International Journal of Computational and Applied Mathematics.
ISSN 1819-4966 Volume 12, Number 3 (2017), pp. 843-856

http://www.ripublication.com

τ −Convex Properties of Bivariate q-Bleimann, Butzer and Hahn-Type Operators

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Abstract

In this paper, we give Approximation properties of q-BBH operator and we show that this operator preserve some properties of the original function *f*, such as Lipschitz constant and convergence properties for n when *f* is τ-convex.

[0]**2010 Mathematics SubjectClassification:** 41A25; 41A36.

[0]**Keywords:** Linear operators, Szasz-Mirakyan Operators, Bernstein Polynomials .

1. INTRODUCTION

In 1987,Lupas[3] introduced the first q-analogue of Bernstein operators and another q-generalization of the classical Bernstein polynomials is due to Phillips[6].After that many generalizations of positive linear operators based on q-integers were introduced and studied by several authors. Some are in [2,12,9].

Now, we recall some notations from q-analysis, the q-integer [n] and the q-factorial [n]! are defined by

$$[n] := [n]_q = \begin{cases} \frac{1-q^n}{1-q} & q \neq 1 \\ n & q = 1 \end{cases} \text{ for } n \in N$$

$[0]! = 1$, and

$$[n]! := \begin{cases} [1]_q [2]_q \ldots\ldots [n]_q, & n = 1,2,\ldots\ldots \\ 1 & n = 0 \end{cases} \text{ for } n \in N$$

respectively, where $q > 0$. For integers $n \geq r \geq 0$ the q- binomial coefficient is defined as

$$\begin{bmatrix} n \\ r \end{bmatrix}_q = \frac{[n]_q\,!}{[r]_q\,![n-r]_q\,!}$$

Bleimann [7] proposed a sequence of positive linear operator L_n defined by

$$L_n = \frac{1}{(1+x)^n} \sum_{k=0}^{n} f\left(\frac{k}{n-k+1}\right) \binom{n}{k} x^k \forall \ x \geq 0, n \in N \qquad (1)$$

For $f \in C\,[0,\infty)$ where $C[0,\infty)$ denote the space of all continuous and real valued functions defined on $[0,\infty)$. Also the authors proved that $L_n\,(f; x) \to (x)$as

$n \to \infty$ point wise on $[0,\infty)$ for any $f \in C_B[0,\infty)$where

$f \in C_B[0, \infty)$denote the space of all bounded functions from $C[0, \infty)$. It is well known that

$$\|f\|\, C_B = \sup_{x \geq 0} |f(x)| \quad \text{defines a norm on } C_B\,[0,\infty) \qquad (2)$$

Aral and Dogru [1] constructed the q-Bleimann, Butzer and Hahn operators as

$$L_{n,q}(f; x) = \frac{1}{\ell n(x)} \sum_{k=0}^{n} f\left(\frac{[k]}{[n-k+1]q^k}\right) q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix} x^k \qquad (3)$$

$$\text{Where } \ell n(x) = \prod_{s-o}^{n-1} (1 + q^s x) \qquad (4)$$

And $f$is defined on the semi axis$[0,\infty)$. The authors studied Korovkin-type approximation properties by using the test functions $(t)\,/((1+t))^v$for $v = 0,1,2$.

In [4] the authors introduced a new generalization of Bernstein polynomials denoted by $B\tau^n$and defined as

$$B\tau^n(f;x) = \sum_{k=0}^{n} \binom{n}{k} (f o \tau^{-1}) f\left(\frac{k}{n}\right) (1 - \tau(x))^{n-k} \tau(x)^k \qquad (5)$$

Where$Bn$is the n^{th} Bernstein polynomial, $f \in [0, 1]$, $x \in [0, 1]$,τ is a function defined on [0, 1] and having the properties:

(τ_1) τ is∞-times continuously differentiable on [0, 1]

(τ_2) $\tau(0) = 0$, $\tau(1) = 1$ and $\tau'(x) > 0$ on [0, 1].

These conditions ensure that τ is strictly increasing and the inverse τ^{-1} of τ exists on [0, 1].

Were call by [7] some usual notations and definitions which are essential for our work.

For x = (x_1, x_2) ∈ R^2, k = k_1, k_2 ∈ N_0^2 and n∈N

we will write |x| := x_1+ x_2, x^k:= $x^{k1}x^{k2}$, |k| := k_1+ k_2, k!:= k_1!k_2!

Now we define a new generalization of q-Bleimann, Butzer, and Hahn operators for $f\in[0,\infty)$ by [5]

$$L_{n,q}\big(f;\ \tau(y)\big) = \frac{1}{\ell n(y)}\sum_{k=0}^{n}(fo\tau^{-1})\left(\frac{[k]}{[n-k+1]q^k}\right)q^{k(k-1)/2}\begin{bmatrix}n\\k\end{bmatrix}\tau(y)^k \qquad (6)$$

Where $\ell n(y) = \prod_{s=o}^{n-1}(1+q^s\tau(y))$

and τ is a function that is continuously differentiable of infinite order on $[0,\infty)$such that $\tau(0)=0$, $\tau(1)=1$, and $inf_{x\in[0,\infty)}\tau'(x)\geq 1$

Definition 1 : Let $f$be a real valued function continuous defined on D ⊆ R^2 and let τ be a function satisfying the conditions (τ_1) and (τ_2) .We say that $f$is a Lipchitz continuous function of order on D , if

$$|f(x) - f(y)| \leq \left|\sum_{i=1}^{2}|\tau(x_i) - \tau(y_i)|\right|^{\mu}$$

Definition2: A continuous real valued function f is said to be convex in D ⊆$[0,\infty)$, if $f(\sum_{i=1}^{m}\propto_i x_i) \leq \sum_{i=1}^{m}\propto_i f(x_i)$for every $x_1 x_2 ....,x_m \in D$

and for every non negative number of $\alpha_1\alpha_2 ....,\alpha_m \in D$ such that $\alpha_1+\alpha_{2+} ....+,\alpha_m = 1$.

2. MAIN RESULT:-

Theorem3:- If f ∈ $Lip_A^{\tau}(\mu,S)$, then $L_n^{\tau} \in Lip_A^{\tau}(\mu,S)$.for all n ∈ N.

Proof: Let x, y ∈ S and x ≤ y which means that $x_1 \leq y_1$ and $x_2 \leq y_2$ .Using of the operators $L_{n,q}^{\tau}$and

We get

$$L_{n,q}\big(f;\ \tau(y)\big) = \frac{1}{\ell n(y)}\sum_{k=0}^{n}(fo\tau^{-1})\left(\frac{[k]}{[n-k+1]q^k}\right)q^{k(k-1)/2}\begin{bmatrix}n\\k\end{bmatrix}\tau(y)^k$$

Where $\ell n(y) = \prod_{s=o}^{n-1}(1+q^s\tau(y))$

$$L_{n,q}\big(f;\ \tau(y)\big) = \sum_{k_1=0}^{n}\sum_{k_2=0}^{n-k_1}\begin{bmatrix}n\\k\end{bmatrix}\tau(y)^k\left(\frac{1}{\prod_{s=o}^{n-1}(1+q^s\tau(y))}\right)(fo\tau^{-1})\left(\frac{[k]}{[n-k+1]q^k}\right)q^{k(k-1)/2}$$

$$= \sum_{k_1=0}^{n}\sum_{k_2=0}^{n-k_1}\begin{bmatrix}n\\k\end{bmatrix}(\tau(y)-\tau(x)+\tau(x))^k\left(\frac{1}{\prod_{s=o}^{n-1}(1+q^s\tau(y))}\right)(fo\tau^{-1})\left(\frac{[k]}{[n-k+1]q^k}\right)q^{k(k-1)/2}$$

$$=\sum_{k_1=0}^{n}\sum_{k_2=0}^{n-k_1}\sum_{i=0}^{k}\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}k\\i\end{bmatrix}(\tau(x))^{i}(\tau(y)+\tau(x))^{k-i}\left(\frac{1}{\prod_{s=o}^{n-1}(1+q^{s}\tau(y))}(fo\tau^{-1})\left(\frac{[k]}{[n-k+1]q^{k}}\right)q^{k(k-1)/2}\right.$$

