reservoir systems as well as the hydrological systems. The governing equation of fingero-imbibition is the nonlinear partial differential equation and its approximate analytical solution is obtained by optimal homotopy analysis method with appropriate boundary conditions.

2. Problem Formulation

Let injected phase (i) and native phase (n) be two immiscible phases governed by Darcy's law and the velocities of water and oil be expressed as [11]

$$V_i = -\frac{k_i}{\mu_i} K \frac{\partial p_i}{\partial x} \tag{2.1}$$

$$V_n = -\frac{k_n}{\mu_n} K \frac{\partial p_n}{\partial x} \tag{2.2}$$

where V_i and V_n are the velocities of water and oil respectively, K is the permeability of the homogeneous porous medium which is constant, k_i and k_n are the relative permeabilities of water and oil respectively, μ_i and μ_n are the constant viscosities of water and oil respectively, p_i and p_n are the pressures of water and oil respectively.

Regarding phase densities as constant, the equations of continuity (mass balance equations) are

$$P\frac{\partial S_i}{\partial t} + \frac{\partial V_i}{\partial x} = 0 \tag{2.3}$$

$$P\frac{\partial S_n}{\partial t} + \frac{\partial V_n}{\partial x} = 0 \tag{2.4}$$

where P is the porosity of the medium regarded as constant. It is obvious that in fingero-imbibition phenomenon [13]

$$V_i + V_n = 0 \tag{2.5}$$

The capillary pressure p_c is a function of the phase saturation. According to Scheidegger [13], it may be written as

$$p_c(S_i) = p_n - p_i \tag{2.6}$$

and

$$p_c = -\beta S_i \tag{2.7}$$

where β is a constant.

For definiteness of the mathematical analysis, the relationship between phase saturation and relative permeability as given by Scheidegger and Johnson [16] is used here.

$$k_i = S_i \quad and \quad k_n = S_n \tag{2.8}$$

From the definition of phase saturation,

$$S_i + S_n = 1 \tag{2.9}$$

Using Eqs.(2.1) and (2.2) in (2.5), we have

$$\frac{k_i}{\mu_i}K\frac{\partial p_i}{\partial x} + \frac{k_n}{\mu_n}K\frac{\partial p_n}{\partial x} = 0 \tag{2.10}$$

Using Eq. (2.6), this becomes

$$\left(\frac{k_i}{\mu_i} + \frac{k_n}{\mu_n}\right)\frac{\partial p_i}{\partial x} + \frac{k_n}{\mu_n}\frac{\partial p_c}{\partial x} = 0 \tag{2.11}$$

Simplyfying, we obtain

$$\frac{\partial p_i}{\partial x} = -\frac{k_n}{\mu_n}\left(\frac{k_i}{\mu_i} + \frac{k_n}{\mu_n}\right)^{-1}\frac{\partial p_c}{\partial x} \tag{2.12}$$

On substituting the value of $\frac{\partial p_i}{\partial x}$ in Eq. (2.1), we get

$$V_i = K\frac{k_i}{\mu_i}\frac{k_n}{\mu_n}\left(\frac{k_i}{\mu_i}+\frac{k_n}{\mu_n}\right)^{-1}\frac{\partial p_c}{\partial x} \tag{2.13}$$

Using Eq. (2.13) in (2.3), we get

$$P\frac{\partial S_i}{\partial t}+\frac{\partial}{\partial x}\left[K\frac{k_i}{\mu_i}\frac{k_n}{\mu_n}\left(\frac{k_i}{\mu_i}+\frac{k_n}{\mu_n}\right)^{-1}\frac{\partial p_c}{\partial x}\right]=0 \tag{2.14}$$

According to Scheidegger [13], it is assumed that

$$\frac{k_i}{\mu_i}\frac{k_n}{\mu_n}\left(\frac{k_i}{\mu_i}+\frac{k_n}{\mu_n}\right)^{-1}\approx\frac{k_n}{\mu_n}$$

Hence Eq. (2.14) reduces to

$$P\frac{\partial S_i}{\partial t}+\frac{\partial}{\partial x}\left[K\frac{k_n}{\mu_n}\frac{dp_c}{dS_i}\frac{\partial S_i}{\partial x}\right]=0 \tag{2.15}$$

