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On Certain Subclass of Analytic Functions Associated with Hurwitz-Lerch Zeta Function

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Abstract

In this paper, we consider a new subclass $T\mathcal{J}_{\mu,b}(\alpha, \beta)$ consisting of analytic univalent functions with negative coefficients defined by Hurwitz-Lerch Zeta function. Coefficient inequalities, extreme points, integral means inequalities are obtained for functions in the class $T\mathcal{J}_{\mu,b}(\alpha, \beta)$. Also, subordination results for functions in the class $\mathcal{J}_{\mu,b}(\alpha, \beta)$ are derived.

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1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions f defined on the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ with normalization $f(0) = f'(0) - 1 = 0$. Such a function has the Taylor series expansion about the origin in the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in U \tag{1}$$

Denote by $\mathcal{S}$, the subclass of $\mathcal{A}$ consisting of functions that are univalent. Also, denote by T a subclass of $\mathcal{A}$ consisting functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0, z \in U \tag{2}$$

introduced and studied by Silverman [1].

For $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n; \quad z \in U \tag{3}$$

The general Hurwitz-Lerch zeta function $\Phi(z, s, a)$ is defined by (cf., e.g., [2], p. 121).

$$\Phi(z, s, a) = \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s}, \tag{4}$$

$$(a \in \mathbb{C} \backslash \{\mathbb{Z}_0^-\} : s \in \mathbb{C}, \mathbb{R}(s) > 1 \text{ when } |z| = 1\}$$

where $\mathbb{Z}_0^- = \mathbb{Z} \backslash \{\mathbb{N}\}$, $(\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}; \mathbb{N} = \{1, 2, 3, \ldots\}$.

Choi and Srivastava [3], Ferreira and Lopez [4], Garg et al. [5], Lin and Srivastava [6], Lin et al. [7] studied several interesting properties and characteristics of the Hurwitz-Lerch Zeta function $\Phi(z, s, a)$.

Srivastava and Attiya [8] (See also Raducanu and Srivastava [9], and Prajapat and Goyal [10]) introduced and investigated the linear operator $\mathcal{J}_{\mu,b} : \mathcal{A} \to \mathcal{A}$ defined interms of the Hadamard product by

$$\mathcal{J}_{\mu,b} f(z) = \mathcal{G}_{\mu,b} * f(z); \quad (z \in U; b \in \mathbb{C} \backslash \{\mathbb{Z}_0^-\}; \mu \in \mathbb{C}; f \in A) \tag{5}$$

where $\mathcal{G}_{\mu,b}(z)$ is given by

$$\mathcal{G}_{\mu,b}(z) = (1+b)^{\mu}[\Phi(z, \mu, b) - b^{-\mu}] \;\; (z \in U) \tag{6}$$

The operator $\mathcal{J}_{\mu,b}$ is called the Srivastava-Attiya operator. Using (1), (5), (6) we have

$$\mathcal{J}_{\mu,b} f(z) = z + \sum_{n=2}^{\infty} C_n(\mu, b) a_n z^n \tag{7}$$

where

$$C_n(\mu, b) = \left| \left(\frac{1+b}{n+b} \right)^{\mu} \right| \text{ and } b \in \mathbb{C} \backslash \{\mathbb{Z}_0^-\}; \mu \in \mathbb{C}; z \in U. \tag{8}$$

A class $UCD(\alpha)$, $\alpha \geq 0$ consisting of functions $f \in A$ satisfying

$$Re[f'(z)] \geq \alpha |f''(z)|, \;\; z \in U$$

was introduced and investigated in [11].

A related class $SD(\alpha)$ have been introduced in [12] and also studied by [13]. A function f of the form (1) is said to be in the class $SD(\alpha)$ if

$$Re\left\{\frac{f(z)}{z}\right\} \geq \alpha\left|f'(z) - \frac{f(z)}{z}\right|, \text{ for } \alpha \geq 0.$$

For $\alpha \geq 0, 0 \leq \beta < 1$, we let $\mathcal{J}_{\mu,b}(\alpha, \beta)$ be the subclass of $\mathcal{A}$ consisting of functions of the form (1) and satisfy

$$Re\left(\frac{\mathcal{J}_{\mu,b}f(z)}{z}\right) \geq \alpha\left|(\mathcal{J}_{\mu,b}f(z))' - \frac{\mathcal{J}_{\mu,b}f(z)}{z}\right| + \beta \tag{9}$$

where $\mathcal{J}_{\mu,b}f(z)$ is given by (7).