If we change the order of the above summation, then we can write

$$L_{n,q}(f;\ \tau(y))=\sum_{i_1=0}^{n}\sum_{k_2=i_1}^{n}\sum_{i_2=0}^{n-k_1}\sum_{k_2=i_2}^{n-k_1}\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}k\\i\end{bmatrix}((\tau(x))^{i}(\tau(y)-\tau(x))^{k-i}\left(\frac{1}{\prod_{s=o}^{n-1}(1+q^{s}\tau(y))}\right)(fo\tau^{-1})\left(\frac{[k]}{[n-k+1]q^{k}}\right)q^{k(k-1)/2}$$

Now letting k-i=m, one gets

$$L_{n,q}(f;\ \tau(y))=\sum_{i_1=0}^{n}\sum_{m_1=0}^{n-i_1}\sum_{i_2=0}^{n-i_1-m_1}\sum_{m_2=0}^{n-|i|-m_1}\begin{bmatrix}n\\i+m\end{bmatrix}\begin{bmatrix}i+m\\i\end{bmatrix}((\tau(x))^{i}(\tau(y)-\tau(x))^{m}\left(\frac{1}{\prod_{s=o}^{n-1}(1+q^{s}\tau(y))}\right)(fo\tau^{-1})\left(\frac{[i+m]}{[n-i-m+1]q^{i+m}}\right)q^{i+m(i+m-1)/2}$$

$$=\sum_{i_1=0}^{n}\sum_{m_1=0}^{n-i_1}\sum_{i_2=0}^{n-i_1-m_1}\sum_{m_2=0}^{n-|i|-m_1}\frac{n!}{i!m!(n-|i|-|m|)!}((\tau(x))^{i}(\tau(y)-\tau(x))^{m}\left(\frac{1}{\prod_{s=o}^{n-1}(1+q^{s}\tau(y))}\right)(fo\tau^{-1})\left(\frac{[i+m]}{[n-i-m+1]q^{i+m}}\right)q^{i+m(i+m-1)/2}$$

.2.1

Similarly

$$L_{n,q}(f;\ \tau(x))=\sum_{i_1=0}^{n}\sum_{i_2=0}^{n-i_1}\begin{bmatrix}n\\i\end{bmatrix}\tau(x))^{i}(\tau(y)-\tau(x))^{m}\left(\frac{1}{\prod_{s=o}^{n-1}(1+q^{s}\tau(y))}\right)(fo\tau^{-1})\left(\frac{[i]}{[n-i+1]q^{i}}\right)q^{i+m(i+m-1)/2}$$

$$=\sum_{i_1=0}^{n}\sum_{i_2=0}^{n-i_1}\sum_{m_1=0}^{n-|i|}\sum_{m_2=0}^{n-|i|-m_1}\begin{bmatrix}n\\i\end{bmatrix}\begin{bmatrix}n-|i|-m_1\\m_2\end{bmatrix}\times((\tau(x))^{i}(\tau(y)-\tau(x))^{m}\left(\frac{1}{\prod_{s=o}^{n-1}(1+q^{s}\tau(y))}\right)(fo\tau^{-1})\left(\frac{[i]}{[n-i+1]q^{i}}\right)q^{i+m(i+m-1)/2}$$

Change of the order of the second and third summations gives

$$L_{n,q}(f;\ \tau(x))=\sum_{i_1=0}^{n}\sum_{m_1=0}^{n-i_1}\sum_{i_2=0}^{n-i_1-m_1}\sum_{m_2=0}^{n-|i|-m_1}\begin{bmatrix}n\\i\end{bmatrix}\begin{bmatrix}n-|i|\\m_1\end{bmatrix}\begin{bmatrix}n-|i|-m_1\\m_2\end{bmatrix}((\tau(x))^{i}(\tau(y)-\tau(x))^{m}\left(\frac{1}{\prod_{s=o}^{n-1}(1+q^{s}\tau(y))}\right)(fo\tau^{-1})\left(\frac{[i]}{[n-i+1]q^{i}}\right)q^{i+(i+m-1)/2}$$

$$=\sum_{i_1=0}^{n}\sum_{m_1=0}^{n-i_1}\sum_{i_2=0}^{n-i_1-m_1}\sum_{m_2=0}^{n-|i|-m_1}\frac{n!}{i!m!(n-|i|-|m|)!}\times((\tau(x))^{i}(\tau(y)-\tau(x))^{m}\left(\frac{1}{\prod_{s=o}^{n-1}(1+q^{s}\tau(y))}\right)(fo\tau^{-1})\left(\frac{[i]}{[n-i+1]q^{i}}\right)q^{i+(i+m-1)/2}$$

2.2

Thus, from (2.1) and (2.2), it follows that

$$|L_{n,q}(f;\ \tau(y)) - L_{n,q}(f;\ \tau(x))| \leq \sum_{i_1=0}^{n} \sum_{m_1=0}^{n-i_1} \sum_{i_2=0}^{n-i_1-m_1} \sum_{m_2=0}^{n-|i|-m_1} \frac{n!}{i!m!(n-|i|-|m|)!} \times ((\tau(x))^i(\tau(y)-\tau(x))^m \frac{1}{\prod_{s=o}^{n-1}(1+q^s\tau(y))}) q^{i+\frac{i+m-1}{2}} \left|fo\tau^{-1}\left(\frac{[i+m]}{[n-i-m+1]q^{i+m}}\right) - fo\tau^{-1}\left(\frac{[i]}{[n-i+1]q^i}\right)\right|$$

Since f ∈ $Lip_A^\tau(\mu, S)$,we have

$$\left|fo\tau^{-1}\left(\frac{[i+m]}{[n-i-m+1]q^{i+m}}\right) - fo\tau^{-1}\left(\frac{[i]}{[n-i+1]q^i}\right)\right| =$$

$$\left|f(\tau^{-1}\left(\frac{[i+m_1]}{[n-i-m_1+1]q^{i+m_1}}\right).\tau^{-1}\left(\frac{[i+m_2]}{[n-i-m_2+1]q^{i+m_2}}\right) f(\tau^{-1}\left(\frac{[i_1]}{[n-i_1+1]q^{i_1}}\right).\tau^{-1}\left(\frac{[i_2]}{[n-i_2+1]q^{i_2}}\right)\right|$$

$$\leq A\left[\left(\frac{m_1}{n}\right)^\mu + \left(\left(\frac{m_2}{n}\right)^\mu\right)\right]$$

And so

$$|L_{n,q}(f;\ \tau(y)) - L_{n,q}(f;\ \tau(x))| \leq A\sum_{i_1=0}^{n} \sum_{m_1=0}^{n-i_1} \sum_{i_2=0}^{n-i_1-m_1} \sum_{m_2=0}^{n-|i|-m_1} \frac{n!}{i!m!(n-|i|-|m|)!} \times ((\tau(x))^i(\tau(y)-\tau(x))^m (\frac{1}{\prod_{s=o}^{n-1}(1+q^s\tau(y))}) q^{i+\frac{i+m-1}{2}} \left[\left(\frac{m_1}{n}\right)^\mu + \left(\left(\frac{m_2}{n}\right)^\mu\right)\right]$$

