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Results on Global Exponential Stability of Impulsive Retarded Functional Differential Equation

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Abstract

In this paper the global exponential stability of impulsive retarded functional differential system is investigated. By means of Lyapunov functions combined with the Razumikhin technique the effect of delay at the time of impulses for global exponential stability is obtained. This result extends some existing results in the literature.

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1. Introduction

The concept of impulsive delay differential equations is used in many problems like in the field of communication network, control theory, agricultural sciencs, medical sciences and many more [1, 3, 4]. In the recent years the theory of stability analysis has been extensively studied. In these studies major interest has been shown on exponential stability because it has a very significant role in many areas like synchronization of coupled oscillator, neural networks etc [2, 5, 6]. In [7] the criteria on exponential stability is obtained for impulsive delay differential equations by using Lyapunov function along with Razumikhin technique. In [8, 9] the stability of impulsive differential equations with delay is considered and various results are obtained. In the obtained results researchers predominantly consider the relation amongst state variables on impulses to present state variables. But very less work is done in which state variables at the time of impulses are related to the time delay, so in this letter impulsive retarded delay differential equations where the state variables are related with past and present state variables is considered and some criteria as a theorem for global exponential stability is obtained in which the effect of delay at the time of impulses is obtained.

This paper is organized as follows. In section 2, we introduce some notations and definitions. In sections 3, we establish the several criteria about the global exponential stability of the systems of impulsive retarded differential equations. Concluding remarks are given in section 4.

2. Preliminaries

Consider Impulsive retarded functional differential equations

$$\begin{aligned} x' &= f(t, x_t), \quad t \neq \quad t_j, \quad t \geq t_0 \\ \Delta x(t_j) &= I_j(x(t_j^-)) + J_j(x(t_j^- - \tau)) \\ x_{t_0} &= \eta \end{aligned} \tag{1}$$

we assume the function $f : R_+ \times PC([-\tau, 0], R^n) \to R^n$;$\eta \in PC([-\tau, 0], R^n)$; $I_j, J_j \in C[R^n, R^n]$ satisfy all the required conditions for the global existence and uniqueness of solutions $\forall \quad t \geq t_0$. $[t_j]_{j=1}^{\infty}$ satisfies $0 = t_0 < t_1 < \cdots < t_j < \cdots, lim_{j\to\infty} t_j = \infty$.$\Delta x(t_j) = x(t_j) - x(t_j^-)$, where $x_t, x_{t^-} \in R^n$. For the given constant $\tau > 0$, we equip the linear space $PC([-\tau, 0], R^n)$ with the norm $||.||$ defined by $||.|| = sup_{-\tau<j<0}||\psi(j)||$. Denote $x(t) = x(t, t_0, \eta)$, the unique solution of equation (1), where $x_{t_0} = \eta$, we further suppose that all the solutions of $x(t)$ of (1) are continuous except at t_j, where $j \in N$, at which $x(t)$ is right continuous i.e. $x(t_j^+) = x(t_j)$, $j \in N$.

Definition 2.1. For a function $V : R_+ \times R^n \longrightarrow R_+$, the upper right-hand derivative of the function V with respect to system (1) is defined by
$D_+V(t, \vartheta(0)) = \lim_{\kappa\longrightarrow 0^+} sup\frac{1}{\kappa}[V(t + \kappa, \vartheta(0) + \kappa f(t, \phi)) - V(t, \vartheta(0))]$
for $(t, \phi) \in R_+ \times PC([-\tau, 0], R^n)$.

Definition 2.2. The function $V : R_+ \times R^n \to R_+$ belongs to class υ_0 if it holds following conditions:

(i) the function V is continuous on each of the sets $[t_{k-1}, t_k) \times R^n$ and for each $x \in R^n, t \in [t_{k-1}, t_k), k \in N, lim_{(t,r)\to(t_k^-,x)} V(t, r) = V(t_k^-, x)$ exists.

(ii) V is locally Lipschitzian with respect to $x \in R^n$ and $\forall t \geq t_0$, $V(t, 0) \equiv 0$.