We further let $T\mathcal{J}_{\mu,b}(\alpha, \beta) = \mathcal{J}_{\mu,b}(\alpha, \beta) \cap T$ For $\mu = 0; \beta = 0$, the class $\mathcal{J}_{\mu,b}(\alpha, \beta)$ reduces to the class $SD(\alpha)$ studied by Sunil Varma et al. [13].

Motivated by the earlier works of [1, 14, 15, 16], in this paper, we investigate various properties and characterization of the class $T\mathcal{J}_{\mu,b}(\alpha, \beta)$. The results include integral means inequalities for the functions in the class $T\mathcal{J}_{\mu,b}(\alpha, \beta)$ and subordination results for the class of functions $f \in \mathcal{J}_{\mu,b}(\alpha, \beta)$.

2. Coefficient Estimates

Theorem 2.1. A function $f(z)$ be the form (1) is in $\mathcal{J}_{\mu,b}(\alpha, \beta)$ if

$$\sum_{n=2}^{\infty}[1 + \alpha(n-1)]C_n(\mu, b)|a_n| \leq 1 - \beta, \tag{10}$$

where $\alpha \geq 0, 0 \leq \beta < 1$ and $C_n(\mu, b)$ is given by (8).

Proof. It suffices to show that

$$\alpha\left|(\mathcal{J}_{\mu,b}f(z))' - \frac{(\mathcal{J}_{\mu,b}f(z))}{z}\right| - Re\left\{\frac{(\mathcal{J}_{\mu,b}f(z))}{z} - 1\right\} \leq 1 - \beta.$$

We have

$$\begin{aligned}
&\alpha\left|(\mathcal{J}_{\mu,b}f(z))' - \frac{(\mathcal{J}_{\mu,b}f(z))}{z}\right| - Re\left\{\frac{(\mathcal{J}_{\mu,b}f(z))}{z} - 1\right\}\\
&\quad \le \alpha\left|(\mathcal{J}_{\mu,b}f(z))' - \frac{(\mathcal{J}_{\mu,b}f(z))}{z}\right| - Re\left\{\frac{(\mathcal{J}_{\mu,b}f(z))}{z} - 1\right\}\\
&\quad \le \alpha\left|\frac{\sum_{n=2}^{\infty}(n-1)C_n(\mu,b)a_n z^n}{z}\right| + \left|\frac{\sum_{n=2}^{\infty}C_n(\mu,b)a_n z^n}{z}\right|\\
&\quad \le \alpha\sum_{n=2}^{\infty}(n-1)C_n(\mu,b)|a_n| + \sum_{n=2}^{\infty}C_n(\mu,b)|a_n|\\
&\quad = \sum_{n=2}^{\infty}(1+\alpha(n-1))C_n(\mu,b)|a_n|.
\end{aligned}$$

The last expression is bounded above by $(1-\beta)$ if
$\sum_{n=2}^{\infty}(1+\alpha(n-1))C_n(\mu,b)|a_n| \le 1-\beta$ and hence the proof. ■

In the following theorem, we obtain necessary and sufficient conditions for functions in $T\mathcal{J}_{\mu,b}(\alpha,\beta)$.

Theorem 2.2. For $\alpha \ge 0$, $0 \le \beta < 1$, a function f of the form (2) to be in the class $T\mathcal{J}_{\mu,b}(\alpha,\beta)$ if and only if

$$\sum_{n=2}^{\infty}C_n(\mu,b)(1+\alpha(n-1))|a_n| \le 1-\beta$$

Proof. Suppose $f(z)$ of the form (2) is in the class $T\mathcal{J}_{\mu,b}(\alpha,\beta)$. Then

$$Re\left[\frac{T\mathcal{J}_{\mu,b}f(z)}{z}\right] - \alpha\left|(T\mathcal{J}_{\mu,b}f(z))' - \frac{(T\mathcal{J}_{\mu,b}f(z))}{z}\right| \ge \beta.$$

Equivalently,

$$Re\left[1 - \sum_{n=2}^{\infty}C_n(\mu,b)|a_n|z^{n-1}\right] - \alpha\left|\sum_{n=2}^{\infty}(n-1)C_n(\mu,b)a_n z^{n-1}\right| \ge \beta.$$