Using (2.7), (2.8) and (2.9) into (2.15), we get

$$P\frac{\partial S_i}{\partial t}=\frac{\beta K}{\mu_n}\frac{\partial}{\partial x}\left[(1-\ S_i)\frac{\partial S_i}{\partial x}\right] \tag{2.16}$$

Using dimensionless variables

$$X=\frac{x}{L},\ \ T=\frac{\beta K t}{P\mu_n L^2},$$

Eq. (2.16) reduces to

$$\frac{\partial S_i}{\partial T}=\frac{\partial}{\partial X}\left[(1-S_i)\frac{\partial S_i}{\partial X}\right] \tag{2.17}$$

Eq. (2.17) is the nonlinear partial differential equation of the fingero-imbibition phenomenon for the flow of two immiscible phases in the homogeneous porous medium. Let at the common interface, the saturation of injected water be linear function of time, that is

$$S_i(0,T)=aT \quad for\ T>0 \tag{2.18}$$

where a is constant.

Since, it is assumed that the porous medium is completely surrounded by an impermeable surface except for one end, we consider

$$\frac{\partial S_i}{\partial X}(1,T)=0 \quad for\ T>0 \tag{2.19}$$

We solve Eq. (2.17) together with boundary conditions (2.18) and (2.19) using optimal homotopy analysis method.

3. Solution using the Optimal Homotopy Analysis Method

We choose

$$S_{i_0}(X, T) = aT[e^{-X} + Xe^{-1}] \tag{3.1}$$

as the initial approximation of $S_i(X, T)$ which satisfies boundary conditions (2.18) and (2.19).

Besides we choose the linear operator as

$$\mathcal{L}[\phi(X, T; q)] = \frac{\partial^2 \phi(X, T; q)}{\partial X^2} \tag{3.2}$$

with the property

$$\mathcal{L}[f] = 0 \quad \text{when } f = 0. \tag{3.3}$$

Furthermore, based on governing Eq. (2.17), we define such a nonlinear operator

$$\begin{aligned} \mathcal{N}[\phi(X, T; q)] = {} & \frac{\partial^2 \phi(X, T; q)}{\partial X^2} - \phi(X, T; q)\frac{\partial^2 \phi(X, T; q)}{\partial X^2} \\ & - \left\{\frac{\partial \phi(X, T; q)}{\partial X}\right\}^2 - \frac{\partial \phi(X, T; q)}{\partial T} \end{aligned} \tag{3.4}$$

Let c_0 denote a nonzero auxiliary parameter. According to Liao [23], the zeroth order deformation equation is

$$(1 - q)\mathcal{L}[\phi(X, T; q) - S_{i_0}(X, T)] = c_0 q H(X, T)\mathcal{N}[\phi(X, T; q)] \tag{3.5}$$

where $q \in [0, 1]$ is the embedding parameter, $H(X, T)$ is nonzero auxiliary function and $\phi(X, T; q)$ is an unknown function. Obviously, when $q = 0$ and $q = 1$, we have from Eq. (3.3) and Eq. (3.5),

$$\phi(X, T; 0) = S_{i_0}(X, T) \tag{3.6}$$

and

$$\phi(X, T; 1) = S_i(X, T) \tag{3.7}$$

Therefore, the solution $\phi(X, T; q)$ varies from the initial guess $S_{i_0}(X, T)$ to the solution $S_i(X, T)$ of the Eq. (2.17) as the embedding parameter q increases from 0 to 1. Obviously, $\phi(X, T; q)$ is determined by the auxiliary linear operator $\mathcal{L}$, the initial guess $S_{i_0}(X, T)$ and the auxiliary parameter c_0. We have great freedom to select all of them. Assuming that all of them are so properly chosen that the Taylor series

$$\phi(X, T; q) = S_{i_0}(X, T) + \sum_{m=1}^{\infty} S_{i_m}(X, T) q^m \tag{3.8}$$

exists and converges at $q = 1$, we have the homotopy-series solution

$$S_i(X,T) = S_{i_0}(X,T) + \sum_{m=1}^{\infty} S_{i_m}(X,T) \tag{3.9}$$

where

$$S_{i_m}(X,T) = \frac{1}{m!}\frac{\partial^m \phi(X,T;q)}{\partial q^m}\bigg|_{q=0} \tag{3.10}$$

