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Product of multiplication, composition and differentiation operators on weighted Hardy spaces

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Abstract

Let φ be an analytic self-map of the open unit disc $\mathbb{D}$ in the finite complex plane $\mathbb{C}$. Let C_φ , M_ψ and D be the composition, multiplication and Differentiation operators defined by $C_\varphi f = f o\varphi$, $M_\psi f = \psi f$ and $Df = f'$ respectively. In this paper, we shall study the boundedness and compactness of the operator $M_\psi C_\varphi D$ defined by $M_\psi C_\varphi D f = \psi(f' o \Phi)$ on weighted Hardy spaces by using the orthonormal basis of the weighted Hardy spaces.

Keywords: Composition operator, Multiplication operator, Differentiation operator and Weighted Hardy spaces.

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1. Introduction

Throughout this paper, by $\mathbb{D}$ we shall denote the open unit disc of the finite complex plane $\mathbb{C}$; by $\partial\mathbb{D}$ the boundary of $\mathbb{D}$; by $H(\mathbb{D})$ the set of all complex valued analytic functions on $\mathbb{D}$ and by φ, the analytic self-map of $\mathbb{D}$. Let $\beta = \{\beta_n\}_{n=0}^{\infty}$ be the sequence of positive numbers such that $\beta_0 = 1$ and $lim_{n\to\infty} \frac{\beta_{n+1}}{\beta_n} = 1$. Then the weighted Hardy spaces $H^2(\beta)$ is the Banach space of all analytic functions f on the open unit disk $\mathbb{D}$:

$$H^2(\beta) = \left\{ f: z \to \sum_{n=0}^{\infty} a_n z^n \quad such\ that\ \|f\|^2{}_{H^2(\beta)} = \sum_{n=0}^{\infty} |a_n|^2 \beta^2{}_n < \infty \right\}$$

where $\|f\|_{H^2(\beta)}$ is a norm on $H^2(\beta)$.

If $\beta \equiv 1$, then $H^2(\beta)$ becomes the classical Hardy space $H^2(D)$. Also $H^2(\beta)$ is a Hilbert space w.r.t the inner product.

$$\langle f, g \rangle = \sum_{n=0}^{\infty} a_n \overline{b_n} \beta_n^2 ,$$

where $f, g \in H^2(\beta)$. For a detailed discussion on $H^2(\beta)$ one can see [15]. Associated with φ the classical linear operator $C_\varphi : H(\mathbb{D}) \to H(\mathbb{D})$ defined by $f \to f\, o\varphi$ is known as composition operator induced by the self-map φ. If ψ is an analytic function from the open unit disc $\mathbb{D}$ to $\mathbb{C}$, then associated with ψ the multiplication operator M_ψ is defined as $M_\psi f = \psi f$. It has been known that the composition operator C_φ is bounded on almost all spaces of analytic functions, for example see [3, 4, 5], and D (the differentiation operator) is usually unbounded on spaces of analytic functions. Recently, the above defined operators has received the attention of many researcher see, for example [1, 2, 7, 9, 10, 11, 12, 17, 21, 22, 23, 24]. In [7], Hibschweiles and Portony defined the product $C_\varphi D$ and DC_φ and studied the boundedness and compactness of these operators between Bergman and Hardy spaces by using the Carleson-type measure, where as in [12], the author studied the boundedness and compactness of $C_\varphi D$ and DC_φ between Hardy type spaces.

This paper is organized as follows. In the second section, we shall discuss the boundedness of the operator $M_\psi C_\varphi D$ on weighted Hardy spaces $H^2(\beta)$. In the third section, we shall study the compactness of the operators $M_\psi C_\varphi D$ on weighted Hardy spaces $H^2(\beta)$, and in the final section, we shall give necessary and sufficient condition for the operator $M_\psi C_\varphi D$ to be the Hilbert-Schmidt operator on weighted Hardy spaces $H^2(\beta)$.

2. Boundedness of $M_\psi C_\varphi D$

In this section, we shall give the necessary and sufficient condition for the boundedness of the operators $M_\psi C_\varphi D$. Recall that a linear operator T on a Hilbert space X is bounded if it takes every bounded set in X into a bounded set in X.