Definition 2.3. The zero solution of the system (1) is said to be globally exponentially stable if $\exists$ some constants $c > 0$ and $\Re \geq 1$ such that for any initial value $x_{t_0} = \phi$ $||x(t, t_0, \phi)|| \leq \Re||\phi||e^{-c(t-t_0)}, t \geq t_0$,where $(t_0, \phi) \in R_+ \times PC([-\tau, 0], R^n)$.

3. Main Results

Theorem 3.1. Consider a function $V \in \upsilon_0$ and some constants $w, w_1, w_2, q, \alpha, \beta > 0, \gamma \geq 1$ and $\alpha - \beta \geq w$ such that

(i) $w_1||x||^q \leq V(t, x) \leq w_2||x||^q$,for any $t \in R_+, x \in R^n$.

(ii) $D^+V(t, \eta(0)) \leq wV(t, \eta(0)), \forall t \in [t_{j-1}, t_j), j \in N$, whenever $\rho V(t, \eta(0)) \geq V(t + u, \eta(u))$ for $u \in [-\tau, 0]$, where $\rho \geq \gamma e^{\beta\tau}$ is a constant.

(iii) $V(t_j, \eta(0)+I_j(\eta(0))+J_j(\eta(0))) \leq \xi_j(V(t_j^-, \eta(0))+supV(t_j^-+u, \eta(0))$ where $\xi_j \in (0, 1], \forall j \in N$ are constants.

(iv) $\beta \geq \dfrac{1}{\xi_{j-1}}, \xi_j = \dfrac{e^{-(\alpha+\beta)(t_{j+1}-t_j)}}{1 + e^{\beta\tau}}$,where $j \in N$.

Then the zero solution of impulsive differential equation(1) is globally exponential stable whose convergence rate is β/q for any time delay τ where $\tau \in (0, \infty)$.

Proof. Let $x(t) = x(t, t_0, \eta)$ be the solution of the impulsive differential system (1) with initial condition $x_{t_0} = \eta$ and $\upsilon(t) = V(t, x)$.
We shall prove that

$$\upsilon(t) \leq \wp||\eta||^q e^{-\beta(t-t_0)}, t \in [t_{j-1}, t_j), j \in N \tag{2}$$

Let us consider that $\gamma \geq sup_{j\in N}(\dfrac{1}{\xi_{j-1}})$. From condition (iv), we can select a constant $\wp > 0$, such that

$$0 < w_2||\eta||^q < w_2||\eta||^q e^{(t_1-t_0)} \leq \wp \leq w_2\gamma e^{\beta\tau-(\alpha+\beta)(t_1-t_0)} e^{(\alpha+\beta)(t_1-t_0)} \tag{3}$$

then it implies that

$$0 < w_2||\eta||^q < w_2||\eta||^q e^{\alpha(t_1-t_0)} \leq \wp||\eta||^q e^{-\beta(t_1-t_0)} \tag{4}$$

we will first prove that

$$\upsilon(t) \leq \wp||\eta||^q e^{-\beta(t-t_0)}, t \in [t_0, t_1) \tag{5}$$

To prove this we are only required to prove that

$$\upsilon(t) \leq \wp||\eta||^q e^{-\beta(t_1-t_0)}, t \in [t_0, t_1) \tag{6}$$

If the equation (6) is false, then by equation (4), there exist $\hat{t} \in [t_0, t_1)$ such that $\upsilon(\hat{t} > \wp||\eta||^q e^{-\beta(t_1-t_0)} \geq w_2||\eta||^q e^{\alpha(t_1-t_0)} > w_2||\eta||^q \geq \upsilon(t_0+u), u \in [-\tau, 0]$ which means that there exist $t^{'} \in (t_0, \hat{t})$, such that

$$\upsilon(t^{'}) = \wp||\eta||^q e^{-\beta(t_1-t_0)}, \upsilon(t) \leq \upsilon(t'), t \in [t_0-\tau, t^{'}] \tag{7}$$

also there exist $t^{''} \in (t_0, \hat{t})$ such that

$$\upsilon(t^{''}) = w_2||\eta||^q, \upsilon(t^{''}) \leq \upsilon(t) \leq \upsilon(t'), t \in [t^{''}, t^{'}] \tag{8}$$

