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On the First and Second Harary Index of Generalized Transformation Graphs G^{ab}

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Abstract

The Harary index is defined as the sum of distances between all pairs of vertices of connected graph *G*. In this article, we obtain the expression for the first and second Harary index of generalized transformation graph G^{ab} of graph *G*, which are are G^{++}, G^{+-}, G^{-+} and G^{--}. We consider line splitting graph $L_s(G)$ of a graph *G* as transformation graph G^{++} of G^{ab}.

Keywords: First Harary index, second Harary index, transformation graph G^{ab}, line splitting graph.

1. INTRODUCTION

Throughout this paper, we consider finite, un-directed, simple, connected, *r*-regular graphs with vertex set $V(G)=\{v_1,v_2,v_3,...,v_n\}$ and edge set $E(G)=\{e_1,e_2,e_3,...,e_m\}$. For the undefined terminologies we refer[1].

The degree of vertex in a graph *G* is denoted by $deg_G(v)$ or $d_G(v)$ and the distance between two vertices v_i and v_j, denoted by $dist_G(v_i,v_j)$ or $d_G(v_i,v_j)$, is the length of a shortest path between the vertices v_i and v_j in *G*. The shortest v_i - v_j path is often called a geodesic. The diameter of a connected graph *G* is the length of any longest geodesic. The graphs considered in this construction are with $diam \leq 2$. The degree of an edge e_i in *G* is the number of edges adjacent to e_i and is denoted by $deg_G(e_i)$ .The degree of edge in a graph G is,

$$deg_G(e_i) = deg_G(uv) = deg_G(u) + deg_G(v) - 2.$$

A chemical graph or molecular graph is a graph related to the structure of a chemical compound. Each vertex of this graph represents an atom of the molecule and covalent bonds between atoms are represented by edges between the corresponding vertices. In theoretical chemistry, the physico chemical properties of chemical compounds are often modelled by the molecular graph based on molecular structure descriptors, which are also refered to as topological indices[12]. Among the variety of those indices which are designed to capture the different aspects of molecular structure Wiener index[13][14] is the best known one.Wiener index is the first reported distance based topological index, which was introduced by the chemist, Harold Wiener, in 1947 and is defined as,

$$W(G) = \sum_{u,v \subseteq V(G)} d_G(u,v) \qquad (1)$$

The Harary index or the first Harary index a distance-based topological index, was introduced by Plašić et. al.[11] and by Ivanciuc et. al.[7] in 1993. It has been named in honor of Professor Frank Harary on the occasion of his 70^{th} birthday and also due to his influence in development of graph theory and especially to its application in chemistry. Harary index of G is denoted by $H(G)$ and is defined as follows.

$$H(G) = \sum_{u,v \subseteq V(G)} \frac{1}{d_G(u,v)} \qquad (2)$$

where the summation goes over all un-ordered pairs of vertices of G. Mathematical properties and applications of $H(G)$ are reported in [2,3,5,6,9,15-17].

K. C. Das et al.[4] introduced the second Harary index, which is defined as,

$$H_1(G) = \sum_{u,v \subseteq V(G)} \frac{1}{d_G(u,v)+1} \qquad (3)$$

Where the summation goes over all un-ordered pairs of vertices of G. The open neighborhood $N(e_i)$ of an edge e_i in $E(G)$ is the set of edges adjacent to e_i i.e., $N(e_i)=\{e_i / e_i, e_j$ are adjacent in $G\}$. For each edge e_i of G, a new vertex e'_i is taken and the resulting set of vertices is denoted by $E'(G)$.

The line splitting graph $L_s(G)$ of a graph G is defined as the graph having vertex set $E(G) \cup E'(G)$ with two vertices adjacent if they correspond to adjacent edges of G or one correspond to an element e'_i of $E'(G)$ and the other to an element e_j of $E(G)$, where e_j is $N(e_i)$ in G. This concept was introduced by Kulli and Biradar in [8].

2. GENERALIZED TRANSFORMATION GRAPHS G^{ab}

Let $G = (V, E)$ be a graph. Let α, β and α', β' be the element of $E(G)$ and $E'(G)$ respectively. We say that the associativity of α and β is +, if they are adjacent in G otherwise is - and the associativity of α and β' or α' and β is +, if α is the neighborhood point of β or β is neighborhood point of α in G, otherwise is - .

Let ab be a 2-permutation of the set $\{+,-\}$. We say that α and β corresponds to the first term a of ab, and α, $\beta \in E(G)$. Whereas α and β' or β and α' corresponds to the both first and second term of ab and $\alpha', \beta' \in E'(G)$.

The transformation graph G^{ab} of a graph G is the graph with vertex set $E(G) \cup E'(G)$. $\alpha \; and \; \beta$ or $\alpha \; and \; \beta'$ or $\beta \; and \; \alpha'$ are adjacent if and only if the following conditions holds;

* $\alpha, \beta \in E(G)$, α and β are adjacent in G if a = + otherwise a= -.

** $\alpha, \beta \in E(G)$ and $\alpha', \beta' \in E'(G)$, if α neighborhood points of β or β is neighborhood point of α in G then b = + otherwise b= -.

Since there are four distinct 2-permutations of $\{+,-\}$, we obtain 4-graphical transformations of G. Here we consider G^{++}, which is nothing but line splitting graph of G and the other generalized transformation graphs are G^{+-}, G^{-+} and G^{--}.

Note that, in this paper we consider graphs with $n \geq 5$ for G^{++} and G^{-+} and in particular for G^{+-} and G^{--} we consider graphs with $n > 5$ and having atleast three edges e_i , e_j and $e_w \in E(G)$; $i,j,w=1,2,3,...,m$ and $i \neq j \neq w$ such that e_i and e_j are non adjacent edges and e_w is non adjacent to e_i and e_j.

The aim of present work is to obtain the expression for the first and second Harary index of the generalized transformation graphs G^{ab}.

3. RESULTS

In this section we obtain the sum degree distance and product degree distance of the transformation graphs G^{ab}, which is line splitting graph i.e., G^{++}, and its generalized transformation graphs G^{+-}, G^{-+}, G^{--}.