Theorem 2.1. *Let* $\varphi : \mathbb{D} \to \mathbb{D}$ *be an analytic map such that* $\{\varphi^n : n \geq 0\}$ *be an orthogonal family. Then the composition operator* $M_\psi C_\varphi D : H^2(\beta) \to H^2(\beta)$ *is bounded iff*

$$\|\psi\varphi^{n-1}\| \leq \frac{M}{n}\beta(n).$$

Proof: Suppose that the operator $M_\psi C_\varphi D : H^2(\beta) \to H^2(\beta)$ Is bounded. Then, there exists $M > 0$ such that

$$\left\|M_\psi C_\varphi Df\right\|_{H^2(\beta)} \leq M\|f\|_{H^2(\beta)} \quad \forall f \in H^2(\beta) \tag{2.1}$$

Let $(z) = z^n$. Then $f \in H^2(\beta)$ so from equation (2.1), we have

$$\left\|M_\psi C_\varphi Dz^n\right\|_{H^2(\beta)} = \|\psi n\varphi^{n-1}\| < M\|z^n\|_{H^2(\beta)}.$$

This implies that

$$\|\psi\varphi^{n-1}\| \leq \frac{M}{n}\beta(n).$$

Conversely, suppose that

$$\|\psi\varphi^{n-1}\| \leq \frac{M}{n}\beta(n).$$

To prove that $M_\psi C_\varphi D$ is bounded, let $f \in H^2(\beta)$ be any element such that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Than, we have

$$\left\|M_\psi C_\varphi Df\right\|_{H^2(\beta)}^2 = \|\textstyle\sum_{n=0}^{\infty} a_n \psi n \varphi^{n-1}\|_{H^2(\beta)}^2$$

$$= \sum_{n=0}^{\infty} |a_n|^2 n^2 \|\psi.\varphi^{n-1}\|_{H^2(\beta)}^2$$

$$\leq \sum_{n=0}^{\infty} |a_n|^2 n^2 \frac{M^2}{n^2}\beta^2(n)$$

$$= M^2 \sum_{n=0}^{\infty} |a_n|^2 \left(\beta(n)\right)^2$$

$$= M^2\|f\|^2.$$

This implies that $\left\|M_\psi C_\varphi Df\right\|_{H^2(\beta)} \leq M\|f\|_{H^2(\beta)}$ and so, the operator $M_\psi C_\varphi D$ is bounded.

Corollary 2.2. *Let* $\varphi : \mathbb{D} \to \mathbb{D}$ *be an analytic map such that* $\{\varphi^n : n \geq 0\}$ *be an orthogonal family. Then the composition operator* $C_\varphi D : H^2(\beta) \to H^2(\beta)$ *is bounded iff*

$$\|\varphi^{n-1}\| \leq \frac{M}{n}\beta(n).$$

Proof: Let $\psi(z) = 1$, then $M_\psi C_\varphi D = C_\varphi D$. From theorem (2.1) in [10. p.581], we find that

$$C_\varphi D : H^2(\beta) \to H^2(\beta) \text{ is bounded}$$

iff

$$\|\varphi^{n-1}\| \leq \frac{M}{n}\beta(n).$$

Corollary 2.3. *Let* $\varphi : \mathbb{D} \to \mathbb{D}$ *be an analytic map such that* $\{\varphi^n : n \geq 0\}$ *be an orthogonal family. Then the composition operator* $DC_\varphi : H^2(\beta) \to H^2(\beta)$ *is bounded iff*

$$\|\varphi^{n-1}\phi'\| \leq \frac{M}{n}\beta(n).$$

Proof: Let $\psi(z) = \varphi'(z)$. Then $M_\psi C_\varphi D = D\, C_\varphi$. From theorem (2.3) in [10.p.581], we find that

$$DC_\varphi : H^2(\beta) \to H^2(\beta) \text{ is bounded}$$

iff

$$\|\varphi^{n-1}\varphi'\| \leq \frac{M}{n}\beta(n).$$

Corollary 2.4. *Let* $\varphi : \mathbb{D} \to \mathbb{D}$ *be an analytic map such that* $\{\varphi^n : n \geq 0\}$ *be an orthogonal family. Then the composition operator* $M_\psi D : H^2(\beta) \to H^2(\beta)$ *is bounded iff*

$$\|\varphi z^{n-1}\varphi'\| \leq \frac{M}{n}\beta(n).$$

Proof : Let $\psi(z) = z$. Then $M_\psi C_\varphi D = M_\psi D$. From theorem (2.1) we find that

$$M_\psi D : H^2(\beta) \to H^2(\beta) \text{ is bounded}$$

iff

$$\|z\varphi^{n-1}\| \leq \frac{M}{n}\beta(n).$$

3 Compactness of the operators $\mathbf{M_\psi C_\varphi D}$

In this section, we shall study the compactness of the operator $M_\psi C_\varphi D$. For this, we need the following Lemma.