Letting z to take real values and as $|z| \to 1$, we have

$$1 - \sum_{n=2}^{\infty}C_n(\mu,b)|a_n| - \alpha\sum_{n=2}^{\infty}(n-1)C_n(\mu,b)|a_n| \ge \beta,$$

which implies

$$\sum_{n=2}^{\infty}(1+\alpha(n-1))C_n(\mu,b)|a_n| \leq 1-\beta,$$

where $\alpha \geq 0$, $0 \leq \beta < 1$, $C_n(\mu, b)$ is given by (8) and the sufficiency follows from Theorem 2.1. ■

Corollary 2.3. If $f \in T\mathcal{J}_{\mu,b}(\alpha,\beta)$, then $|a_n| \leq \dfrac{(1-\beta)}{[1+\alpha(n-1)]C_n(\mu,b)}$, where $\alpha \geq 0$, $0 \leq \beta < 1$, $C_n(\mu, b)$ is given by (8).
Equality holds for the function $f(z) = z - \dfrac{(1-\beta)}{[1+\alpha(n-1)]C_n(\mu,b)}z^n$, $\alpha \geq 0$, $0 \leq \beta < 1$, $C_n(\mu, b)$ is given by (8).

3. Extreme points

Theorem 3.1. Let $f_1(z) = z$ and $f_n(z) = z - \dfrac{(1-\beta)}{[1+\alpha(n-1)]C_n(\mu,b)}z^n$, $n \geq 2$ for $\alpha \geq 0$, $0 \leq \beta < 1$, $C_n(\mu, b)$ are given by (8). Then $f(z)$ is in the class $T\mathcal{J}_{\mu,b}(\alpha,\beta)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=1}^{\infty}\lambda_n f_n(z),$$

where $\lambda_n \geq 0$ and $\sum_{n=1}^{\infty}\lambda_n = 1$.

Proof. Let $f(z)$ be expressible in the form $f(z) = \sum_{n=1}^{\infty}\lambda_n f_n(z)$. Then

$$\begin{aligned} f(z) &= \lambda_1 f_1(z) + \sum_{n=2}^{\infty}\lambda_n f_n(z) = \lambda_1 z + \sum_{n=2}^{\infty}\lambda_n\left[z - \frac{(1-\beta)}{[1+\alpha(n-1)]C_n(\mu,b)}z^n\right] \\ &= z - \sum_{n=2}^{\infty}\frac{(1-\beta)}{[1+\alpha(n-1)]C_n(\mu,b)}z^n. \end{aligned}$$

Now,

$$\sum_{n=2}^{\infty}\frac{[1+\alpha(n-1)]C_n(\mu,b)}{(1-\beta)}\frac{(1-\beta)}{[1+\alpha(n-1)]C_n(\mu,b)}\lambda_n = \sum_{n=1}^{\infty}\lambda_n = 1-\lambda_1 \leq 1.$$

Thus $f(z) \in T\mathcal{J}_{\mu,b}(\alpha,\beta)$.

Conversely, suppose $f(z) \in T\mathcal{J}_{\mu,b}(\alpha, \beta)$. Then Corollary 2.3 gives

$$a_n \leq \frac{(1-\beta)}{[1+\alpha(n-1)]C_n(\mu, b)}, \; n \geq 2.$$

Set $\lambda_n = \dfrac{[1+\alpha(n-1)]C_n(\mu, b)}{(1-\beta)} a_n$, $n \geq 2$, where $\lambda_1 = 1 - \sum_{n=2}^{\infty} \lambda_n$. Then

$$\begin{aligned} f(z) &= z - \sum_{n=2}^{\infty} a_n z^n \\ &= z - \sum_{n=2}^{\infty} \lambda_n \frac{(1-\beta)}{[1+\alpha(n-1)]C_n(\mu, b)} z^n \\ &= z - \sum_{n=2}^{\infty} \lambda_n z + \sum_{n=2}^{\infty} \lambda_n f_n(z) \\ &= z - \left[1 - \sum_{n=2}^{\infty} \lambda_n\right] + \sum_{n=2}^{\infty} \lambda_n f_n(z) \\ &= \lambda_1 f_1(z) + \sum_{n=2}^{\infty} \lambda_n f_n(z) \\ &= \sum_{n=1}^{\infty} \lambda_n f_n(z). \end{aligned}$$

Hence the proof. ∎

4. Integral Means inequalities

Definition 4.1. [Subordintion Principle] For analytic functions g and h with $g(0) = h(0)$, g is said to be subordinate to h, denoted by $g \prec h$, if there exists an analytic function ω suchthat $\omega(0) = 0$, $|\omega(z)| < 1$ and $g(z) = h(\omega(z))$, for all $z \in U$.