We start by stating the following propositions and observations, needed for proving our main results.

Proposition 3.1[10] Let G be an (n, m) graph. Then by the definition order of G^{ab} is $2m$ and

(i) The size of G^{++} is $-m+\frac{1}{2}nr^2+2m(r-1)$.

(ii) The size of G^{+-} is $-m^2+\frac{1}{2}(nr^2)-2mr$.

(iii)The size of G^{-+} is $\frac{m}{2}[m+2r-3]$.

(iv)The size of G^{--} is $\frac{3}{2}m[m-2r+1]$.

Observation A.

1. Let G be any (n, m) graph.

If $d_{G^{++}}(e_i,e_j)=1$, then

$$\sum_{(e_i,e_j)\subseteq V(G^{++})}\frac{1}{d_{G^{++}}(e_i,\, e_j)} \text{ in } G^{++} = -m+\frac{1}{2}nr^2.$$

If $d_{G^{++}}(e_i,e_j)=2$, then

$$\sum_{(e_i,e_j)\subseteq V(G^{++})}\frac{1}{d_{G^{++}}(e_i,\, e_j)} \text{ in } G^{++} = \frac{1}{4}m(m-2r+1).$$

2. Let G be any (n, m) graph.

If $d_{G^{++}}(e_i,e'_j)=1$, then

$$\sum_{(e_i,e'_j)\subseteq V(G^{++})}\frac{1}{d_{G^{++}}(e_i,\, e'_j)} \text{ in } G^{++} = 2m(r-1).$$

If $d_{G^{++}}(e_i,e'_j)=2$, then

$$\sum_{(e_i,e'_j)\subseteq V(G^{++})}\frac{1}{d_{G^{++}}(e_i,\, e'_j)} \text{ in } G^{++} = \frac{1}{2}m(m-2r+2).$$

3. Let G be any (n, m) graph.

When r =2.

If $d_{G^{++}}(e'_i, e'_j) = 2$, then

$$\sum_{(e'_i,e'_j)\subseteq V(G^{++})} \frac{1}{d_{G^{++}}(e'_i, e'_j)} \text{ in } G^{++} = \frac{1}{4}m(m-2r+1).$$

If $d_{G^{++}}(e'_i, e'_j) = 3$, then

$$\sum_{(e'_i,e'_j)\subseteq V(G^{++})} \frac{1}{d_{G^{++}}(e_i, e_j)} \text{ in } G^{++} = \frac{1}{3}m(r-1).$$

When r >2,

If $d_{G^{++}}(e'_i, e'_j) = 2$, then

$$\sum_{(e'_i,e'_j)\subseteq V(G^{++})} \frac{1}{d_{G^{++}}(e_i, e_j)} \text{ in } G^{++} = \frac{1}{2}\sum_{k=2}^{m}(k-1)$$

Theorem 3.2. For any (n, m) graph G, the first Harary index is

$$H(G^{++}) = \frac{1}{2}nr^2 + m^2 - \frac{11}{6}m + \frac{1}{3}mr \qquad when\ \ r = 2$$

$$H(G^{++}) = \frac{1}{2}nr^2 + \frac{3}{4}m^2 + \frac{1}{2}mr - \frac{7}{4}m + \frac{1}{2}\sum_{k=2}^{m}(k-1) \qquad when\ \ r > 2$$

Proof. Let G be any (n, m)-graph.

From Eq.(2), we have

$$H(G) = \sum_{u,v\subseteq V(G)} \frac{1}{d_G\ (u,v)}$$

Therefore,

$$H(G^{++}) = \sum_{(e_i,e_j)\subseteq V(G^{++})} \frac{1}{d_{G^{++}}(e_i, e_j)} + \sum_{(e_i,e'_j)\subseteq V(G^{++})} \frac{1}{d_{G^{++}}(e_i, e'_j)} + \sum_{(e'_i,e'_j)\subseteq V(G^{++})} \frac{1}{d_{G^{++}}(e'_i, e'_j)}.$$

Applying observation A to the above equation,

when r = 2,

$$H(G^{++}) = -m + \frac{1}{2}nr^2 + \frac{1}{4}m(m-2r+1) + 2m(r-1) + \frac{1}{2}m(m-2r+2) + \frac{1}{4}m(m-2r+1) + \frac{1}{3}m(r-1).$$

and r >2,

$$H(G^{++}) = -m + \frac{1}{2}nr^2 + \frac{1}{4}m(m-2r+1) + 2m(r-1) + \frac{1}{2}m(m-2r+2) + \frac{1}{2}\sum_{k=1}^{m}(k-1)$$

On simplification, we get

$$H(G^{++}) = \frac{1}{2}nr^2 + m^2 - \frac{11}{6}m + \frac{1}{3}mr \qquad when\ \ r = 2$$

$$H(G^{++}) = \frac{1}{2}nr^2 + \frac{3}{4}m^2 + \frac{1}{2}mr - \frac{7}{4}m + \frac{1}{2}\sum_{k=2}^{m}(k-1) \qquad when\ \ r > 2.$$

Observation B.

1. Let G be any (n, m) graph.

If $d_{G^{++}}(e_i, e_j) = 1$, then

$$\sum_{(e_i,e_j)\subseteq V(G^{++})} \frac{1}{d_{G^{++}}(e_i,\, e_j)+1} \ \text{ in } G^{++} = \frac{1}{2}(-m + \frac{1}{2}nr^2).$$

If $d_{G^{++}}(e_i, e_j) = 2$, then

$$\sum_{(e_i,e_j)\subseteq V(G^{++})} \frac{1}{d_{G^{++}}(e_i,\, e_j)+1} \ \text{ in } G^{++} = \frac{1}{6}m(m-2r+1).$$

2. Let G be any (n, m) graph.

If $d_{G^{++}}(e_i, e'_j) = 1$, then

$$\sum_{(e_i,e'_j)\subseteq V(G^{++})} \frac{1}{d_{G^{++}}(e_i,\, e'_j)+1} \ \text{ in } G^{++} = m(r-1).$$

If $d_{G^{++}}(e_i, e'_j) = 2$, then

$$\sum_{(e_i, e'_j)\subseteq V(G^{++})} \frac{1}{d_{G^{++}}(e_i, e'_j)+1} \text{ in } G^{++} = \frac{1}{3}m(m-2r+2).$$

3. Let G be any (n, m) graph.

   When $r = 2$.