Lemma 3.1. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic self-map of $\mathbb{D}$. Then the composition operators $M_\psi C_\varphi D : H^2(\beta) \to H^2(\beta)$ is compact iff for any bounded sequence $\{f_n\}_{n=0}^{\infty}$ converging to zero locally uniformly on $\mathbb{D}$, we have*

$$\left\|M_\psi C_\varphi D f_n\right\|_{H^2(\beta)} \to 0.$$

Proof. The proof of this Lemma can be written by using the similar arguments as in [4, p.128].

Theorem 3.2. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic self map $\mathbb{D}$ of such that $\{\varphi^n : n \geq 0\}$ be an orthogonal family. Then*

$$M_\psi C_\varphi D : H^2(\beta) \to H^2(\beta) \text{ is compact}$$

iff

$$\frac{\|\psi n \varphi^{n-1}\|}{\beta_n} \to 0 \; as \; n \to 0.$$

Proof: Suppose that $M_\psi C_\varphi D : H^2(\beta) \to H^2(\beta)$ is compact and $\left\{\frac{\boldsymbol{z^n}}{\boldsymbol{\beta_n}}\right\}_{\boldsymbol{n=0}}^{\infty}$ converges uniformly to zero on $\mathbb{D}$. Then, by using the above lemma, we have

$$\left\|M_\psi C_\varphi D\left(\frac{z^n}{\beta_n}\right)\right\| \to 0 \;\; as \; n \to 0.$$

That is

$$\frac{\|\psi n \varphi^{n-1}\|}{\beta_n} \to 0 \; as \; n \to 0.$$

Conversely, suppose that

$$\frac{\|\psi n \varphi^{n-1}\|}{\beta_n} \to 0 \; as \; n \to 0.$$

Then, for every $\varepsilon > 0$, there exist a positive integer M such that

$$\frac{\|\psi n \varphi^{n-1}\|}{\beta_n} < \epsilon \; \forall n \geq M.$$

Let $f \in H^2(\beta)$ be such that

$$f = \sum_{n=0}^{\infty} a_n z^n.$$

Define an operator T_k as

$$T_k f = \sum_{n=0}^{k} a_n M_\psi C_\varphi D z^n$$

$$= \sum_{n=0}^{k} a_n \psi n \varphi^{n-1}.$$

Then T_k is a finite rank operator on $H^2(\beta)$. Now

$$\left\|(M_\psi C_\varphi D - T_k)f\right\|_{H^2(\beta)}^2 = \left\|\sum_{n=k+1}^{\infty} a_n \psi n \varphi^{n-1}\right\|_{H^2(\beta)}^2$$

$$= \sum_{n=k+1}^{\infty} |a_n|^2 n^2 \|\psi\varphi^{n-1}\|_{H^2(\beta)}^2$$

$$\leq \sum_{n=k+1}^{\infty} |a_n|^2 n^2 \frac{\epsilon^2}{n^2} \beta^2 n$$

$$\leq \sum_{n=k+1}^{\infty} |a_n|^2 \frac{\epsilon^2}{n^2} \beta^2 n$$

$$= \epsilon^2 \|f\|_{H^2(\beta)}^2.$$

Thus

$$\left\|(M_\psi C_\varphi D - T_k)f\right\| < \epsilon \quad \forall\, n \geq m.$$

Corollary 3.3. *Let* $\varphi : \mathbb{D} \to \mathbb{D}$ *be an analytic map such that* $\{\varphi^n : n \geq 0\}$ *be an orthogonal family. Then*

$C_\varphi D : H^2(\beta) \to H^2(\beta)$ *is compact*

iff

$$\frac{\|n\varphi^{n-1}\|}{\beta_n} \to 0 \; as \; n \to \infty.$$

Proof: Let $\psi(z) = 1$. Then $M_\psi C_\varphi D = C_\varphi D$. From theorem (3.1) in [10.p.583], we find that $C_\varphi D : H^2(\beta) \to H^2(\beta)$ is compact iff

$$\frac{\|n\varphi^{n-1}\|}{\beta_n} \to 0 \; as \; n \to \infty.$$

Corollary 3.4. *Let* $\varphi : \mathbb{D} \to \mathbb{D}$ *be an analytic map such that* $\{\varphi^n : n \geq 0\}$ *be an orthogonal family. Then the operator* $C_\varphi D : H^2(\beta) \to H^2(\beta)$ *is bounded iff*

$$\|\varphi^{n-1}\varphi'\| \leq \frac{M}{n}\beta(n).$$

Proof: Let $\psi(z) = \varphi'(z)$. Then $M_\psi C_\varphi D = DC_\varphi$. From theorem (3.2) in [10.p.584], we conclude that $DC_\varphi : H^2(\beta) \to H^2(\beta)$ is bounded iff

$$\frac{\|\varphi' n\varphi^{n-1}\|}{\beta_n} \to 0 \; as \; n \to \infty.$$

4. Necessary and sufficient conditions for the operator $\mathbf{M_\psi C_\varphi D}$ to be Hilbert-Schmidt operator on $\boldsymbol{H^2(\beta)}$