Now, changing order of the summations, one has

$$|L_{n,q}(f;\ \tau(y)) - L_{n,q}(f;\ \tau(x))| \leq A\sum_{m_1=0}^{n} \sum_{i_1=0}^{n-m_1} \sum_{m_2=0}^{n-i_1-m_1} \sum_{i_2=0}^{n-i_1-|m|} \frac{n!}{i!m!(n-|i|-|m|)!} \times ((\tau(x))^i(\tau(y)-\tau(x))^m (\frac{1}{\prod_{s=o}^{n-1}(1+q^s\tau(y))}) q^{i+\frac{i+m-1}{2}} \left[\left(\frac{m_1}{n}\right)^\mu + \left(\left(\frac{m_2}{n}\right)^\mu\right)\right]$$

$$= A\sum_{m_1=0}^{n} \sum_{m_2=0}^{n-m_1} \sum_{i_1=0}^{n-|m|} \sum_{i_2=0}^{n-i_1-|m|} \frac{n!}{m!(n-|m|)!} \frac{(n-|m|)!}{i!m!(n-|i|-|m|)!} \times ((\tau(x))^i(\tau(y) - \tau(x))^m \frac{1}{\prod_{s=o}^{n-1}(1+q^s\tau(y))} q^{i+\frac{i+m-1}{2}} \left[\left(\frac{m_1}{n}\right)^\mu + \left(\left(\frac{m_2}{n}\right)^\mu\right)\right]$$

$$= A\sum_{m_1=0}^{n} \sum_{m_2=0}^{n-m_1} \begin{bmatrix} n \\ m \end{bmatrix} (\tau(y)-\tau(x))^m \left[\left(\frac{m_1}{n}\right)^\mu + \left(\frac{m_2}{n}\right)^\mu\right]$$

$$\times \sum_{i_1=0}^{n-|m|} \sum_{i_2=0}^{n-i_1-|m|} \begin{bmatrix} n-|m| \\ i \end{bmatrix} (\tau(x))^i ((\frac{1}{\prod_{s=o}^{n-1}(1+q^s\tau(y))}) q^{i+\frac{i+m-1}{2}}$$

By direct computations, we obtain

$$|L_{n,q}(f;\ \tau(y)) - L_{n,q}(f;\ \tau(x))| \leq A\sum_{m_1=0}^{n} \sum_{m_2=0}^{n-m_1} \begin{bmatrix} n \\ m \end{bmatrix} (\tau(y)-\tau(x))^m \left[\left(\frac{m_1}{n}\right)^\mu + \left(\frac{m_2}{n}\right)^\mu\right] (\tau(x))^i \frac{1}{\prod_{s=o}^{n-1}(1+q^s\tau(y))} q^{i+\frac{i+m-1}{2}}$$

$$= AL_n((\tau(t_1))^\mu + (\tau(t_2))^\mu;\ (\tau(y)-\tau(x)))$$

$$= A\{L_n((\tau(t_1))^\mu; ; \tau(y_1)-\tau(x_1)) + L_n(\tau(t_2))^\mu; \tau(y_2)-\tau(x_2)))\}$$

$$= A\{ \sum_{m_1=0}^{n} \begin{bmatrix} n \\ m_1 \end{bmatrix} (\tau(y_1)-\tau(x_1)^{m_1}[1-(\tau(y_1)-\tau(x_1)]^{n-m_1} \left(\frac{m_1}{n}\right)^\mu$$

$$+ \sum_{m_2=0}^{n} \begin{bmatrix} n \\ m_2 \end{bmatrix} (\tau(y_2)-\tau(x_2)^{m_2}[1-(\tau(y_2)-\tau(x_2)]^{n-m_2} \left(\frac{m_2}{n}\right)^\mu \}$$

Application of the Holder inequality with p = $\frac{1}{\mu}$ and q = $\frac{1}{1-\mu}$yields

$$|L_{n,q}(f;\ \tau(y)) - L_{n,q}(f;\ \tau(x))| \leq\ A\{ [L_n(\tau(t_1); (\tau(y_1)-(\tau(x_1))]^\mu [L_n(1; \tau(t_1)-\tau(x_1))]^{1-\mu}$$

$$+ [L_n(\tau(t_2); (\tau(y_2)-(\tau(x_2))]^\mu [L_n(1; \tau(t_2)-\tau(x_2))]^{1-\mu} \}$$

We have $L_n(\tau(t); \tau(x)) = \tau(x) and\ L_n(1; \tau(x)) = 1$

$$|L_{n,q}^\tau(\mathrm{f};\ \tau(y)) - L_{n,q}^\tau(\mathrm{f};\ \tau(x))| \leq\ A\{[(\tau(y_1)) - (\tau(x_1))]^\mu + [(\tau(y_2)) - (\tau(x_2))]^\mu$$

Which gives $L_n^\tau \in Lip_A^\tau\ (\mu, S)$

Theorem 4:- Let τ −convex function defined on S. Then $L_{n,q}^\tau(\mathrm{f};\ \tau(x))$ is monotonically non increasing in n.

Proof: Let x, y ∈ S and x ≤ y which means that $x_1 \leq y_1$ and $x_2 \leq y_2$ .Using of the operators $L_{n,q}^\tau$and

We get

$$L_{n,q}^\tau(\mathrm{f};\ \tau(x)) = \frac{1}{\ell n^\tau(x)} \sum_{k=0}^{n} (fo\tau^{-1}) \left(\frac{[k]}{[n-k+1]q^k}\right) q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix} \tau(x)^k$$

Where $\ell n^\tau(x) = \prod_{s=o}^{n-1}(1+q^s\tau(x))$

$$L_{n,q}^\tau = \sum_{k_1=0}^{n} \sum_{k_2=0}^{n-k_1} \begin{bmatrix} n \\ k \end{bmatrix} \tau(x)^k \left(\frac{1}{\prod_{s=o}^{n-1}(1+q^s\tau(x))}\right)(fo\tau^{-1}) \left(\frac{[k]}{[n-k+1]q^k}\right) q^{k(k-1)/2}$$

$$= \sum_{k_1=0}^{n} \sum_{k_2=0}^{n-k_1} \begin{bmatrix} n \\ k \end{bmatrix} (|(\tau(x)|+1-|\tau(x)|)(\tau(x))^k \left(\frac{1}{\prod_{s=o}^{n-1}(1+q^s\tau(x))}\right)(fo\tau^{-1}) \left(\frac{[k]}{[n-k+1]q^k}\right) q^{k(k-1)/2}$$

$$= \sum_{k_1=0}^{n} \sum_{k_2=0}^{n-k_1} \begin{bmatrix} n \\ k \end{bmatrix} \tau(x_1))^{k_1+1}\ \tau(x_2))^{k_2} \left(\frac{1}{\prod_{s=o}^{n-1}(1+q^s\tau(x))}\ (fo\tau^{-1}) \left(\frac{[k]}{[n-k+1]q^k}\right) q^{k_1(k_1-1)/2} q^{k_2(k_2-1)/2}\right.$$

$$+ \sum_{k_1=0}^{n} \sum_{k_2=0}^{n-k_1} \begin{bmatrix} n \\ k \end{bmatrix} (\tau(x_1))^{k_1}\ (\tau(x_2))^{k_2+1} \left(\frac{1}{\prod_{s=o}^{n-1}(1+q^s\tau(x))}\ (fo\tau^{-1}) \left(\frac{[k]}{[n-k+1]q^k}\right) q^{k_1(k_1-1)/2} q^{k_2(k_2-1)/2}\right.$$

$$+ \sum_{k_1=0}^{n} \sum_{k_2=0}^{n-k_1} \begin{bmatrix} n \\ k \end{bmatrix} (\tau(x))^k \left(\frac{1}{\prod_{s=o}^{n+1}(1+q^s\tau(x))}\right)(fo\tau^{-1}) \left(\frac{[k]}{[n-k+1]q^k}\right) q^{k(k-1)/2}$$