If $d_{G^{++}}(e'_i, e'_j) = 2$, then

$$\sum_{(e'_i, e'_j)\subseteq V(G^{++})} \frac{1}{d_{G^{++}}(e'_i, e'_j)+1} \text{ in } G^{++} = \frac{1}{6}m(m-2r+1).$$

If $d_{G^{++}}(e'_i, e'_j) = 3$, then

$$\sum_{(e'_i, e'_j)\subseteq V(G^{++})} \frac{1}{d_{G^{++}}(e_i, e_j)+1} \text{ in } G^{++} = \frac{1}{4}m(r-1).$$

When $r > 2$

If $d_{G^{++}}(e'_i, e'_j) = 3$, then

$$\sum_{(e'_i, e'_j)\subseteq V(G^{++})} \frac{1}{d_{G^{++}}(e_i, e_j)+1} \text{ in } G^{++} = \frac{1}{3}\sum_{k=2}^{m}(k-1).$$

Theorem 3.3. For any (n, m) graph G, the second Harary index is,

$$H_1(G^{++}) = \frac{1}{4}nr^2 + \frac{2}{3}m^2 - \frac{3}{4}m - \frac{1}{12}mr, \quad when \ \ r = 2,$$

$$H_1(G^{++}) = \frac{1}{4}nr^2 + \frac{1}{2}m^2 - \frac{2}{3}m + \frac{1}{3}\sum_{k=2}^{m}(k-1), \quad when \ \ r > 2.$$

Proof. Let G be any (n, m)-graph.

From Eq.(3), we have

$$H_1(G) = \sum_{u,v\subseteq V(G)} \frac{1}{d_G(u,v)+1}$$

Therefore,

$$H_1(G^{++}) = \sum_{(e_i,e_j)\subseteq V(G^{++})} \frac{1}{d_{G^{++}}(e_i,e_j)+1} + \sum_{(e_i,e'_j)\subseteq V(G^{++})} \frac{1}{d_{G^{++}}(e_i,e'_j)+1} + \sum_{(e'_i,e'_j)\subseteq V(G^{++})} \frac{1}{d_{G^{++}}(e'_i,e'_j)+1}.$$

Applying observation B to the above equation,

when r = 2,

$$H_1(G^{++}) = \frac{1}{2}(-m+\frac{1}{2}nr^2) + \frac{1}{6}m(m-2r+1) + m(r-1) + \frac{1}{3}m(m-2r+2) + \frac{1}{6}m(m-2r+1) + \frac{1}{4}m(r-1)$$

and r >2,

$$H_1(G^{++}) = \frac{1}{2}(-m+\frac{1}{2}nr^2) + \frac{1}{6}m(m-2r+1) + m(r-1) + +\frac{1}{3}m(m-2r+2) + \frac{1}{3}\sum_{k=1}^{m}(k-1).$$

On simplification, we get

$$H_1(G^{++}) = \frac{1}{4}nr^2 + \frac{2}{3}m^2 - \frac{3}{4}m - \frac{1}{12}mr, \quad when \;\; r = 2,$$

$$H_1(G^{++}) = \frac{1}{4}nr^2 + \frac{1}{2}m^2 - \frac{2}{3}m + \frac{1}{3}\sum_{k=2}^{m}(k-1), \quad when \;\; r > 2.$$

Observation C.

1. Let G be any (n, m) graph.

If $d_{G^{+-}}(e_i,e_j) = 1$, then

$$\sum_{(e_i,e_j)\subseteq V(G^{+-})} \frac{1}{d_{G^{+-}}(e_i,\, e_j)} \;\; \text{in } G^{+-} = -m + \frac{1}{2}nr^2.$$

If $d_{G^{+-}}(e_i,e_j) = 2$, then

$$\sum_{(e_i,e_j)\subseteq V(G^{+-})} \frac{1}{d_{G^{+-}}(e_i,\, e_j)} \;\; \text{in } G^{+-} = \frac{1}{2}m(m-2r+1).$$

2. Let G be any (n, m) graph.

If $d_{G^{+-}}(e_i, e'_j) = 1$, then

$$\sum_{(e_i, e'_j) \subseteq V(G^{+-})} \frac{1}{d_{G^{+-}}(e_i, e'_j)} \text{ in } G^{+-} = m(m - 2r + 1).$$

If $d_{G^{+-}}(e_i, e'_j) = 2$, then

$$\sum_{(e_i, e'_j) \subseteq V(G^{+-})} \frac{1}{d_{G^{+-}}(e_i, e'_j)} \text{ in } G^{+-} = m(r - 1).$$

If $d_{G^{+-}}(e_i, e'_j) = 3$, then

$$\sum_{(e_i, e'_j) \subseteq V(G^{+-})} \frac{1}{d_{G^{+-}}(e_i, e'_j)} \text{ in } G^{+-} = \frac{1}{3}m.$$

3. Let G be any (n, m) graph.

When r=2 and $e_i \in G$, contains only one nonadjacent edge.

If $d_{G^{+-}}(e'_i, e'_j) = 2$, then

$$\sum_{(e'_i, e'_j) \subseteq V(G^{+-})} \frac{1}{d_{G^{+-}}(e'_i, e'_j)} \text{ in } G^{+-} = \frac{1}{16}m(m - 2r + 1).$$

If $d_{G^{+-}}(e'_i, e'_j) = 3$, then

$$\sum_{(e'_i, e'_j) \subseteq V(G^{+-})} \frac{1}{d_{G^{+-}}(e'_i, e'_j)} \text{ in } G^{+-} = \frac{1}{3}m(r - 1).$$

When r =2 and $e_i \in G$, contains more than two nonadjacent edge.