Differentiating the zeroth order deformation equation Eq. (3.5) m times with respect to the embedding parameter q and then dividing by $m!$ and finally setting $q = 0$, we have the so called high order deformation equations

$$\mathcal{L}[S_{i_m}(X,T) - \chi_m S_{i_{m-1}}(X,T)] = c_0 H(X,T)\mathcal{R}_m(X,T) \tag{3.11}$$

subject to the conditions

$$S_{i_m}(0,T) = 0, \quad \frac{\partial S_{i_m}}{\partial X}(1,T) = 0, \quad \text{for } T > 0, \quad m \geq 1 \tag{3.12}$$

where

$$\mathcal{R}_m(X,T) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\mathcal{N}[\phi(X,T;q)]}{\partial q^{m-1}}\bigg|_{q=0} \tag{3.13}$$

and

$$\chi_m = \begin{cases} 0 & \text{if } m \leq 1, \\ 1 & \text{if } m > 1. \end{cases} \tag{3.14}$$

It is very important to emphasize that Eq. (3.11) is linear for all $m \geq 1$. For simplicity, assume $H(X,T) = 1$. The solution of the mth order deformation equation (3.11) for $m \geq 1$ is

$$S_{i_m}(X,T) = \chi_m S_{i_{m-1}}(X,T) + c_0\mathcal{L}^{-1}[\mathcal{R}_m(X,T)] + C_1 X + C_2 \tag{3.15}$$

where the constants of integration C_1 and C_2 are determined using (3.12). The solution of (3.15) for $m = 1$ is

$$\begin{aligned} S_{i_1}(X,T) = c_0\Big[& aTe^{-X} - ae^{-X} - \frac{a^2T^2e^{-2X}}{2} \\ & - a^2T^2Xe^{-1}e^{-X} - \frac{a^2T^2e^{-2}X^2}{2} \\ & - \frac{ae^{-1}X^3}{6} + aTe^{-1}X - \frac{aXe^{-1}}{2} \\ & + a + \frac{a^2T^2}{2} - aT\Big] \end{aligned} \tag{3.16}$$

$S_{i_m}(X, T)$ can be obtained similarly. Therefore, the solution of Eq. (2.17) is

$$S_i(X, T) = S_{i_0}(X, T) + S_{i_1}(X, T) + \cdots \tag{3.17}$$

Hence

$$\begin{aligned} S_i(X, T) &= aT[e^{-X} + Xe^{-1}] \\ &\quad + c_0\Big[aTe^{-X} - ae^{-X} - \frac{a^2T^2e^{-2X}}{2} \\ &\quad - a^2T^2Xe^{-1}e^{-X} - \frac{a^2T^2e^{-2}X^2}{2} \\ &\quad - \frac{ae^{-1}X^3}{6} + aTe^{-1}X - \frac{aXe^{-1}}{2} \\ &\quad + a + \frac{a^2T^2}{2} - aT\Big] + \cdots \end{aligned} \tag{3.18}$$

Eq. (3.18) represents the saturation of injected water at distace X and at time T which satisfies boundary conditions.

The convergence of this series depends on the proper selection of auxiliary parameter c_0 which is also known as the convergence-control parameter. The value of c_0 can be determined optimally by minimizing squared residual function E_m.

As given by Liao [26], the discrete squared residual at the mth order of approximation is

$$E_m = \frac{1}{(M+1)(N+1)} \sum_{i=0}^{M} \sum_{j=0}^{N} \left\{ \mathcal{N}\left[\sum_{n=0}^{m} S_{i_n}\left(\frac{i}{M}, \frac{j}{N}\right) \right] \right\}^2 \tag{3.19}$$

Since the squared residual E_m is function of c_0, we can find the optimal value of c_0 from

$$\frac{dE_m(c_0)}{dc_0} = 0 \tag{3.20}$$

This optimization method for obtaining the parameter c_0 has been applied recently to a number of problems for nonlinear ordinary and partial differential equations [17, 19, 20, 24, 27, 28, 29, 30, 31, 32].

Figure 1 shows the curves of discrete squared residual E_m versus c_0.

The optimal value of c_0 is found by the minimum of E_8 using mathematica. According to Table 1, E_8 attains its minimum value 6.86985×10^{-10} at $c_0 = -0.836116$ which can be noticed in Figure 1 also.