Let

$S_1 = \sum_{k_1=0}^{n} \sum_{k_2=0}^{n-k_1} \begin{bmatrix} n \\ k \end{bmatrix} \tau(x_1))^{k_1+1} \tau(x_2))^{k_2} (\frac{1}{\prod_{s=o}^{n-1}(1+q^s\tau(x))} (f o \tau^{-1}) \left(\frac{[k]}{[n-k+1]q^k}\right) q^{k_1(k_1-1)/2} q^{k_2(k_2-1)/2}$

$S_2 = \sum_{k_1=0}^{n} \sum_{k_2=0}^{n-k_1} \begin{bmatrix} n \\ k \end{bmatrix} \left(\tau(x_1)\right)^{k_1} \left(\tau(x_2)\right)^{k_2+1} (\frac{1}{\prod_{s=o}^{n-1}(1+q^s\tau(x))} (f o \tau^{-1}) \left(\frac{[k]}{[n-k+1]q^k}\right)$

$q^{k_1(k_1-1)/2} q^{k_2(k_2-1)/2}$

$S_3 = \sum_{k_1=0}^{n} \sum_{k_2=0}^{n-k_1} \begin{bmatrix} n \\ k \end{bmatrix} (\tau(x))^k (\frac{1}{\prod_{s=o}^{n+1}(1+q^s\tau(x))}) (f o \tau^{-1}) \left(\frac{[k]}{[n-k+1]q^k}\right) q^{k(k-1)/2}$

Since

$S_1 = \sum_{k_1=0}^{n} \sum_{k_2=0}^{n-k_1} \begin{bmatrix} n \\ k \end{bmatrix} \tau(x_1))^{k_1+1} \tau(x_2))^{k_2} (\frac{1}{\prod_{s=o}^{n-1}(1+q^s\tau(x))} (f o \tau^{-1}) \left(\frac{[k]}{[n-k+1]q^k}\right) q^{k_1(k_1-1)/2} q^{k_2(k_2-1)/2}$

$= \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-k_1} \begin{bmatrix} n \\ k \end{bmatrix} \tau(x_1))^{k_1+1} \tau(x_2))^{k_2} (\frac{1}{\prod_{s=o}^{n-1}(1+q^s\tau(x))}) f(\tau^{-1}\left(\frac{[k_1]}{[n-k_1+1]q^{k_1}}\right). \tau^{-1}\left(\frac{[k_2]}{[n-k_2+1]q^{k_2}}\right))$
$q^{k_1(k_1-1)/2} \ q^{k_2(k_2-1)/2} + (\tau(x_1))^{n+1} f(\tau^{-1}(1). \tau^{-1}(0)) \ q^{k(k-1)/2}$

$= \sum_{k_1=0}^{n-1} \sum_{k_2=1}^{n-k_1} \begin{bmatrix} n \\ k \end{bmatrix} \tau(x_1))^{k_1+1} \tau(x_2))^{k_2} (\frac{1}{\prod_{s=o}^{n-1}(1+q^s\tau(x))}) f(\tau^{-1}\left(\frac{[k_1]}{[n-k_1+1]q^{k_1}}\right). \tau^{-1}\left(\frac{[k_2]}{[n-k_2+1]q^{k_2}}\right))$

$q^{k_1(k_1-1)/2} \ q^{k_2(k_2-1)/2} + \sum_{k_1=0}^{n-1} \begin{bmatrix} n \\ k_1 \end{bmatrix} \tau(x_1))^{k_1+1} (\frac{1}{\prod_{s=o}^{n-1}(1+q^s\tau(x))}) f(\tau^{-1}\left(\frac{[k_1]}{[n-k_1+1]q^{k_1}}\right). \tau^{-1}(0))$
$q^{k_1(k_1-1)/2} \ + \ (\tau(x_1))^{n+1} \ f\,(\tau^{-1}(1). \tau^{-1}(0)) \ q^{k(k-1)/2}$

$= \sum_{k_1=0}^{n-2} \sum_{k_2=1}^{n-k_1-1} \begin{bmatrix} n \\ k \end{bmatrix} \tau(x_1))^{k_1+1} \tau(x_2))^{k_2} (\frac{1}{\prod_{s=o}^{n-1}(1+q^s\tau(x))}) f(\tau^{-1}\left(\frac{[k_1]}{[n-k_1+1]q^{k_1}}\right)$

$\tau^{-1}\left(\frac{[k_2]}{[n-k_2+1]q^{k_2}}\right) q^{k_1(k_1-1)/2} q^{k_2(k_2-1)/2}$

$+ \sum_{k_1=0}^{n-1} \begin{bmatrix} n \\ k_1 \end{bmatrix} \tau(x_1))^{k_1+1} \tau(x_2))^{n-k_1} (\frac{1}{\prod_{s=o}^{n-1}(1+q^s\tau(x))}) f(\tau^{-1}\left(\frac{[k_1]}{[n-k_1+1]q^{k_1}}\right)$

$. \tau^{-1}\left(\left(\frac{[n-k_1]}{[n-(n-k_1)+1]q^{n-k_1}}\right)\right) q^{\frac{k_1(k_1-1)}{2}} \quad q^{\frac{k_2(k_2-1)}{2}} +$

$\sum_{k_1=0}^{n-1} \begin{bmatrix} n \\ k_1 \end{bmatrix} \tau(x_1))^{k_1+1} (\frac{1}{\prod_{s=o}^{n-1}(1+q^s\tau(x))}) f(\tau^{-1}\left(\frac{[k_1]}{[n-k_1+1]q^{k_1}}\right). \tau^{-1}((0))$

$q^{\frac{k(k-1)}{2}} q^{k_1(k_1-1)/2} \ + (\tau(x_1))^{n+1} f(\tau^{-1}(1). \tau^{-1}(0)) q^{k(k-1)/2}$

$= \sum_{k_1=1}^{n-1} \sum_{k_2=1}^{n-k_1} \begin{bmatrix} n \\ k_1 - 1, k_2 \end{bmatrix} \tau(x))^k \,(\frac{1}{\prod_{s=o}^{n-1}(1+q^s\tau(x))}) f(\tau^{-1}\left(\frac{[k_1-1]}{[n-(k_1-1)+1]q^{k_1-1}}\right)$

$. \tau^{-1}\left(\frac{[k_2]}{[n-k_2+1]q^{k_2}}\right) q^{k_1(k_1-1)/2} q^{k_2(k_2-1)/2} +$

$\sum_{k_1=1}^{n} \begin{bmatrix} n \\ k_1 - 1 \end{bmatrix} \tau(x_1))^{k_1} \tau(x_2))^{n-k_1+1} (\frac{1}{\prod_{s=o}^{n-1}(1+q^s\tau(x))}) f(\tau^{-1}\left(\frac{[k_1-1]}{[n-(k_1-1)+1]q^{k_1-1}}\right)$

$. \tau^{-1}\left(\left(\frac{[n-k_1+1]}{[n-(n-k_1+1)+1]q^{n-k_1+1}}\right)\right) q^{k_1(k_1-1)/2}$

$+\sum_{k_1=0}^{n} \begin{bmatrix} n \\ k_1-1 \end{bmatrix} \tau(x_1))^{k_1} \left(\frac{1}{\prod_{s=o}^{n-1}(1+q^s\tau(x))}\right) f(\tau^{-1}\left(\frac{[k_1-1]}{[n-(k_1-1)+1]q^{k_1-1}}\right)$

$. \tau^{-1}((0)) q^{k_1(k_1-1)/2} + (\tau(x_1))^{n+1} f(\tau^{-1}(1). \tau^{-1}(0)) q^{k(k-1)/2}$