Lemma 4.2. [17] If the functions $f(z)$ and $g(z)$ are analytic in U with $g(z) \prec f(z)$, then $\int_0^{2\pi} |g(re^{i\theta})|^p d\theta \leq \int_0^{2\pi} |f(re^{i\theta})|^p d\theta$, $(0 \leq r < 1, p > 0)$.

Theorem 4.3. Suppose $f(z) \in T\mathcal{J}_{\mu,b}(\alpha, \beta)$, $p > 0$, $\alpha \geq 0$, $0 \leq \beta < 1$ and $f_2(z)$ is defined by $f_2(z) = z - \dfrac{(1-\beta)}{(1+\alpha)C_2(\mu, b)} z^2$, where $C_2(\mu, b) = \left|\left(\dfrac{1+b}{2+b}\right)^{\mu}\right|$. Then for $z = re^{i\theta}$, $0 \leq r < 1$, we have

$$\int_0^{2\pi} |f(z)|^p d\theta \leq \int_0^{2\pi} |f_2(z)|^p d\theta. \tag{11}$$

Proof. For $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$, (11) is equivalent to proving that

$$\int_0^{2\pi} \left| z - \sum_{n=2}^{\infty} |a_n| z^n \right|^p d\theta \leq \int_0^{2\pi} \left| z - \frac{(1-\beta)}{(1+\alpha)C_2(\mu, b)} z^2 \right|^p d\theta, \ (\rho > 0).$$

By applying Littlewood's subordination theorem (Lemma 4.2), it would be sufficient to show that

$$1 - \sum_{n=2}^{\infty} |a_n| z^{n-1} \prec 1 - \frac{(1-\beta)}{(1+\alpha)C_2(\mu, b)} z. \tag{12}$$

Setting

$$1 - \sum_{n=2}^{\infty} |a_n| z^{n-1} = 1 - \frac{(1-\beta)}{(1+\alpha)C_2(\mu, b)} \omega(z),$$

we obtain $\omega(z) = \frac{(1+\alpha)C_2(\mu, b)}{(1-\beta)} \sum_{n=2}^{\infty} a_n z^{n-1}$ and $\omega(z)$ is analytic in U with $\omega(0) = 0$. Moreover it suffices to prove that $\omega(z)$ satisfies $|\omega(z)| < 1, z \in U$. Now

$$\begin{aligned} |\omega(z)| = \left| \sum_{n=2}^{\infty} \frac{(1+\alpha)C_2(\mu, b)}{(1-\beta)} a_n z^{n-1} \right| &\leq |z| \sum_{n=2}^{\infty} \frac{(1+\alpha)C_2(\mu, b)}{(1-\beta)} |a_n| \\ &\leq |z| < 1. \end{aligned} \tag{13}$$

Thus in view of the inequality (13), the subordination (12) follows, which proves the theorem. ■

5. Subordination Results

Definition 5.1. [Subordination Factor Sequence] A sequence $\{b_n\}_{n=1}^{\infty}$ of complex numbers is said to be a subordinating sequence if, whenever $f(z) = \sum_{n=1}^{\infty} a_n z^n$, $a_1 = 1$ is regular, univalent and convex in U, we have $\sum_{n=1}^{\infty} b_n a_n z^n \prec f(z), z \in U$.

Lemma 5.2. [18] The sequence $\{b_n\}_{n=1}^{\infty}$ is a subordinating factor sequence if and only if $Re \left\{ 1 + 2 \sum_{n=1}^{\infty} b_n z^n \right\} > 0, z \in U$.