If $d_{G^{+-}}(e'_i, e'_j) = 2$, then

$$\sum_{(e'_i, e'_j) \subseteq V(G^{+-})} \frac{1}{d_{G^{+-}}(e'_i, e'_j)} \text{ in } G^{+-} = \frac{1}{2}m(r - 1).$$

If $d_{G^{+-}}(e'_i, e'_j) = 2$, then

$$\sum_{(e'_i, e'_j) \subseteq V(G^{+-})} \frac{1}{d_{G^{+-}}(e'_i, e'_j)} \text{ in } G^{+-} = \frac{m}{6}(m - 2r + 1).$$

When $r>2$.

If $d_{G^{+-}}(e'_i, e'_j) = 2$, then

$$\sum_{(e'_i, e'_j)\subseteq V(G^{+-})} \frac{1}{d_{G^{+-}}(e'_i, e'_j)} \text{ in } G^{+-} = \frac{1}{2}\sum_{k=2}^{m}(k-1).$$

Theorem 3.4. For any (n, m) graph G, the first Harary index is,

$$H(G^{+-}) = \frac{1}{2}nr^2 + \frac{25}{16}m^2 - \frac{7}{16}m - \frac{43}{24}mr,$$
$$\text{when } r = 2, e_i \in G \text{ contains only one non-adjacent edge},$$
$$H(G^{+-}) = \frac{1}{2}nr^2 + \frac{5}{3}m^2 - \frac{1}{2}m - \frac{11}{6}mr,$$
$$\text{when } r = 2, e_i \in G \text{ contains morethan two non-adjacent edge},$$

$$H(G^{+-}) = \frac{1}{2}nr^2 + \frac{3}{2}m^2 - \frac{1}{6}m - 2mr + \frac{1}{2}\sum_{k=2}^{m}(k-1), \qquad \text{when } r>2.$$

Proof. Let G be any (n, m)-graph.

From Eq.(2), we have

$$H(G) = \sum_{u,v\subseteq V(G)} \frac{1}{d_G(u,v)}$$

Therefore,

$$H(G^{+-}) = \sum_{(e_i,e_j)\subseteq V(G^{+-})} \frac{1}{d_{G^{+-}}(e_i,e_j)} + \sum_{(e_i,e'_j)\subseteq V(G^{+-})} \frac{1}{d_{G^{+-}}(e_i,e'_j)} + \sum_{(e'_i,e'_j)\subseteq V(G^{+-})} \frac{1}{d_{G^{+-}}(e'_i,e'_j)}.$$

Applying the observation C, we get

when $r = 2$ and $e_i \in G$ contains only one non-adjacent edge.

$$H(G^{+-}) = -m + \frac{1}{2}nr^2 + \frac{m}{2}(m-2r+1) + m(m-2r+1) + m(r-1) + \frac{1}{3}m + \frac{m}{2}(r-1)$$
$$+ \frac{m}{6}(m-2r+1).$$

when $r = 2$ and $e_i \in G$ contains more than two non-adjacent edge.

$$H(G^{+-}) = -m + \frac{1}{2}nr^2 + \frac{m}{2}(m-2r+1) + m(m-2r+1) + m(r-1) + \frac{1}{3}m + \frac{m}{2}(r-1) + \frac{m}{6}(m-2r+1) + \frac{m}{2}(r-1) + \frac{m}{6}(m-2r+1).$$

When r > 2,

$$H(G^{+-}) = -m + \frac{1}{2}nr^2 + \frac{m}{2}(m-2r+1) + m(m-2r+1) + m(r-1) + \frac{1}{3}m + \frac{1}{2}\sum_{k=2}^{m}(k-1).$$

On simplification, we get

$$H(G^{+-}) = \frac{1}{2}nr^2 + \frac{25}{16}m^2 - \frac{7}{16}m - \frac{43}{24}mr,$$
$$when \;\; r = 2, e_i \in G \, contains \;\; only \;\; one \;\; non-adjacent \;\; edge,$$

$$H(G^{+-}) = \frac{1}{2}nr^2 + \frac{5}{3}m^2 - \frac{1}{2}m - \frac{11}{6}mr,$$
$$when \;\; r = 2, e_i \in G \, contains \;\; morethan \;\; two \;\; non-adjacent \;\; edge,$$

$$H(G^{+-}) = \frac{1}{2}nr^2 + \frac{3}{2}m^2 - \frac{1}{6}m - 2mr + \frac{1}{2}\sum_{k=2}^{m}(k-1), \qquad when \;\; r > 2.$$

Observation D.

1. Let G be any (n, m) graph.

If $d_{G^{+-}}(e_i, e_j) = 1$, then

$$\sum_{(e_i, e_j) \subseteq V(G^{+-})} \frac{1}{d_{G^{+-}}(e_i, e_j) + 1} \;\; \text{in } G^{+-} = \frac{1}{2}(-m + \frac{1}{2}nr^2).$$

If $d_{G^{+-}}(e_i, e_j) = 2$, then

$$\sum_{(e_i, e_j) \subseteq V(G^{+-})} \frac{1}{d_{G^{+-}}(e_i, e_j) + 1} \;\; \text{in } G^{+-} = \frac{m}{6}(m - 2r + 1).$$

2. Let G be any (n, m) graph.

If $d_{G^{+-}}(e_i, e'_j) = 1$, then

$$\sum_{(e_i, e'_j) \subseteq V(G^{+-})} \frac{1}{d_{G^{+-}}(e_i, e'_j) + 1} \;\; \text{in } G^{+-} = \frac{m}{2}(m - 2r + 1).$$

If $d_{G^{+-}}(e_i, e'_j) = 2$, then

$$\sum_{(e_i, e'_j) \subseteq V(G^{+-})} \frac{1}{d_{G^{+-}}(e_i, e'_j) + 1} \text{ in } G^{+-} = \frac{2}{3} m(r-1).$$

If $d_{G^{+-}}(e_i, e'_j) = 3$, then

$$\sum_{(e_i, e'_j) \subseteq V(G^{+-})} \frac{1}{d_{G^{+-}}(e_i, e'_j) + 1} \text{ in } G^{+-} = \frac{1}{4} m.$$

3. Let G be any (n, m) graph.

When r=2 and $e_i \in G$ contains only one nonadjacent edge.