Since $\tau^{-1}(1) = 1, \tau^{-1}(0) = 0$

$S_1 = \sum_{k_1=1}^{n-1} \sum_{k_2=1}^{n-k_1} \begin{bmatrix} n \\ k_1-1, k_2 \end{bmatrix} \tau(x))^k \left(\frac{1}{\prod_{s=o}^{n-1}(1+q^s\tau(x))}\right) +$

$f(\tau^{-1}\left(\frac{[k_1-1]}{[n-(k_1-1)+1]q^{k_1-1}}\right). \tau^{-1}\left(\frac{[k_2]}{[n-k_2+1]q^{k_2}}\right) q^{\frac{k(k-1)}{2}}$

$\sum_{k_1=1}^{n} \begin{bmatrix} n \\ k_1-1 \end{bmatrix} \tau(x_1))^{k_1} \tau(x_2))^{n-k_1+1} \left(\frac{1}{\prod_{s=o}^{n-1}(1+q^s\tau(x))}\right) f(\tau^{-1}\left(\frac{[k_1-1]}{[n-(k_1-1)+1]q^{k_1-1}}\right)$

$. \tau^{-1}\left(\left(\frac{[n-k_1+1]}{[n-(n-k_1+1)+1]q^{n-k_1+1}}\right)\right) q^{k_1(k_1-1)/2} + \sum_{k_1=0}^{n} \begin{bmatrix} n \\ k_1-1 \end{bmatrix} \tau(x_1))^{k_1} \left(\frac{1}{\prod_{s=o}^{n-1}(1+q^s\tau(x))}\right)$

$(\tau^{-1}\left(\frac{[k_1-1]}{[n-(k_1-1)+1]q^{k_1-1}}\right), 0) q^{k_1(k_1-1)/2} + (\tau(x_1))^{n+1} f(1,0) q^{k(k-1)/2}$

$S_2 = \sum_{k_1=1}^{n-1} \sum_{k_2=1}^{n-k_1} \begin{bmatrix} n \\ k_1, k_2-1 \end{bmatrix} \tau(x))^k \left(\frac{1}{\prod_{s=o}^{n-1}(1+q^s\tau(x))}\right) f(\tau^{-1}\left(\frac{[k_1]}{[n-k_1+1]q^{k_1}}\right)$

$. \tau^{-1}\left(\frac{[k_2-1]}{[n-(k_2-1)+1]q^{k_2-1}}\right) q^{k(k-1)/2}$

$+\sum_{k_1=1}^{n} \begin{bmatrix} n \\ k_1 \end{bmatrix} \tau(x_1))^{k_1} \tau(x_2))^{n-k_1+1} \left(\frac{1}{\prod_{s=o}^{n-1}(1+q^s\tau(x))}\right) f(\tau^{-1}\left(\frac{[k_1]}{[n-k_1+1]q^{k_1}}\right)$

$. \tau^{-1}\left(\left(\frac{[n-k_1]}{[n-(n-k_1)+1]q^{n-k_1}}\right)\right) q^{k_1(k_1-1)/2}$

$+\sum_{k_1=1}^{n} \begin{bmatrix} n \\ k_2-1 \end{bmatrix} \tau(x_2))^{k_2} \left(\frac{1}{\prod_{s=o}^{n-1}(1+q^s\tau(x))}\right) f(\tau^{-1}\left(0, \frac{[k_2-1]}{[n-(k_2-1)+1]q^{k_2-1}}\right)) q^{k_2(k_2-1)/2}$

$S_3 = \sum_{k_1=1}^{n-1} \sum_{k_2=1}^{n-k_1} \begin{bmatrix} n \\ k_1, k_2 \end{bmatrix} \tau(x))^k \left(\frac{1}{\prod_{s=o}^{n-1}(1+q^s\tau(x))}\right) f(\tau^{-1}\left(\frac{[k_1]}{[n-k_1+1]q^{k_1}}\right)$

$. \tau^{-1}\left(\frac{[k_2]}{[n-k_2+1]q^{k_2}}\right) q^{k_1(k_1-1)/2} q^{k_2(k_2-1)/2} +$

$\sum_{k_1=1}^{n} \begin{bmatrix} n \\ k_1 \end{bmatrix} \tau(x_1))^{k_1} \tau(x_2))^{n-k_1+1} \left(\frac{1}{\prod_{s=o}^{n-1}(1+q^s\tau(x))}\right) f(\tau^{-1}\left(\frac{[k_1]}{[n-k_1+1]q^{k_1}}\right). \tau^{-1}(0) q^{k_1(k_1-1)/2}$

$+\sum_{k_1=1}^{n} \begin{bmatrix} n \\ k_2 \end{bmatrix} \tau(x_2))^{k_2} \left(\frac{1}{\prod_{s=o}^{n-1}(1+q^s\tau(x))}\right) f(\tau^{-1}\left(0, \frac{[k_2]}{[n-(k_2)+1]q^{k_2}}\right)) q^{\frac{k_2(k_2-1)}{2}} + f(0,0) q^{k(k-1)/2}$

Therefore, we have

$L_{n,q}^{\tau}(f;\ \tau(x)) = \sum_{k_1=1}^{n-1} \sum_{k_2=1}^{n-k_1} \begin{bmatrix} n \\ k_1-1, k_2 \end{bmatrix} \tau(x))^k \frac{1}{\prod_{s=o}^{n-1}(1+q^s\tau(x))} f(\tau^{-1}\left(\frac{[k_1-1]}{[n-(k_1-1)+1]q^{k_1-1}}\right)$

$. \tau^{-1}\left(\frac{[k_2]}{[n-k_2+1]q^{k_2}}\right) q^{k(k-1)/2} \sum_{k_1=1}^{n-1} \sum_{k_2=1}^{n-k_1} \begin{bmatrix} n \\ k_1-1, k_2 \end{bmatrix} \tau(x))^k \frac{1}{\prod_{s=o}^{n-1}(1+q^s\tau(x))}$

$f(\tau^{-1}\left(\frac{[k_1-1]}{[n-(k_1-1)+1]q^{k_1-1}}\right).\tau^{-1}\left(\left(\frac{[n-k_1+1]}{[n-(n-k_1+1)+1]q^{n-k_1+1}}\right)\right)q^{k_1(k_1-1)/2}$ +

$+\ \sum_{k_1=0}^{n}\begin{bmatrix} n \\ k_1-1\end{bmatrix}\tau(x_1))^{k_1}\frac{1}{\prod_{s=o}^{n-1}(1+q^s\tau(x))}f(\tau^{-1}\left(\frac{[k_1-1]}{[n-(k_1-1)+1]q^{k_1-1}}\right).\tau^{-1}((0))$

$q^{\frac{k_1(k_1-1)}{2}}(\tau(x_1))^{n+1}f\ (1{,}0)q^{k(k-1)/2}$

$\sum_{k_1=1}^{n-1}\sum_{k_2=1}^{n-k_1}\begin{bmatrix} n \\ k_1,k_2-1\end{bmatrix}\tau(x))^{k}\frac{1}{\prod_{s=o}^{n-1}(1+q^s\tau(x))}f(\tau^{-1}\left(\frac{[k_1]}{[n-k_1+1]q^{k_1}}\right)$

$.\tau^{-1}\left(\frac{[k_2-1]}{[n-(k_2-1)+1]q^{k_2-1}}\right)q^{k(k-1)/2}+\sum_{k_1=1}^{n}\begin{bmatrix} n \\ k_1\end{bmatrix}\tau(x_1))^{k_1}\tau(x_2))^{n-k_1+1}\frac{1}{\prod_{s=o}^{n-1}(1+q^s\tau(x))}$