Therefore for any $u \in [-\tau, 0]$, using equation (3) and (8), we get
$\upsilon(t+u) \leq \wp||\eta||^q e^{-\beta(t_1-t_0)} \leq w_2\gamma e^{\beta\tau-(\alpha+\beta)(t_1-t_2)} e^{(\alpha+\beta)(t_1-t_0)}||\eta||^q e^{-\beta(t_1-t_0)}$

$$\leq \gamma e^{\beta\tau} w_2||\eta||^q = \gamma e^{\beta\tau}\upsilon(t^{''}) \leq q\upsilon(t), t \in [t'', t'] \tag{9}$$

So taking equation (9) and condition (ii), for $t \in [t'', t']$, we get $D^+\upsilon(t) \leq w\upsilon(t) \leq (\alpha-\beta)\upsilon(t)$. It follows from equations (3), (7) and (8) that
$\upsilon(t') \leq \upsilon(t'')e^{(\alpha-\beta)(t^{'}-t^{''})} < w_2||\eta||^q e^{(\alpha-\beta)(t_1-t_0)} < w_2||\eta||^q e^{\alpha(t_1-t_0)}$
$= w_2||\eta||^q e^{(\alpha+\beta)(t_1-t_0)} e^{-\beta(t_1-t_0)} \leq \wp||\eta||^q e^{-\beta(t_1-t_0)} = \upsilon(t')$
which is a contradiction.So equation (5) holds and equation (2) is true for j=1.
Now we assume that result (2) is true for $j = 1, 2, \ldots, n$, $(n \in N)$, i.e.,

$$\upsilon(t) \leq \wp||\eta||^q e^{-\beta(t-t_0)}, t \in [t_{j-1}, t_j), j = 1, 2, ..., n. \tag{10}$$

Now we shall show that result (2) hold for $j = n+1$, i.e.,

$$\upsilon(t) \leq \wp||\eta||^q e^{-\beta(t-t_0)}, t \in [t_n, t_{n+1}) \tag{11}$$

Let us consider that the result of equation (11) is false. Then define $\hat{t} = inf[t \in [t_n, t_{n+1}); \upsilon(t) > \wp||\eta||^q e^{-\beta(t-t_0)}]$. From conditions (iii), (iv) and equation (10), we get

$$\begin{aligned}
\upsilon(t_n) &\leq \xi_n[V(t_n^-, \eta(0)) + supV(t_n^- + u, \eta(u))] \\
&\leq \xi_n[\wp||\eta||^q e^{-\beta(t_n-t_0)} + \wp||\eta||^q e^{-\beta(t_n+u-t_0)}] \\
&\leq \xi_n\wp||\eta||^q e^{-\beta(t_n-t_0)}(1+e^{-\beta u}) \\
&\leq \xi_n\wp||\eta||^q e^{-\beta(t_n-t_0)}(1+e^{\beta\tau}) \\
&\leq \xi_n\wp||\eta||^q e^{\beta(\hat{t}-t_n)} e^{-\beta(\hat{t}-t_0)}(1+e^{\beta\tau}) \\
&< \xi_n e^{\beta(t_{n+1}-t_n)}\wp||\eta||^q e^{-\beta(\hat{t}-t_0)}(1+e^{\beta\tau}) \\
&< e^{-(\alpha+\beta)(t_{n+1}-t_n)} e^{\beta(t_{n+1}-t_n)}\wp||\eta||^q e^{-\beta(\hat{t}-t_0)} \\
&< \wp||\eta||^q e^{-\beta(\hat{t}-t_0)}
\end{aligned}$$

Therefore $\hat{t} \neq t_n$. From the continuity of $\upsilon(t)$ on the interval $[t_n, t_{n+1})$, we have

$$\upsilon(\hat{t}) = \wp||\eta||^q e^{-\beta(\hat{t}-t_0)}, \upsilon(t) \leq \upsilon(\hat{t}), t \in [t_n, \hat{t}] \tag{12}$$

From equation (12), we know that $\exists t^{'} \in (t_n, \hat{t})$ such that

$$\upsilon(t^{'}) = \xi_n e^{\beta(t_{n+1}-t_n)} \wp||\eta||^q e^{-\beta(\hat{t}-t_0)}, \upsilon(t^{'}) \leq \upsilon(t) \leq \upsilon(\hat{t}), t \in [t^{'}, \hat{t}] \tag{13}$$