Theorem 5.3. Let $f \in \mathcal{J}_{\mu,b}(\alpha, \beta)$ and $g(z)$ be any function in the usual class of convex functions C, then

$$\frac{(1+\alpha)C_2(\mu, b)}{2[(1-\beta)+(1+\alpha)C_2(\mu, b)]}(f * g)(z) \prec g(z) \tag{14}$$

where $\alpha \geq 0$, $0 \leq \beta < 1$, with $C_2(\mu, b) = \left(\frac{1+b}{2+b}\right)^{\mu}$ and

$$Re\{f(z)\} > -\frac{(1-\beta)+(1+\alpha)C_2(\mu, b)}{(1+\alpha)C_2(\mu, b)}; \ z \in U. \tag{15}$$

The constant $\frac{(1+\alpha)C_2(\mu, b)}{2[(1-\beta)+(1+\alpha)C_2(\mu, b)]}$ is the best estimate.

Proof. Let $f \in \mathcal{J}_{\mu,b}(\alpha, \beta)$ and suppose that $g(z) = z + \sum_{n=2}^{\infty} c_n z^n \in C$. Then

$$\frac{(1+\alpha)C_2(\mu, b)}{2[(1-\beta)+(1+\alpha)C_2(\mu, b)]}(f * g)(z) = \frac{(1+\alpha)C_2(\mu, b)}{2[(1-\beta)+(1+\alpha)C_2(\mu, b)]}\left(z + \sum_{n=2}^{\infty} c_n a_n z^n\right)$$

Thus, by Definition 5.1, the subordination result holds true if $\left\{\frac{(1+\alpha)C_2(\mu, b)}{2[(1-\beta)+(1+\alpha)C_2(\mu, b)]}\right\}_{n=1}^{\infty}$ is a subordinating factor sequence, with $a_1 = 1$. In view of Lemma 5.2, this is equivalent to the following inequality

$$Re\left\{1 + \sum_{n=1}^{\infty} \frac{(1+\alpha)C_2(\mu, b)}{[(1-\beta)+(1+\alpha)C_2(\mu, b)]} a_n z^n\right\} > 0, \ z \in U. \tag{16}$$

Now, for $|z| = r < 1$, we have

$$\begin{aligned}
&Re\left\{1 + \sum_{n=1}^{\infty} \frac{(1+\alpha)C_2(\mu, b)}{[(1-\beta)+(1+\alpha)C_2(\mu, b)]} a_n z^n\right\} \\
&= Re\left\{1 + \frac{(1+\alpha)C_2(\mu, b)}{(1-\beta)+(1+\alpha)C_2(\mu, b)} z + \frac{\sum_{n=2}^{\infty}(1+\alpha)C_2(\mu, b) a_n z^n}{(1-\beta)+(1+\alpha)C_2(\mu, b)}\right\} \\
&\geq 1 - \frac{(1+\alpha)C_2(\mu, b)}{(1-\beta)+(1+\alpha)C_2(\mu, b)} r - \frac{\sum_{n=2}^{\infty}(1+\alpha)C_2(\mu, b) a_n r^n}{(1-\beta)+(1+\alpha)C_2(\mu, b)} \\
&\geq 1 - \frac{(1+\alpha)C_2(\mu, b)}{(1-\beta)+(1+\alpha)C_2(\mu, b)} r - \frac{(1-\beta)}{(1-\beta)+(1+\alpha)C_2(\mu, b)} r \\
&> 0,
\end{aligned}$$

using (10) and the fact that $(1+\alpha(n-1))C_n(\mu, b)$ is increasing function for $n \geq 2$.

This proves inequality (16) and hence also the subordination result (14) asserted by Theorem 5.3. The inequality (15) follows from (14) by taking

$$g(z) = \frac{z}{1-z} = z + \sum_{n=2}^{\infty} z^n \in C.$$

Now, we consider the function $F(z) := z - \dfrac{1-\beta}{(1+\alpha)C_2(\mu, b)} z^2$, where $\alpha \geq 0, 0 \leq \beta < 1$. Clearly $F \in \mathcal{J}_{\mu,b}(\alpha, \beta)$. For this function (14) becomes

$$\frac{(1+\alpha)C_2(\mu, b)}{2[(1-\beta)+(1+\alpha)C_2(\mu, b)]} F(z) \prec \frac{z}{1-z}.$$

It is easily verified that

$$\min \left\{ Re \left(\frac{(1+\alpha)C_2(\mu, b)}{2[(1-\beta)+(1+\alpha)C_2(\mu, b)]} F(z) \right) \right\} = -\frac{1}{2}, \; z \in U.$$

This shows that the constant $\dfrac{(1+\alpha)C_2(\mu, b)}{2[(1-\beta)+(1+\alpha)C_2(\mu, b)]}$ is best possible. ■

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