$f\left(\tau^{-1}\left(\frac{[k_1]}{[n-k_1+1]q^{k_1}}\right).\tau^{-1}\left(\frac{[n-k_1]}{[n-(n-k_1)+1]q^{n-k_1}}\right)\right)q^{\frac{k_1(k_1-1)}{2}}+\sum_{k_1=1}^{n}\begin{bmatrix} n \\ k_2-1\end{bmatrix}\tau(x_2))^{k_2}\frac{1}{\prod_{s=o}^{n-1}(1+q^s\tau(x))}$

$f(.\tau^{-1}(0),\tau^{-1}\left(\frac{[k_2-1]}{[n-(k_2-1)+1]q^{k_2-1}}\right)))q^{k_2(k_2-1)/2}+(\tau(x_2))^{n+1}f\ (0{,}1)q^{k(k-1)/2}$

$+\sum_{k_1=1}^{n-1}\sum_{k_2=1}^{n-k_1}\begin{bmatrix} n \\ k_1,k_2\end{bmatrix}\tau(x))^{k}\frac{1}{\prod_{s=o}^{n-1}(1+q^s\tau(x))}\frac{(n)^k}{k!}f(\tau^{-1}\left(\frac{[k_1]}{[n-k_1+1]q^{k_1}}\right).\tau^{-1}\left(\frac{[k_2]}{[n-k_2+1]q^{k_2}}\right)q^{k(k-1)/2}$

$+\sum_{k_1=1}^{n}\begin{bmatrix} n \\ k_1\end{bmatrix}\tau(x_1))^{k_1}\tau(x_2))^{n-k_1+1}(\frac{1}{\prod_{s=o}^{n-1}(1+q^s\tau(x))}f\left(\tau^{-1}\left(\frac{[k_1]}{[n-k_1+1]q^{k_1}}\right),0\right)q^{k_1(k_1-1)/2}$

$+\sum_{k_1=0}^{n}\begin{bmatrix} n \\ k_2\end{bmatrix}\tau(x_2))^{k_2}\frac{1}{\prod_{s=o}^{n-1}(1+q^s\tau(x))}f(.\tau^{-1}(0),\tau^{-1}(\frac{[k_2]}{[n-k_2+1]q^{k_2}}))q^{k_2(k_2-1)/2}+f(0{,}0)q^{k(k-1)/2}$ 3.1

Similarly

$L_{n+1,q}^{\tau}(f;\ \tau(x))=\sum_{k_1=0}^{n+1}\sum_{k_2=0}^{n+1-k_1}\begin{bmatrix} n+1 \\ k\end{bmatrix}(\tau(x))^{k}\ \frac{1}{\prod_{s=o}^{n-1}(1+q^s\tau(x))})(fo\tau^{-1})\left(\frac{[k]}{[(n+1)-k+1]q^{k}}\right)q^{k(k-1)/2}$

$=\sum_{k_1=1}^{n-1}\sum_{k_2=1}^{n-k_1}\begin{bmatrix} n+1 \\ k_1,k_2\end{bmatrix}\tau(x))^{k}\frac{1}{\prod_{s=o}^{n-1}(1+q^s\tau(x))}f(\tau^{-1}\left(\left(\frac{[k_1]}{[(n+1)-k_1+1]q^{k_1}}\right)\right)$

$.\tau^{-1}\left(\left(\frac{[k_2]}{[(n+1)-k_2+1]q^{k_2}}\right)\right)q^{\frac{k(k-1)}{2}}+\sum_{k_1=1}^{n}\begin{bmatrix} n+1 \\ k_1\end{bmatrix}\tau(x_1))^{k_1}\tau(x_2))^{n-k_1+1}\frac{1}{\prod_{s=o}^{n-1}(1+q^s\tau(x))}$

$\tau^{-1}\left(\frac{[k_1]}{[(n+1)-k_1+1]q^{k_1}}\right).\tau^{-1}\left(\left(\frac{[n-k_1+1]}{[(n-(n-k_1+1)+1]q^{n-k_1+1}}\right)\right)q^{\frac{k_1(k_1-1)}{2}}$

$+\ \sum_{k_1=0}^{n}\begin{bmatrix} n+1 \\ k_1\end{bmatrix}\tau(x_1))^{k_1}\frac{1}{\prod_{s=o}^{n-1}(1+q^s\tau(x))}f(\tau^{-1}\left(\frac{[k_1]}{[(n+1)-k_1+1]q^{k_1}}\right)$

$.\tau^{-1}((0))q^{k_1(k_1-1)/2}+(\tau(x_1))^{n+1}f(1{,}0)q^{k(k-1)/2}\sum_{k_2=0}^{n}\begin{bmatrix} n+1 \\ k_2\end{bmatrix}\tau(x_2))^{k_2}\frac{1}{\prod_{s=o}^{n-1}(1+q^s\tau(x))}$

$f(.\tau^{-1}(0),\tau^{-1}\left(\frac{[k_2]}{[(n+1)-k_2+1]q^{k_2}}\right))q^{k_2(k_2-1)/2}+(\tau(x_2))^{n+1}f(0{,}1)+\ f(0{,}0)(1{,}0)q^{k(k-1)/2}$

3.2

Thus, we can write $\ell n^{\tau}(x)=\prod_{s=o}^{n-1}(1+q^s\tau(x))$

$L_{n,q}^{\tau}(f;\ \tau(x))-L_{n+1,q}^{\tau}(f;\ \tau(x))=$

$\sum_{k_1=1}^{n-1}\sum_{k_2=1}^{n-k_1}\tau(x))^{k}q^{k(k-1)/2}\frac{1}{\ell n^{\tau}(x)}\ \{\begin{bmatrix} n \\ k_1-1,k_2\end{bmatrix}(f(\tau^{-1}\left(\frac{[k_1-1]}{[n-(k_1-1)+1]q^{k_1-1}}\right)$

$.\tau^{-1}\left(\left(\frac{[k_2]}{[n-k_2+1]q^{k_2}}\right)\right)) + \begin{bmatrix} n \\ k_1, k_2-1 \end{bmatrix} (f(\tau^{-1}\left(\frac{[k_1]}{[n-k_1+1]q^{k_1}}\right).\tau^{-1}\left(\left(\frac{[k_2-1]}{[n-(k_2-1)+1]q^{k_2-1}}\right)\right)) +$

$\begin{bmatrix} n \\ k_1, k_2 \end{bmatrix} (f(\tau^{-1}\left(\frac{[k_1]}{[n-k_1+1]q^{k_1}}\right).\tau^{-1}\left(\left(\frac{[k_2]}{[n-k_2+1]q^{k_2}}\right)\right))$

$-\begin{bmatrix} n+1 \\ k_1, k_2 \end{bmatrix} f\left(\tau^{-1}\left(\left(\frac{[k_1]}{[(n+1)-k_1+1]q^{k_1}}\right)\right).\tau^{-1}\left(\frac{[k_2]}{[(n+1)-k_2+1]q^{k_2}}\right)\right\}$

$+\sum_{k_1=1}^{n} \frac{1}{\ell n^{\tau}(x)} \tau(x_1))^{k_1} \tau(x_2))^{n-k_1-1} q^{\frac{k_1(k_1-1)}{2}} \{\begin{bmatrix} n \\ k_1-1 \end{bmatrix} (f(\tau^{-1}\left(\frac{[k_1-1]}{[n-(k_1-1)+1]q^{k_1-1}}\right)$

$.\tau^{-1}\left(\left(\frac{[n-k_1+1]}{[(n-(n-k_1+1)+1]q^{n-k_1+1}}\right)\right) + \begin{bmatrix} n \\ k_1 \end{bmatrix} f(\tau^{-1}\left(\left(\frac{[k_1]}{[n-k_1+1]q^{k_1}}\right)\right).\tau^{-1}(\frac{[n-k_1]}{[n-(n-k_1)+1]q^{n-k_1}}))$