4. Numerical and Graphical Representation

The numerical values of the saturation of injected water are obtained upto 8th order of approximation using $c_0 = -0.836116$ and $a = 0.005$. Table 3 indicates the numerical

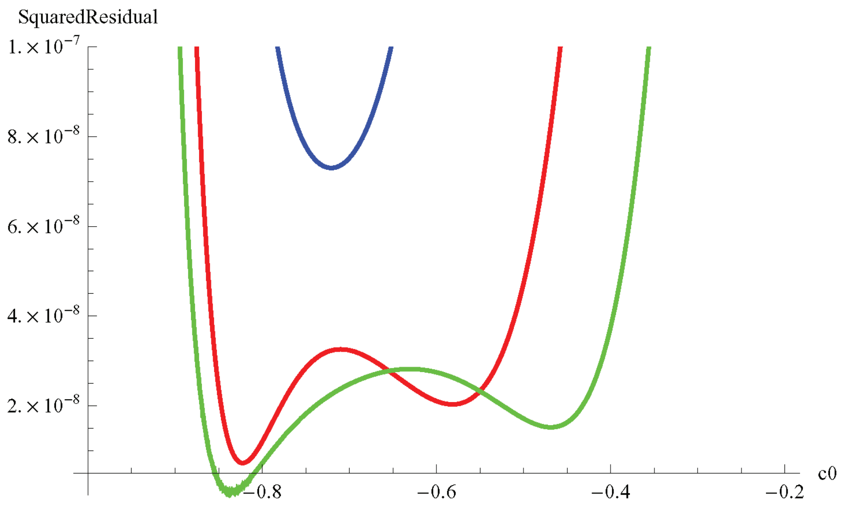

Figure 1: E_m versus c_0. Blue: E_4, Red: E_6, Green: E_8.

Table 1: Optimal value of c_0 at different order of approximations

m, Order of approximation	Optimal value of c_0	Minimum value of E_m
2	−0.63154	2.63105×10^{-6}
4	−0.720796	7.30153×10^{-8}
6	−0.82238	7.24932×10^{-9}
8	−0.836116	6.86985×10^{-10}

Table 2: Discrete squared residual E_m of governing equation Eq. (2.17) by means of $c_0 = -0.836116$

m, Order of approximation	Discrete squared residual E_m
2	4.76732×10^{-6}
4	2.51007×10^{-7}
6	1.17267×10^{-8}
8	1.23155×10^{-9}

values of saturation of injected water for different distance X and time T upto 8th order approximation. Figure 2 represents the graph of $S_i(X, T)$ versus time T for fixed values of distance $X = 0, 0.1, \dots, 1$. Numerical values of Table 3 are used for figure 2.

5. Conclusion

The optimal homotopy analysis method is applied to find the approximate analytical solution of the nonlinear partial differential equation (2.17) with boundary conditions (2.18) and (2.19). The optimal value of the convergence-control parameter is obtained

Table 3: Numerical values of the Saturation $S_i(X, T)$ of injected water

X	$T = 0.5$	$T = 0.6$	$T = 0.7$	$T = 0.8$	$T = 0.9$	$T = 1$
0	2.5E-3	3E-3	3.5E-3	4E-3	4.5E-3	5E-3
0.1	2.0248E-3	2.5245E-3	3.0243E-3	3.5240E-3	4.0238E-3	4.5236E-3
0.2	1.5998E-3	2.0993E-3	2.5989E-3	3.0984E-3	3.5980E-3	4.0975E-3
0.3	1.2250E-3	1.7244E-3	2.2237E-3	2.7231E-3	3.2225E-3	3.7218E-3
0.4	9.0036E-4	1.3996E-3	1.8988E-3	2.3980E-3	2.8977E-3	3.3396E-3
0.5	6.2576E-4	1.1125E-3	1.6239E-3	2.1230E-3	2.3622E-3	3.1211E-3
0.6	4.0116E-4	9.0012E-4	1.3991E-3	1.8980E-3	2.3970E-3	2.8959E-3
0.7	2.2652E-4	7.2539E-4	1.2243E-3	1.7231E-3	2.2220E-3	2.7208E-3
0.8	1.0181E-4	6.0061E-4	1.0994E-3	1.5982E-3	2.0970E-3	2.5958E-3
0.9	2.6985E-5	5.2573E-4	1.0245E-3	1.5233E-3	2.0220E-3	2.5208E-3
1	4.731E-7	5.0080E-4	9.9956E-4	1.4983E-3	1.9971E-3	2.4958E-3