In this section, we shall give necessary and sufficient condition for the operator $M_\psi C_\varphi D$ to be Hilbert-Schmidt operator on $H^2(\beta)$. Recall that a linear operator T on a Hilbert space H is said to be Hilbert-Schmidt operator if $\sum_{n=0}^{\infty}\|Te_n\|^2 < \infty$ for some orthonormal basis $\{e_n: n \geq 0\}$ of Hilbert space H.

Theorem 4.1. *Let φ be analytic self-map of $\mathbb{D}$. Then the operator $M_\psi C_\varphi D$: $H^2_\beta(\mathbb{D}) \to H^2_\beta(\mathbb{D})$ is Hilbert-Schmidt operator iff*

$$\sum_{n=0}^{\infty}(n+1)^2 \, \|\psi.\varphi^{n-1}\|^2_{H^2_\beta(\mathbb{D})} < \infty.$$

Proof: Let the operator $M_\psi C_\varphi D : H^2_\beta(\mathbb{D}) \to H^2_\beta(\mathbb{D})$ be Hilbert-Schmidt operator. Then for the orthonormal basis $\{z^n : n \geq 0\}$ of $H^2_\beta(\mathbb{D})$,
we have

$$\sum_{n=0}^{\infty}\left\|M_\psi C_\varphi D \, z^n\right\|^2_{H^2_\beta(\mathbb{D})} = \sum_{n=1}^{\infty}\|\psi n\varphi^{n-1}\|^2_{H^2_\beta(\mathbb{D})}$$
$$= \sum_{n=0}^{\infty}(n+1)^2 \, \|\psi\varphi^n\|^2_{H^2_\beta(\mathbb{D})}$$
$$< \infty.$$

That is,

$$\sum_{n=0}^{\infty}(n+1)^2 \, \|\psi\varphi^{n-1}\|^2_{H^2_\beta(\mathbb{D})} < \infty.$$

To prove $M_\psi C_\varphi D : H^2_\beta(\mathbb{D}) \to H^2_\beta(\mathbb{D})$ is Hilbert-Schmidt operator, consider the orthonormal basis, $\{z^n : n \geq 0\}$. Then

$$\sum_{n=0}^{\infty} \left\| M_\psi C_\varphi D\, z^n \right\|^2_{H^2_\beta(\mathbb{D})} = \sum_{n=1}^{\infty} \|\psi n \varphi^{n-1}\|^2_{H^2_\beta(\mathbb{D})}$$
$$= \sum_{n=0}^{\infty} (n+1)^2 \, \|\psi . \varphi^n\|^2_{H^2_\beta(\mathbb{D})}$$
$$< \infty.$$

This prove that $M_\psi C_\varphi D$ is Hilbert-Schmidt operator on $H^2_\beta(\mathbb{D})$.

Corollary 4.2. *Let* φ *be analytic self-map of* $\mathbb{D}$. *Then the operator* $C_\varphi D : H^2_\beta(\mathbb{D}) \to H^2_\beta(\mathbb{D})$ *is Hilbert-Schmidt operator iff*

$$\sum_{n=0}^{\infty} n^2 \, \|\varphi^{n-1}\|^2_{H^2_\beta(\mathbb{D})} < \infty.$$

Corollary 4.3. *Let* φ *be analytic self-map of* $\mathbb{D}$. *Then the operator* $D C_\varphi : H^2_\beta(\mathbb{D}) \to H^2_\beta(\mathbb{D})$ *is Hilbert-Schmidt operator iff*

$$\sum_{n=0}^{\infty} n^2 \, \|\varphi^n \varphi^{n-1}\|^2_{H^2_\beta(\mathbb{D})} < \infty.$$

REFERENCES

[1] **Z. Abbas, P. Kumar,** "*Product of multiplication, composition and differentiation operators from weighted Bergman-Nevanlinna spaces to Zygmund spaces*", SS International Journal of Pure and Applied Mathematics, Vol. 1, Issue 2, 2015

[2] **A. Bhat, Z. Abbas, A. K. Sharma,** *Composition followed by differentiation between Weighted Bergman Nevanlinna Spaces*, Mathematica Aeterna, Vol. 2, no. 5, 379-388, 2012

[3] **D. M. Boyd**,: *Composition Operators Bergman space*,Coll.Math,24:1975, 127-136.