If $d_{G^{+-}}(e'_i, e'_j) = 2$, then

$$\sum_{(e'_i, e'_j) \subseteq V(G^{+-})} \frac{1}{d_{G^{+-}}(e'_i, e'_j) + 1} \text{ in } G^{+-} = \frac{m}{4}(r-1).$$

If $d_{G^{+-}}(e'_i, e'_j) = 3$, then

$$\sum_{(e'_i, e'_j) \subseteq V(G^{+-})} \frac{1}{d_{G^{+-}}(e'_i, e'_j) + 1} \text{ in } G^{+-} = \frac{m}{20}(m-2r+1).$$

When r =2 and $e_i \in G$ contains only one nonadjacent edge.

If $d_{G^{+-}}(e'_i, e'_j) = 2$, then

$$\sum_{(e'_i, e'_j) \subseteq V(G^{+-})} \frac{1}{d_{G^{+-}}(e'_i, e'_j) + 1} \text{ in } G^{+-} = \frac{m}{3}(r-1).$$

If $d_{G^{+-}}(e'_i, e'_j) = 3$, then

$$\sum_{(e'_i, e'_j) \subseteq V(G^{+-})} \frac{1}{d_{G^{+-}}(e'_i, e'_j) + 1} \text{ in } G^{+-} = \frac{m}{8}(m-2r+1).$$

When r >2,

If $d_{G^{+-}}(e'_i, e'_j) = 2$, then

$$\sum_{(e'_i, e'_j) \subseteq V(G^{+-})} \frac{1}{d_{G^{+-}}(e'_i, e'_j) + 1} \text{ in } G^{+-} = \frac{1}{3} \sum_{k=2}^{m} (k-1).$$

Theorem 3.5. For any (*n*,*m*) graph *G*, the second Harary index is,

$$H_1(G^{+-}) = \frac{1}{4}nr^2 + \frac{43}{60}m^2 - \frac{9}{20}m - \frac{31}{60}mr, \quad when \;\; r = 2,$$
$$and \;\; e_i \in G\, contins \;\; only \;\; one\, non-adjacent \;\; edge,$$

$$H_1(G^{+-}) = \frac{1}{4}nr^2 + \frac{19}{24}m^2 - \frac{11}{24}m - \frac{7}{12}mr, \quad when \;\; r = 2,$$
$$and \;\; e_i \in G\, contins \;\; more\; than\; two\, non-adjacent \;\; edge,$$

$$H_1(G^{+-}) = \frac{1}{4}nr^2 + \frac{2}{3}m^2 - \frac{1}{4}m - \frac{2}{3}mr + \frac{1}{3}\sum_{k=2}^{m}(k-1), \quad when \;\; r > 2.$$

Proof. Let *G* be any (*n*, *m*) graph.

From Eq.(3), we have

$$H_1(G) = \sum_{(u,v)\in V(G)} \frac{1}{d_G(u,v)+1}$$

Therefore,

$$H_1(G^{+-}) = \sum_{(e_i,e_j)\subseteq V(G^{+-})} \frac{1}{d_{G^{+-}}(e_i,e_j)+1} + \sum_{(e_i,e'_j)\subseteq V(G^{+-})} \frac{1}{d_{G^{+-}}(e_i,e'_j)+1} + \sum_{(e'_i,e'_j)\subseteq V(G^{++})} \frac{1}{d_{G^{+-}}(e'_i,e'_j)+1}.$$

Applying observation D to the above equation,

When r=2 and e_i∈G contains only one edge.

$$H_1(G^{+-}) = \frac{1}{2}(-m + \frac{1}{2}nr^2) + \frac{m}{6}(m-2r+1) + \frac{m}{2}(m-2r+1) + \frac{2}{3}m(r-1) + \frac{1}{4}m + \frac{m}{4}(r-1)$$
$$+ \frac{m}{20}(m-2r+1).$$

When r=2 and e_i∈G contains more than two edges.

$$H_1(G^{+-}) = \frac{1}{2}(-m + \frac{1}{2}nr^2) + \frac{m}{6}(m-2r+1) + \frac{m}{2}(m-2r+1) + \frac{2}{3}m(r-1) + \frac{1}{4}m + \frac{m}{3}(r-1)$$
$$+ \frac{m}{8}(m-2r+1).$$

when r > 2,

$$H_1(G^{+-}) = \frac{1}{2}(-m + \frac{1}{2}nr^2) + \frac{m}{6}(m-2r+1) + \frac{m}{2}(m-2r+1) + \frac{2}{3}m(r-1) + \frac{1}{4}m + \frac{1}{3}\sum_{k=2}^{m}(k-1).$$

On Simplification, we get

$$H_1(G^{+-}) = \frac{1}{4}nr^2 + \frac{43}{60}m^2 - \frac{9}{20}m - \frac{31}{60}mr, \quad when \;\; r=2,$$
$$and \;\; e_i \in G\, contins \;\; only \;\; one\, non-adjacent \;\; edge,$$

$$H_1(G^{+-}) = \frac{1}{4}nr^2 + \frac{19}{24}m^2 - \frac{11}{24}m - \frac{7}{12}mr, \quad when \;\; r=2,$$
$$and \;\; e_i \in G\, contins \;\; more\; than \;\; two\, non-adjacent \;\; edge,$$

$$H_1(G^{+-}) = \frac{1}{4}nr^2 + \frac{2}{3}m^2 - \frac{1}{4}m - \frac{2}{3}mr + \frac{1}{3}\sum_{k=2}^{m}(k-1), \quad when \;\; r>2.$$

Observation E.