$-\begin{bmatrix} n+1 \\ k_1 \end{bmatrix} f\left(\tau^{-1}\left(\frac{[k_1]}{[(n+1)-k_1+1]q^{k_1}}\right).\tau^{-1}\left(\left(\frac{[n-k_1+1]}{[(n-(n-k_1+1)+1]q^{n-k_1+1}}\right)\right)\right\}$

$+\sum_{k_1=0}^{n} \frac{1}{\ell n^{\tau}(x)} \tau(x_1))^{k_1} q^{k_1(k_1-1)/2} \{\begin{bmatrix} n \\ k_1-1 \end{bmatrix} f(\tau^{-1} \frac{[k_1-1]}{[n-(k_1-1)+1]q^{k_1-1}}.\tau^{-1}((0))+$

$\begin{bmatrix} n \\ k_1 \end{bmatrix} f\left(\tau^{-1}\left(\left(\frac{[k_1]}{[n-k_1+1]q^{k_1}}\right)\right), 0\right)$

$-\begin{bmatrix} n+1 \\ k_1 \end{bmatrix} f\left(\tau^{-1}\left(\left(\frac{[k_1]}{[(n+1)-k_1+1]q^{k_1}}\right)\right), 0\right)\} +$

$\sum_{k_2=1}^{n} \frac{1}{\ell n^{\tau}(x)} \tau(x_2))^{k_2} q^{k_2(k_2-1)/2} \{\begin{bmatrix} n \\ k_2-1 \end{bmatrix} f\left(.\tau^{-1}(0), \tau^{-1}\left(\frac{[k_2-1]}{[n-(k_2-1)+1]q^{k_2-1}}\right)\right) +$

$+\begin{bmatrix} n \\ k_2 \end{bmatrix} f\left(.\tau^{-1}(0), \tau^{-1}\left(\left(\frac{[k_2]}{[n-k_2+1]q^{k_2}}\right)\right)\right) -$

$\begin{bmatrix} n+1 \\ k_2 \end{bmatrix} f\left(.\tau^{-1}(0), \tau^{-1}\left(\frac{[k_2]}{[(n+1)-k_2+1]q^{k_2}}\right)\right\}\right\} + \frac{1}{\ell n^{\tau}(x))} \{f(1,0) - f(1,0)\} q^{k(k-1)/2} (\tau(x_1))^{n+1}$

$+\frac{1}{\ell n^{\tau}(x)} q^{k(k-1)/2} \{f(0,1) - f(0,1)\} - \tau(x_1))^{n+1} \{f(0,0)\} e^{-n|\tau(x)|} q^{k(k-1)/2}$

$L_n(f;\ \tau(x)) - L_{n+1}(f;\ \tau(x)) =$

$\tau(x_2))\tau(x))^k \sum_{k_1=0}^{n-2} \sum_{k_2=0}^{n-2-k_1} \begin{bmatrix} n \\ k_1, k_2+1 \end{bmatrix} \{ (f o \tau^{-1})\left(\left(\frac{[k_1]}{[n-k_1+1]q^{k_1}}\right), \frac{[k_2+1]}{[n-(k_2-1)+1]q^{k_2+1}}\right)$

$+\begin{bmatrix} n \\ k_1+1, k_2 \end{bmatrix} (f o \tau^{-1})\left(\frac{[k_1+1]}{[n-(k_1-1)+1]q^{k_1+1}}, \frac{[k_2]}{[n-k_2+1]q^{k_2}}\right) +$

$\begin{bmatrix} n \\ k_1+1, k_2+1 \end{bmatrix} (f o \tau^{-1})\left(\frac{[k_1+1]}{[n-(k_1-1)+1]q^{k_1+1}}, \frac{[k_2+1]}{[n-(k_2-1)+1]q^{k_2+1}}\right)$

$-\begin{bmatrix} n+1 \\ k_1+1, k_2+1 \end{bmatrix}(fo\tau^{-1})\left(\frac{[k_1+1]}{[(n+1)-(k_1-1)+1]q^{k_1+1}}, \frac{[k_2+1]}{[(n+1)-(k_2-1)+1]q^{k_2+1}}\right)\}q^{k(k-1)/2}\frac{1}{\ell_{n+1}^{\tau}}+$
$\tau(x_1))\tau(x_2)\sum_{k_1=0}^{n-1}\{\begin{bmatrix} n \\ k_1 \end{bmatrix}(fo\tau^{-1})\left(\frac{[k_1]}{[n-k_1+1]q^{k_1}}, \frac{[n-k_1]}{[n-(n+k_1)+1]q^{n-k_1}}\right)+$
$\begin{bmatrix} n \\ k_1+1 \end{bmatrix}(fo\tau^{-1})\left(\frac{[k_1+1]}{[n-(k_1-1)+1]q^{k_1+1}}, \frac{[n-k_1-1]}{[n-(n-k_1-1)+1]q^{n-k_1}}\right)$

$-\begin{bmatrix} n+1 \\ k_1+1 \end{bmatrix}(fo\tau^{-1})\left(\frac{[k_1+1]}{[(n+1)-(k_1-1)+1]q^{k_1+1}}, \frac{[n-k_1]}{[n-(n-k_1)+1]q^{n-k_1}}\right)\}q^{k_1(k_1-1)/2}$

$\tau(x_1)^{x_1}\frac{1}{\ell_{n+1}^{\tau}}+\tau(x_1)\sum_{k_1=0}^{n-1}\{\begin{bmatrix} n \\ k_1 \end{bmatrix}(fo\tau^{-1})\left(\frac{[k_1]}{[n-k_1+1]q^{k_1}}, 0\right)$

$+\begin{bmatrix} n \\ k_1+1 \end{bmatrix}(fo\tau^{-1})\left(\frac{[k_1+1]}{[n-(k_1-1)+1]q^{k_1+1}}, 0\right)-$
$\begin{bmatrix} n+1 \\ k_1+1 \end{bmatrix}(fo\tau^{-1})\left(\frac{[k_1+1]}{[n-(k_1-1)+1]q^{k_1+1}}, 0\right)\}q^{k_1([k_1-1)/2}\tau(x_1)^{k_1}\frac{1}{\ell_{n+1}^{\tau}}$

$+\tau(x_2)\sum_{k_2=0}^{n-1}\{\begin{bmatrix} n \\ k_2 \end{bmatrix}(fo\tau^{-1})\left(0, \frac{[k_2]}{[n-k_2+1]q^{k_2}}\right)+\begin{bmatrix} n \\ k_2+1 \end{bmatrix}(fo\tau^{-1})$

$\left(0, \frac{[k_2+1]}{[n-(k_2-1)+1]q^{k_2+1}},\right)-\begin{bmatrix} n+1 \\ k_2+1 \end{bmatrix}(fo\tau^{-1})\left(0, \frac{[k_2+1]}{[n-(k_2-1)+1]q^{k_2+1}}\right)$