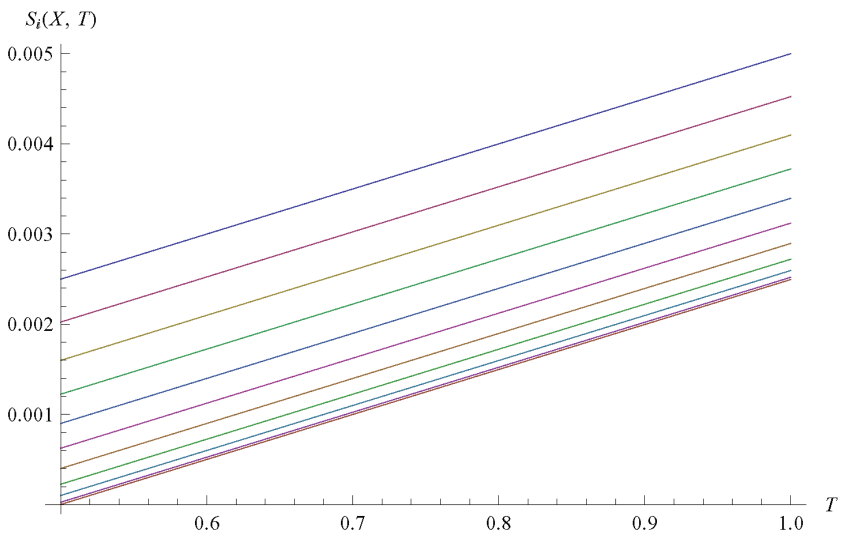

Figure 2: $S_i(X, T)$ versus T for fixed values of distance $X = 0$ (lowermost graph), $0.1, \ldots, 1$ (uppermost graph).

by the minimum of the discrete squared residual. It is found that the saturation of the injected water $S_i(X, T)$ at distance X increases smoothly with increase in time T.

References

[1] Patel, K., Mehta, M. N., Patel, T., Power series solution of fingero-imbibition in double phase flow through homogeneous porous media, *International Journal of Applied Mathematics and Mechanics*, **7**(8) (2011), 65–77.

[2] Nikookar, M., Mohammad, A.H., Study of oil recovery in naturally fractured reser-

voirs considering impact of parameters included in performance of spontaneous imbibition mechanism, *Advances in Environmental Research*, **40** (2015).

[3] Patel, A.V., Rabari, N.S., Bhathawala P.H., A mathematical model for fingero-imbibition phenomenon in a cracked porous media with magnetic fluid, *International Journal of Advanced Technology in Engineering and Science*, **4**(4) (2016).

[4] Yadav, S.R., Mehta, M.N., Analytical approximate expression for cocurrent imbibition during immiscible two-phase flow through porous media, *Mathematical Problems in Engineering*, (2014).

[5] Patel, A.V., Rabari, N.S., Bhathawala P.H., Numerical solution of imbibition phenomenon in a homogeneous medium with magnetic fluid, *IOSR Journal of Mathematics*, **11**(4) (2015).

[6] Mishra, D., Pradhan V., Mehta, M., Solution of counter current imbibition phenomenon in homogeneous porous media by homotopy perturbation transform method (HPTM), *Advances in Water Science and Technology*, **2**(2) (2015).

[7] Melean, Y., Broseta, D., Blossey, R., Imbibition fronts in porous media: effects of initial wetting fluid saturation and flow rate, *Journal of Petroleum Science and Engineering*, **39** (2003), 327–336.

[8] Verma, A. P., Statistical behaviour of fingering in a displacement process in heterogeneous porous medium with capillary pressure, *Canadian Journal of Physics*, **47**(3) (1969), 319–324.

[9] Mehta, M. N., Verma, A. P., Composite expansion solution of fingero-imbibition in double phase flow through underground porous medium, *Proc. Indian Acad. Sci.* **87** A(2) (1978), 65–70.