1. Let G be any (n, m) graph.

If $d_{G^{-+}}(e_i, e_j) = 1$, then

$$\sum_{(e_i,e_j)\subseteq V(G^{-+})} \frac{1}{d_{G^{-+}}(e_i,\, e_j)} \quad \text{in } G^{-+} \quad = \frac{m}{2}(m-2r+1).$$

If $d_{G^{-+}}(e_i, e_j) = 2$, then

$$\sum_{(e_i,e_j)\subseteq V(G^{-+})} \frac{1}{d_{G^{-+}}(e_i,\, e_j)} \quad \text{in } G^{-+} \quad = \frac{m}{3}(r-1).$$

2. Let G be any (n, m) graph.

If $d_{G^{-+}}(e_i, e'_j) = 1$, then

$$\sum_{(e_i,e'_j)\subseteq V(G^{-+})} \frac{1}{d_{G^{-+}}(e_i,\, e'_j)} \quad \text{in } G^{-+} \quad = 2\,m(r-1)\,.$$

If $d_{G^{-+}}(e_i, e'_j) = 2$, then

$$\sum_{(e_i,e'_j)\subseteq V(G^{-+})} \frac{1}{d_{G^{-+}}(e_i,\, e'_j)} \quad \text{in } G^{-+} \quad = \frac{m}{2}(m-2r+1)\,.$$

If $d_{G^{-+}}(e_i, e'_j) = 3$, then

$$\sum_{(e_i, e'_j) \subseteq V(G^{-+})} \frac{1}{d_{G^{-+}}(e_i, e'_j)} \text{ in } G^{-+} = \frac{m}{3}.$$

3. Let G be any (n, m) graph.

 When r=2.

If $d_{G^{-+}}(e'_i, e'_j) = 2$, then

$$\sum_{(e'_i, e'_j) \subseteq V(G^{-+})} \frac{1}{d_{G^{-+}}(e'_i, e'_j)} \text{ in } G^{-+} = \frac{m}{4}(m - 2r + 1).$$

If $d_{G^{-+}}(e'_i, e'_j) = 3$, then

$$\sum_{(e'_i, e'_j) \subseteq V(G^{-+})} \frac{1}{d_{G^{-+}}(e'_i, e'_j)} \text{ in } G^{-+} = \frac{m}{3}(r - 1).$$

When r>2

If $d_{G^{-+}}(e'_i, e'_j) = 2$, then

$$\sum_{(e'_i, e'_j) \subseteq V(G^{-+})} \frac{1}{d_{G^{-+}}(e_i, e_j)} \text{ in } G^{-+} = \frac{1}{2}\sum_{k=2}^{m}(k - 1).$$

Theorem 3.6. For any (n, m) graph G, the first Harary index is

$$H(G^{-+}) = \frac{5}{2}m^2 + \frac{1}{6}mr - \frac{13}{12}m, \qquad when \ r = 2,$$

$$H(G^{-+}) = m^2 + \frac{1}{3}mr - m + \frac{1}{2}\sum_{k=2}^{m}(k-1), \qquad when \ r > 2.$$

Proof. Let G be any (n, m) graph.

From Eq.(2), we have

$$H(G) = \sum_{(u,v) \in V(G)} \frac{1}{d_G(u,v)}$$

Therefore,

$$H(G^{-+}) = \sum_{(e_i, e_j) \subseteq V(G^{-+})} \frac{1}{d_{G^{-+}}(e_i, e_j)} + \sum_{(e_i, e'_j) \subseteq V(G^{-+})} \frac{1}{d_{G^{-+}}(e_i, e'_j)} + \sum_{(e'_i, e'_j) \subseteq V(G^{-+})} \frac{1}{d_{G^{-+}}(e'_i, e'_j)}.$$

Applying observation E to the above equation, when r = 2,

$$H(G^{-+}) = \frac{m}{2}(m-2r+1) + \frac{m}{3}(r-1) + 2m(r-1) + \frac{m}{2}(m-2r+1) + \frac{m}{3} + \frac{m}{4}(m-2r+1) + \frac{m}{3}(r-1).$$

and r >2,

$$H(G^{-+}) = \frac{m}{2}(m-2r+1) + \frac{m}{3}(r-1) + 2m(r-1) + \frac{m}{2}(m-2r+1) + \frac{1}{2}\sum_{k=2}^{m}(k-1)..$$

On simplification, we get

$$H(G^{-+}) = \frac{5}{2}m^2 + \frac{1}{6}mr - \frac{13}{12}m, \qquad when \;\; r = 2,$$

$$H(G^{-+}) = m^2 + \frac{1}{3}mr - m + \frac{1}{2}\sum_{k=2}^{m}(k-1), \qquad when \;\; r > 2.$$

Observation F.

1. Let *G* be any (*n*, *m*) graph.

If $d_{G^{-+}}(e_i, e_j) = 1$, then

$$\sum_{(e_i, e_j) \subseteq V(G^{-+})} \frac{1}{d_{G^{-+}}(e_i,\, e_j)+1} \;\; \text{in } G^{-+} \;\; = \frac{m}{4}(m-2r+1).$$

If $d_{G^{-+}}(e_i, e_j) = 2$, then

$$\sum_{(e_i, e_j) \subseteq V(G^{-+})} \frac{1}{d_{G^{-+}}(e_i,\, e_j)+1} \;\; \text{in } G^{-+} \;\; = \frac{m}{4}(r-1).$$

2. Let *G* be any (*n*, *m*) graph.

If $d_{G^{-+}}(e_i, e'_j) = 1$, then

$$\sum_{(e_i, e'_j) \subseteq V(G^{-+})} \frac{1}{d_{G^{-+}}(e_i,\, e'_j)+1} \;\; \text{in } G^{-+} \;\; = m(r-1).$$

If $d_{G^{-+}}(e_i, e'_j) = 2$, then

$$\sum_{(e_i, e'_j) \subseteq V(G^{-+})} \frac{1}{d_{G^{-+}}(e_i,\, e'_j)+1} \;\; \text{in } G^{-+} \;\; = \frac{m}{3}(m-2r+1).$$

If $d_{G^{-+}}(e_i, e'_j) = 3$, then

$$\sum_{(e_i, e'_j) \subseteq V(G^{-+})} \frac{1}{d_{G^{-+}}(e_i,\, e'_j)+1} \quad \text{in } G^{-+} \quad = \frac{m}{4}.$$

3. Let G be any (n, m) graph.

When r=2.