$q^{k_2([k_2-1)/2}\tau(x_1)^{k_2}\frac{1}{\ell_{n+1}^{\tau}}$

Now set

$I_1:=\begin{bmatrix} n \\ k_1, k_2+1 \end{bmatrix}(fo\tau^{-1})\left(\left(\frac{[k_1]}{[n-k_1+1]q^{k_1}}\right), \frac{[k_2+1]}{[n-(k_2-1)+1]q^{k_2+1}}\right)+$
$\begin{bmatrix} n \\ k_1+1, k_2 \end{bmatrix}(fo\tau^{-1})\left(\frac{[k_1+1]}{[n-(k_1-1)+1]q^{k_1+1}}, \frac{[k_2]}{[n-k_2+1]q^{k_2}}\right)$
$+\begin{bmatrix} n \\ k_1+1, k_2+1 \end{bmatrix}(fo\tau^{-1})\left(\frac{[k_1+1]}{[n-(k_1-1)+1]q^{k_1+1}}, \frac{[k_2+1]}{[n-(k_2-1)+1]q^{k_2+1}}\right)-$
$\begin{bmatrix} n+1 \\ k_1+1, k_2+1 \end{bmatrix}(fo\tau^{-1})\left(\frac{[k_1+1]}{[(n+1)-(k_1-1)+1]q^{k_1+1}}, \frac{[k_2+1]}{[(n+1)-(k_2-1)+1]q^{k_2+1}}\right)$

$I_2:=\begin{bmatrix} n \\ k_1 \end{bmatrix}(fo\tau^{-1})\left(\frac{[k_1]}{[n-k_1+1]q^{k_1}}, \frac{[n-k_1]}{[n-(n+k_1)+1]q^{n-k_1}}\right)+$
$\begin{bmatrix} n \\ k_1+1 \end{bmatrix}(fo\tau^{-1})\left(\frac{[k_1+1]}{[n-(k_1-1)+1]q^{k_1+1}}, \frac{[n-k_1-1]}{[n-(n-k_1-1)+1]q^{n-k_1}}\right)$
$-\begin{bmatrix} n+1 \\ k_1+1 \end{bmatrix}(fo\tau^{-1})\left(\frac{[k_1+1]}{[(n+1)-(k_1-1)+1]q^{k_1+1}}, \frac{[n-k_1]}{[n-(n-k_1)+1]q^{n-k_1}}\right)$

$I_3:=\sum_{k_1=0}^{n-1}\{\begin{bmatrix} n \\ k_1 \end{bmatrix}(fo\tau^{-1})\left(\frac{[k_1]}{[n-k_1+1]q^{k_1}}, 0\right)$
$+\begin{bmatrix} n \\ k_1+1 \end{bmatrix}(fo\tau^{-1})\left(\frac{[k_1+1]}{[n-(k_1-1)+1]q^{k_1+1}}, 0\right)$
$-\begin{bmatrix} n+1 \\ k_1+1 \end{bmatrix}(fo\tau^{-1})\left(\frac{[k_1+1]}{[n-(k_1-1)+1]q^{k_1+1}}, 0\right)$

$$I_4 := \begin{bmatrix} n \\ k_2 \end{bmatrix} (fo\tau^{-1}) \left(0, \frac{[k_2]}{[n-k_2+1]q^{k_2}}\right)$$

$$+ \begin{bmatrix} n \\ k_2+1 \end{bmatrix} (fo\tau^{-1}) \left(0, \frac{[k_2+1]}{[n-(k_2-1)+1]q^{k_2+1}}, \right)$$

$$- \begin{bmatrix} n+1 \\ k_2+1 \end{bmatrix} (fo\tau^{-1}) \left(0, \frac{[k_2+1]}{[n-(k_2-1)+1]q^{k_2+1}}\right)$$

We firstly consider I_1. Let

$$\propto_1 = \frac{\begin{bmatrix} n \\ k_1, k_2+1 \end{bmatrix}}{\begin{bmatrix} n+1 \\ k_1+1, k_2+1 \end{bmatrix}} = \frac{k_1+1}{n+1} \geq 0 \ , \quad \propto_2 = \frac{\begin{bmatrix} n \\ k_1+1, k_2 \end{bmatrix}}{\begin{bmatrix} n+1 \\ k_1+1, k_2+1 \end{bmatrix}} = \frac{k_2+1}{n+1} \geq 0 \ ,$$

$$\propto_3 = \frac{\begin{bmatrix} n \\ k_1+1, k_2+1 \end{bmatrix}}{\begin{bmatrix} n+1 \\ k_1+1, k_2+1 \end{bmatrix}} = \frac{n-|k|-1}{n+1} \geq 0$$

And

$$x_1 = \left(\left(\frac{[k_1]}{[n-k_1+1]q^{k_1}}\right), \frac{[k_2+1]}{[n-(k_2-1)+1]q^{k_2+1}}\right),$$

$$x_2 = \left(\frac{[k_1+1]}{[n-(k_1-1)+1]q^{k_1+1}}, \frac{[k_2]}{[n-k_2+1]q^{k_2}}\right),$$

$$x_3 = (\left(\frac{[k_1+1]}{[n-(k_1-1)+1]q^{k_1+1}}, \frac{[k_2+1]}{[n-(k_2-1)+1]q^{k_2+1}}\right)$$

Then it is easily seen that $\propto_1 + \propto_2 + \propto_3 = 1$ and $\propto_1 x_1 + \propto_2 x_2 + \propto_3 x_3 = (\frac{k_1+1}{n+1}, \frac{k_2+1}{n+1})$.

Thus, from the definition of τ – convexity it readily follows that $I_1 \geq 0$.

For I_2 , we set

$$\propto_1 = \frac{\begin{bmatrix} n \\ k_1 \end{bmatrix}}{\begin{bmatrix} n+1 \\ k_1+1 \end{bmatrix}} = \frac{k_1+1}{n+1} \geq 0 \ , \quad \propto_2 = \frac{\begin{bmatrix} n \\ k_1+1 \end{bmatrix}}{\begin{bmatrix} n+1 \\ k_1+1 \end{bmatrix}} = \frac{n-k_1}{n+1} \geq 0$$

And

$$x_1 = \left(\frac{[k_1]}{[n-k_1+1]q^{k_1}}, \frac{[n-k_1]}{[n-(n+k_1)+1]q^{n-k_1}}\right),$$

$$x_2 = \left(\frac{[k_1+1]}{[n-(k_1-1)+1]q^{k_1+1}}, \frac{[n-k_1-1]}{[n-(n-k_1-1)+1]q^{n-k_1}}\right)$$

Thus we have $\propto_1 + \propto_2 = 1$ and $\propto_1 x_1 + \propto_2 x_2 = (\frac{k_1+1}{n+1}, \frac{n-k_1}{n+1})$. Thus, from the definiation of τ – convexity it readily follows that $I_2 \geq 0$

For I_3 , we set

$$\propto_1 = \frac{\begin{bmatrix} n \\ k_1 \end{bmatrix}}{\begin{bmatrix} n+1 \\ k_1+1 \end{bmatrix}} = \frac{k_1+1}{n+1} \geq 0 \quad , \quad \propto_2 = \frac{\begin{bmatrix} n \\ k_1+1 \end{bmatrix}}{\begin{bmatrix} n+1 \\ k_1+1 \end{bmatrix}} = \frac{n-k_1}{n+1} \geq 0$$

And

$$x_1 = \left(\frac{[k_1]}{[n-k_1+1]q^{k_1}}, 0\right), \quad x_2 = \left(\frac{[k_1+1]}{[n-(k_1-1)+1]q^{k_1+1}}, 0\right)$$

Thus we have $\propto_1 + \propto_2 = 1$ and $\propto_1 x_1 + \propto_2 x_2 = (\frac{k_1+1}{n+1}, 0)$. Thus, from the definiation of τ – convexity it readily follows that $I_3 \geq 0$ similarly $I_4 \geq 0$. Therefore, from (3.3)

$$L_n\left((f;\, \tau(x))\right) - L_{n+1}\left((f;\, \tau(x))\right) \geq 0$$

We reach to the desired result $L_n(f; \tau(x) \geq L_{n+1}(f; \tau(x)$for all n $\in$ N.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

ACKNOWLEDGMENT

The authors thank the referee for their helpful suggestions and valuable comments.

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