[10] Desai, N. B., *The study of problems arises in single phase and multiphase flow through porous media*, Ph.D. Thesis, South Gujarat University, Surat, India, 2002.

[11] Bear, J., *Dynamics of fluids in porous media*, American Elsevier Publishing Company, Inc., New York, 1972.

[12] Muskat, M., *The flow of homogeneous fluids through porous media*, First edition, McGraw-Hill Book Company, Inc., New York and London, 1937.

[13] Scheidegger, A. E., *The Physics of flow through porous media*, revised edition, University of Toronto Press, Toronto, 1960.

[14] Verma, A. P., Fingero-imbibition in artificial replenishment ground water through cracked porous medium, *Water Resources Research*, **6**(3) (1970), 906–911.

[15] Mehta, M. N., *Asymptotic expansions of fluid flow through porous media*, Ph.D. Thesis, South Gujarat University, Surat, India, 1977.

[16] Scheidegger, A. E., Johnson, E. F., The statistically behaviour of instabilities in displacement process in porous media, *Canadian Journal of Physics*, **39**(2) (1961), 326–334.

[17] Liao, S. J., An optimal homotopy analysis approach for strongly nonlinear differential equations, *Commun. Nonlinear Sci. Numer. Simul.*, **15** (2010), 2003–2016.

[18] Yabushita, K., Yamashita, M., Tsuboi, K., An analytic solution of projectile motion with the quadratic resistance law using the homotopy analysis method, *J.Phys. A-Math.Theor.*, **40** (2007), 8403–8416.

[19] Nadjafi, J.S., Jafari, J.S., Comparison of Liao's optimal HAM and Niu's one-step optimal HAM for solving integro-differential equations, *Journal of Applied Mathematics and Bioinformatics*, **1**(2) (2011), 85–98.

[20] Baxter, M., Robert, A., Gorder, V., Vajravelu, K., On the choice of auxiliary linear operator in the optimal homotopy analysis of the Cahn-Hilliard initial value problem, *Numer Algor*, **66**(2) (2014), 269–298.

[21] Patel, M. A., Desai, N. B., Homotopy analysis solution of countercurrent imbibition phenomenon in inclined homogeneous porous medium, *Global Journal of Pure and Applied Mathematics*, **12**(1) (2016), 1035–1052.

[22] Liao, S. J., Advances in the homotopy analysis method, *World Scientific*, (2013).

[23] Liao, S. J., *Beyond perturbation: Introduction to the homotopy analysis method*, Chapman and Hall/CRC Press, Boca Raton, 2003.

[24] Saeidian, J., Javadi, S., Some notes on the convergence control parameter in the framework of the homotopy analysis method, *Journal of Mathematics and Computer Science*, **9** (2014), 103–110.

[25] Vajravelu, K., Gorder, V., Robert, A., *Nonlinear flow phenomena and homotopy analysis: Fluid flow and Heat transfer*, Higher Education Press, Beijing and Springer-Verlag Berlin Heidelberg, 2012.

[26] Liao, S. J., *Homotopy analysis method in nonlinear differential equations*, Higher Education Press, Beijing and Springer-Verlag Berlin Heidelberg, 2012.

[27] Prajapati, D. J., Desai, N. B., The solution of immiscible fluid flow by means of optimal homotopy analysis method, *International Journal of Computer and Mathematical Sciences*, **4**(8) (2015).

[28] Prajapati, D. J., Desai, N. B., Application of the basic optimal homotopy analysis method to fingering phenomenon, *Global Journal of Pure and Applied Mathematics*, **12**(3) (2016), 2011–2022.

[29] Fan, T., You, X., Optimal homotopy analysis method for nonlinear differential equations in the boundary layer, *Numer Algor*, **62** (2013), 337–354.

[30] Hassan, H.N., Semary, M.S., Analytic approximate solution for the Bratu's problem by optimal homotopy analysis method, *International Scientific Publications and Consulting Services*, (2013).

[31] Abbasbandy, S., Jalili, M., Determination of optimal convergence-control parameter value in homotopy analysis method, *Numer Algor*, (2013).

[32] Mallory, K., Gorder, V., Optimal homotopy analysis and control of error for solutions to the non-local Whitham equation, *Numer Algor*, (2013).