If $d_{G^{-+}}(e'_i, e'_j) = 2$, then

$$\sum_{(e'_i, e'_j) \subseteq V(G^{-+})} \frac{1}{d_{G^{-+}}(e'_i,\, e'_j)+1} \quad \text{in } G^{-+} \quad = \frac{m}{6}(m-2r+1).$$

If $d_{G^{-+}}(e'_i, e'_j) = 3$, then

$$\sum_{(e'_i, e'_j) \subseteq V(G^{-+})} \frac{1}{d_{G^{-+}}(e'_i,\, e'_j)+1} \quad \text{in } G^{-+} \quad = \frac{m}{4}(r-1).$$

When r>2

If $d_{G^{-+}}(e'_i, e'_j) = 2$, then

$$\sum_{(e'_i, e'_j) \subseteq V(G^{-+})} \frac{1}{d_{G^{-+}}(e_i,\, e_j)+1} \quad \text{in } G^{-+} \quad = \frac{1}{3}\sum_{k=2}^{m}(k-1).$$

Theorem 3.7. For any (n, m) graph G, the second Harary index is,

$$H_1(G^{-+}) = \frac{3}{4}m^2 - \frac{1}{2}m, \quad when \;\; r = 2,$$

$$H_1(G^{-+}) = \left[\frac{7}{12}m^2 + \frac{1}{12}mr - \frac{5}{12}m + \frac{1}{3}\sum_{k=2}^{m}(k-1)\right], \quad when \;\; r > 2.$$

Proof. Let G be any (n, m) graph.

From Eq.(3), we have

$$H_1(G) = \sum_{(u,v)\in V(G)} \frac{1}{d_G(u,v)+1}$$

Therefore,

$$H_1(G^{-+}) = \sum_{(e_i,e_j)\subseteq V(G^{-+})} \frac{1}{d_{G^{-+}}(e_i,e_j)+1} + \sum_{(e_i,e'_j)\subseteq V(G^{-+})} \frac{1}{d_{G^{-+}}(e_i,e'_j)+1} + \sum_{(e'_i,e'_j)\subseteq V(G^{-+})} \frac{1}{d_{G^{++}}(e'_i,e'_j)+1}.$$

Applying observation F to the above equation,

when $r = 2$,

$$H_1(G^{-+}) = \frac{m}{4}(m-2r+1) + \frac{m}{4}(r-1) + m(r-1) + \frac{m}{3}(m-2r+1) + \frac{m}{4} + \frac{m}{6}(m-2r+1) + \frac{m}{4}(r-1).$$

and if $r > 2$,

$$H_1(G^{-+}) = \frac{m}{4}(m-2r+1) + \frac{m}{4}(r-1) + m(r-1) + \frac{m}{3}(m-2r+1) + \frac{m}{4} + \frac{1}{3}\sum_{k=2}^{m}(k-1).$$

On simplification, we get

$$H_1(G^{-+}) = \frac{3}{4}m^2 - \frac{1}{2}m, \quad when\ \ r = 2,$$

$$H_1(G^{-+}) = \left[\frac{7}{12}m^2 + \frac{1}{12}mr - \frac{5}{12}m + \frac{1}{3}\sum_{k=2}^{m}(k-1)\right], \quad when\ \ r > 2.$$

Observation G.

1. Let G be any (n, m) graph

If $d_{G^{--}}(e_i,e_j) = 1$, then

$$\sum_{(e_i,e_j)\subseteq V(G^{--})} \frac{1}{d_{G^{--}}(e_i,\, e_j)} \ \text{ in } G^{--} = \frac{m}{2}(m-2r+1).$$

If $d_{G^{--}}(e_i,e_j) = 2$, then

$$\sum_{(e_i,e_j)\subseteq V(G^{--})} \frac{1}{d_{G^{--}}(e_i,\, e_j)} \ \text{ in } G^{--} = \frac{1}{2}m(r-1).$$

2. Let G be any (n, m) graph

If $d_{G^{--}}(e_i,e'_j) = 1$, then

$$\sum_{(e_i,e'_j)\subseteq V(G^{--})}\frac{1}{d_{G^{--}}(e_i,e'_j)} \text{ in } G^{--} = m(m-2r+1).$$

If $d_{G^{--}}(e_i,e'_j)=2$, then

$$\sum_{(e_i,e'_j)\subseteq V(G^{--})}\frac{1}{d_{G^{--}}(e_i,e'_j)} \text{ in } G^{--} = \frac{m}{2}(2r-1).$$

3.Let G be any (n, m) graph

When r=2 .

If $d_{G^{--}}(e'_i,e'_j)=2$, then

$$\sum_{(e'_i,e'_j)\subseteq V(G^{--})}\frac{1}{d_{G^{--}}(e'_i,e'_j)} \text{ in } G^{--} = \frac{m}{2}(r-1).$$

If $d_{G^{--}}(e'_i,e'_j)=3$, then

$$\sum_{(e'_i,e'_j)\subseteq V(G^{--})}\frac{1}{d_{G^{--}}(e'_i,e'_j)} \text{ in } G^{--} = \frac{m}{6}(m-2r+1).$$

When r >2.

If $d_{G^{--}}(e'_i,e'_j)=2$, then

$$\sum_{(e'_i,e'_j)\subseteq V(G^{--})}\frac{1}{d_{G^{--}}(e'_i,e'_j)} \text{ in } G^{--} = \frac{1}{2}\sum_{k=2}^{m}(k-1).$$

Theorem 3.8. For any (n, m) graph G, the first Harary index is

$$H(G^{--})=\frac{5}{3}m^2-\frac{4}{3}mr+\frac{1}{6}m,\ \text{when } r=2,$$

$$H(G^{--})=\frac{3}{2}m^2-\frac{3}{2}mr+\frac{1}{2}m+\frac{1}{2}\sum_{k=2}^{m}(k-1),\quad \text{When } r>2.$$

Proof. Let G be any (n, m)-graph. .