[4] **C. C. Cowen, B. D. MacCluear** : *Composition operators on spaces of analytic functions,* Stud. Adv. Math., CRC Press. Boca Ration, 1995.

[5] **P. Duren,** *: Theory of H^p spaces*, Academic Press New York, 1973.

[6] **J. B. Garnett:** *Bounded analytic functions* , Revised first edition, Graduate Texts in Mathematics, 236.Springer, New York,2007.

[7] **R. A. Hibschweiler and N. Portnoy,:** *Composition Operators followed by Differentiation between Bergman and Hardy spaces,* Rocky Mountain J.Math.35, 843-855 (2005).

[8] **Pawan Kumar and S.D Sharma :***Weighted composition operators from weighted Bergman-Nevanlinna spaces to zygmund spaces,* Int.J.Mod. Math.Sci.3(1),(2012),31-54.

[9] **Pawan Kumar and S.D Sharma :** *Generalized composition operators from weighted Bergman-Nevanlinna spaces to zygmund spaces,* Int.J.Mod. Math.Sci.1(3),(2012),150-162.

[10] **Pawan Kumar and Zaheer Abbas :** *Composition operators between weighted Hardy type spaces,* Int.J. of Pure and Applied Mathematics,Vol.107 No.3, 2016,579-587.

[11] **P. Kumar, Z. Abbas,** "*Product of multiplication and composition operators on weighted Hardy spaces*", SS International Journal of Pure and Applied Mathematics, Vol. 1, Issue 2, 2015

[12] **S. Li and S. Stevic, :** *Composition followed by Differentiation between* H^{∞} *and α-Bloch spaces,* Houston J. Math. 35(2009), 327-340.

[13] **S. Ohno,:** *Product of composition and Differentiation between Hardy spaces,* Bull Austral Math. Soc. 73(2006), 235-243.

[14] **H.J.Schwartz,:** *Composition Operators on* H^p, Ph.D thesis,University of Toledo,1969.

[15] **A.L. Shield,:** *Weighted shift operator and analytic function theory, in topic in operator theory,* Math surveys, No 13, Amer. Math. soc. Providence 1947

[16] **J. H. Shapiro, :** *Composition operators and classical function theory,* Springer-Verlag New York 1993.

[17] **S. D Sharma and R. Kumar,:***Substitution Operators on Hardy -Orlicz spaces*, Proc.Nat.Acad.Sci.India Sect.A61 (1991),535-541.

[18] **A.K. Sharma,:** *Product of composition Multiplication and Differentiation between Bergman and Bloch type spaces*, Turkish. J. Math. 34 (2010), 117.

[19] **A.K. Sharma**, :*Volterra composition operators between Bergman Nevan-linna and Bloch-type spaces*, Demonstratio Mathematical VolXLII No. 3 2009.

[20] **A.K. Sharma and S.D. Sharma**, : *Weighted composition operators between Bergman type spaces,* Comm. Korean Math. Soc. 21 No. 3(2006), 465-474.

[21] **A. K. Sharma Z. Abbas,** *Weighted Composition Operators between Weighted Bergman-Nevanlinna and Bloch-Type Spaces*, Jour. Appl. Math. Sci., 4(41): 2039-2048, 2010

[22] **A. K. Sharma, Z. Abbas, Z. (2009),** *Composition Proceeded and Followed by Differentiation between Weighted Bergman Nevanlinna and Bloch spaces*,

Journal of Advanced Research in Pure Mathematics, 1(2): 53-62, online ISSN: 1943-2380.

[23] **A. K. Sharma, R. Sharma and Z. Abbas,** *Weighted Composition Operators between Weighted Bergman-Nevanlinna and Growth Spaces*, Int. Journal of Math. Analysis, 4(26): 1291-1298, 2010

[24] **S. D. Sharma, A. K. Sharma, Z. Abbas,** *Weighted Composition Operators on Weighted Vector valued Begman spaces,* Applied Mathematical Sciences, Vol. 4, 2010, no. 42, 2049 – 2063

[25] **K. Zhu**,:*Operator theory in function spaces*, Marcel-Dekker, New York 1990.

[26] **K. Zhu,:** *Spaces of holomorphic functions in the unit ball. Graduate* Text in Mathematics,vol.226. Springer, New York (2005)

[27] **X. Zhu,:** *Weighted composition operators from area Nevanlinna spaces into Bloch spaces*, Applied Mathematics and computation, 215 (2010), 4340-4346.