However, for $t \in [t^{'}, \hat{t}]$ and $u \in [-\tau, 0]$, either $t + u \in [t_0 - \tau, t_n)$ or $t + u \in [t_n, \hat{t}]$. If $t + u \in [t_0 - \tau, t_n)$,then from equation (10), we get

$$\begin{aligned} \upsilon(t+u) &\leq \wp||\eta||^q e^{-\beta(t-t_0)} e^{-\beta u} \\ &\leq \wp||\eta||^q e^{-\beta(\hat{t}-t_0)} e^{\beta(\hat{t}-t)} e^{\beta\tau} \\ &\leq e^{\beta\tau} e^{\beta(t_{n+1}-t_n)} \wp||\eta||^q e^{-\beta(\hat{t}-t_0)} \end{aligned} \tag{14}$$

while, if $t + u \in [t_n, \hat{t}]$, from (13), then

$$\upsilon(t+u) \leq \wp||\eta||^q e^{-\beta(\hat{t}-t_0)} \leq e^{\beta\tau} e^{\beta(t_{n+1}-t_n)} \wp||\eta||^q e^{-\beta(\hat{t}-t_0)} \tag{15}$$

On the other hand, equation (14), (15) and (16) imply that, for any $u \in [-\tau, 0]$, we have

$$\begin{aligned} \upsilon(t+u) &\leq e^{\beta\tau} e^{\beta(t_{n+1}-t_n)} \wp||\eta||^q e^{-\beta(\hat{t}-t_0)} \\ &\leq \gamma e^{\beta\tau} \upsilon(t') \leq \gamma e^{\beta\tau} \upsilon(t) \leq \rho \upsilon(t), t \in [t', \hat{t}] \end{aligned} \tag{16}$$

So by equation (17) and condition (ii), we have $D^+\upsilon(t) \leq (\alpha - \beta)\upsilon(t)$. Therefore, considering condition (iv), we have

$$\begin{aligned} \upsilon(\hat{t}) &\leq \upsilon(t') e^{(\alpha-\beta)(\hat{t}-t')} \\ &\leq \xi_n e^{\beta(t_{n+1}-t_n)} \wp||\eta||^q e^{-\beta(\hat{t}-t_0)} (1 + e^{\beta\tau}) e^{(\alpha-\beta)(\hat{t}-t')} \\ &< \wp||\eta||^q e^{(\alpha+\beta)(t_{n+1}-t_n)} e^{\beta(t_{n+1}-t_n)} e^{-\beta(\hat{t}-t_0)} e^{(\alpha-\beta)(\hat{t}-t')} \\ &= \wp||\eta||^q e^{-\alpha(t_{n+1}-t_n)} e^{(\alpha-\beta)(\hat{t}-t')} e^{-\beta(\hat{t}-t_0)} \\ &< \wp||\eta||^q e^{-\beta(t_{n+1}-t_n)} e^{-\beta(\hat{t}-t_0)} \\ &< \wp||\eta||^q e^{-\beta(\hat{t}-t_0)} = \upsilon(\hat{t}) \end{aligned}$$

which is a contradiction. Therefore the assumed assumption is false. Hence the equation (11) holds. Thererfore by mathematical induction we get the result that equation (2) holds for any $j \in N$. Then by condition (i), we get

$$||x|| \leq \wp^*||\eta||e^{-\frac{\beta}{q}(t-t_0)}, t \in [t_{j-1}, t_j), k \in N$$

where $\wp^* \geq Max\left\{1, \left[\frac{\wp}{w_1}\right]^{\frac{1}{q}}\right\}$. This implies that the zero solution of the impulsive retarded system (1) is globally exponentially stable with the convergence rate $\frac{\beta}{q}$. ■

4. Conclusion

In this paper, we extended the notion of global exponential stability criteria to the systems of impulsive retarded functional differential equations. With the use of Lyapunov function along with Razumikhin technique, we have obtained some results for global exponential stability of the system in which the effect of delay at the time of impulses is obtained.

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