From Eq.(2), we have

$$H(G)=\sum_{(u,v)\in V(G)}\frac{1}{d_G(u,v)}$$

Therefore,

$H(G^{--})$ =

$$\sum_{(e_i, e_j)\subseteq V(G^{--})} \frac{1}{d_{G^{--}}(e_i,\, e_j)} + \sum_{(e_i, e_j')\subseteq V(G^{--})} \frac{1}{d_{G^{--}}(e_i, e_j')} + \sum_{(e_i', e_j')\subseteq V(G^{--})} \frac{1}{d_{G^{--}}(e_i', e_j')}.$$

Applying the observation G to the above equation,

when r = 2.

$$H(G^{--}) = \frac{m}{2}(m-2r+1) + \frac{1}{2}m(r-1) + m(m-2r+1) + \frac{m}{2}(2r-1) + \frac{m}{2}(r-1) + \frac{m}{6}(m-2r+1)$$

When r > 2,

$$H(G^{--}) = \frac{m}{2}(m-2r+1) + \frac{1}{2}m(r-1) + m(m-2r+1) + \frac{m}{2}(2r-1) + \frac{1}{2}\sum_{k=2}^{m}(k-1).$$

On simplification, we get

$$H(G^{--}) = \frac{5}{3}m^2 - \frac{4}{3}mr + \frac{1}{6}m,\ when\ r = 2,$$

$$H(G^{--}) = \frac{3}{2}m^2 - \frac{3}{2}mr + \frac{1}{2}m + \frac{1}{2}\sum_{k=2}^{m}(k-1),\quad When\ r > 2.$$

Observation H.

1. Let G be any (n, m) graph.

If $d_{G^{--}}(e_i, e_j) = 1$, then

$$\sum_{(e_i, e_j)\subseteq V(G^{--})} \frac{1}{d_{G^{--}}(e_i,\, e_j)+1} \quad \text{in } G^{--} = \frac{1}{4}m(m-2r+1).$$

If $d_{G^{--}}(e_i, e_j) = 2$, then

$$\sum_{(e_i, e_j)\subseteq V(G^{--})} \frac{1}{d_{G^{--}}(e_i,\, e_j)+1} \quad \text{in } G^{--} = \frac{m}{3}(r-1).$$

2. Let G be any (n, m) graph.

If $d_{G^{--}}(e_i, e'_j) = 1$, then

$$\sum_{(e_i, e'_j)\subseteq V(G^{--})} \frac{1}{d_{G^{--}}(e_i, e'_j)+1} \text{ in } G^{--} = \frac{m}{2}(m-2r+1).$$

If $d_{G^{--}}(e_i, e'_j) = 2$, then

$$\sum_{(e_i, e'_j)\subseteq V(G^{--})} \frac{1}{d_{G^{--}}(e_i, e'_j)+1} \text{ in } G^{--} = \frac{m}{3}(2r-1).$$

3. Let G be any (n, m) graph.

When r =2.

If $d_{G^{--}}(e'_i, e'_j) = 2$, then

$$\sum_{(e'_i, e'_j)\subseteq V(G^{--})} \frac{1}{d_{G^{--}}(e'_i, e'_j)+1} \text{ in } G^{--} = \frac{m}{3}(r-1).$$

If $d_{G^{+-}}(e'_i, e'_j) = 3$, then

$$\sum_{(e'_i, e'_j)\subseteq V(G^{--})} \frac{1}{d_{G^{--}}(e'_i, e'_j)+1} \text{ in } G^{--} = \frac{m}{8}(m-2r+1).$$

When r >2,

If $d_{G^{--}}(e'_i, e'_j) = 2$, then

$$\sum_{(e'_i, e'_j)\subseteq V(G^{--})} \frac{1}{d_{G^{--}}(e'_i, e'_j)+1} \text{ in } G^{--} = \frac{1}{3}\sum_{k=2}^{m}(k-1).$$

Theorem 3.9. For any (n, m) graph G, the second Harary index is,

$$H_1(G^{--}) = \frac{7}{8}m^2 - \frac{1}{8}m - \frac{5}{12}mr, \quad when \;\; r = 2,$$

$$H_1(G^{--}) = \frac{3}{4}m^2 - \frac{1}{2}mr + \frac{1}{12}m + \frac{1}{3}\sum_{k=2}^{m}(k-1), \quad when \;\; r > 2.$$

Proof. Let G be any (n, m)-graph.

From Eq.(3), we have

$$H_1(G) = \sum_{(u,v)\in V(G)} \frac{1}{d_G(u,v)+1}$$

Therefore,

$$H_1(G^{--}) = \sum_{(e_i,e_j)\subseteq V(G^{--})} \frac{1}{d_{G^{--}}(e_i, e_j)+1} + \sum_{(e_i,e'_j)\subseteq V(G^{--})} \frac{1}{d_{G^{--}}(e_i,e'_j)+1} + \sum_{(e'_i,e'_j)\subseteq V(G^{--})} \frac{1}{d_{G^{--}}(e'_i,e'_j)+1}.$$

Applying observation H to the above equation,

when r=2,

$$H_1(G^{--}) = \frac{1}{4}m(m-2r+1) + \frac{m}{3}(r-1) + \frac{m}{2}(m-2r+1) + \frac{m}{3}(2r-1) + \frac{m}{3}(r-1) + \frac{m}{8}(m-2r+1)$$

When r >2,

$$H_1(G^{--}) = \frac{1}{4}m(m-2r+1) + \frac{m}{3}(r-1) + \frac{m}{2}(m-2r+1) + \frac{m}{3}(2r-1) + \frac{1}{3}\sum_{k=2}^{m}(k-1).$$

On simplificion we get,

$$H_1(G^{--}) = \frac{7}{8}m^2 - \frac{1}{8}m - \frac{5}{12}mr, \quad when \;\; r = 2,$$

$$H_1(G^{--}) = \frac{3}{4}m^2 - \frac{1}{2}mr + \frac{1}{12}m + \frac{1}{3}\sum_{k=2}^{m}(k-1), \quad when \;\; r > 2.$$

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