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# General Study on Laguerre Polynomials of Two Variable $\mathcal{L}_n$ (x, y)

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## Abstract

In this paper we obtain properties, expansion of polynomials involving the generalized associated Laguerre Polynomials which are closely related to generalized Lagguerre polynomials of Dattoli et. al. These results provide useful extensions of the well known results of Lagguerre Polynomials $L_n(x)$.

**Key Words**: Laguerre Polynomials, Dattoli et.al., recurrence relation,
**AMS Subject Classifications (2000)**: 33 C 45, 33 C 80, 33 C 35

## INTRODUCTION

Two variable one index Laguerre polynomials have been given by Dattoli et al [1], [2], [3], [4] and by Pathan, Khan and Yasmin [5].

Two variable one index Laguerre polynomials defined as

$$\mathcal{L}_n(x, y) = n! \sum_{r=0}^{n} \frac{(-1)^r x^r y^{n-r}}{(n-r)!\,(r!)^2} \qquad \ldots (1.1)$$

$$\mathcal{L}_n(x, 0) = (-1)^n \frac{x^n}{n!}, \qquad \ldots (1.2)$$

and are specified by the generating function

$$\sum_{n=0}^{\infty} t^n \mathcal{L}_n(x, y) = \frac{1}{1-yt} \exp\left(\frac{-xt}{1-yt}\right); |yt| < 1 \qquad \ldots (1.3)$$

$\mathcal{L}_n(x, y)$ are linked to the ordinary Laguerre polynomials $L_n(x)$ by

$$\mathcal{L}_n(x, 1) = L_n(x), \qquad \ldots (1.4)$$

$$\mathcal{L}_n(x, y) = y^n L_n\left(\frac{x}{y}\right). \qquad \ldots (1.5)$$

A generalization of (1.1) provided by the following definition of the generalized associated Laguerre polyniomials

$$\mathcal{L}_n^{(\alpha)}(x, y) = \sum_{r=0}^{n} \frac{(-1)^r (1+\alpha)_n \, y^{n-r} x^r}{(n-r)!\, r!\, (1+\alpha)_r} \qquad \ldots (1.6)$$

$$= \sum_{r=0}^{n} \frac{(-1)^r (\alpha+n)!\, y^{n-r} x^r}{r!\, (n-r)!\, (\alpha+r)!},$$

and the generating function, we get

$$\sum_{n=0}^{\infty} \mathcal{L}_n^{(\alpha)}(x, y)\, t^n = (1-yt)^{-\alpha-1} \exp\left(\frac{-xt}{1-yt}\right), \qquad \ldots (1.7)$$

and $$\sum (c)_n \mathcal{L}_n^{(\alpha)}(x,y)\, t^n / (1+\alpha)_n = (1-yt)^{-c} \, {}_1F_1\left[\begin{matrix} c; & \frac{-xt}{1-yt} \\ 1+\alpha; & \end{matrix}\right] \qquad \ldots (1.8)$$

where c be an arbitirary number

so that for $\alpha = 0$, (1.6) and (1.7) reduces to (1.1) and (1.3)

Now using expansion on R.H.s. in (1.7) and after some calculation, we get

$$x^n = \sum_{r=0}^{n} \frac{(-1)^r \, n!\, (1+\alpha)_n \, y^{n-r}}{(n-r)!\, (1+\alpha)_r} \mathcal{L}_r^{(\alpha)}(x, y) \qquad \ldots (1.9)$$

and Kummer's first transformation formula

$${}_1F_1(\alpha; \gamma; z) = e^z \, {}_1F_1(\gamma - \alpha; \gamma; -z) \qquad \ldots (1.10)$$

where $\lambda$ is neither zero nor a negative integer

and $(\lambda)_{m+n} = (\lambda)_m (\lambda+m)_n \qquad \ldots (1.11)$

$$(n-k)! = \frac{(-1)^k\ n!}{(-n)_k};\ 0 \le k \le n \qquad \ldots (1.12)$$

In this paper we shall give some basic relation and properties then obtain some properties, expansion of polynomials involving the generalized associated Laguerre polynomials $\mathcal{L}_n^{(\alpha)}$ (x, y)

**Some Properties:**

**Theorem – 1 :** If α and β are arbitrary and positive integer then

(i) $$\mathcal{L}_n^{(\alpha)}(x, y) = \sum_{k=0}^{n} (\alpha-\beta)_k\, y^k\ \mathcal{L}_{n-k}^{(\beta)}(x, y)/k! \qquad \ldots (2.1)$$

(ii) $$\mathcal{L}_n^{(\alpha+\beta+1)}(x+z, y) = \sum_{k=0}^{n} \mathcal{L}_k^{\alpha}(x, y)\ \mathcal{L}_{n-k}^{(\beta)}(z, y) \qquad \ldots (2.2)$$

(iii) $$\mathcal{L}_n^{(\alpha)}(x, y) = \frac{(1+\alpha)_n}{(c)_n} \sum_{k=0}^{n} \frac{(1+\alpha-c)_k}{(1+\alpha)_k}\ \mathcal{L}_k^{(\alpha)}(-x, y)\ \mathcal{L}_{n-k}^{(2c-\alpha-2)}(x, y) \quad \ldots (2.3)$$

(iv) $$\mathcal{L}_n^{(\alpha)}(xy, z) = \sum_{k=0}^{n} \frac{y^k \left[z\,(1-y)^{n-k}\right](1+\alpha)_n}{(n-k)!\,(1+\alpha)_k}\ \mathcal{L}_k^{(\alpha)}(x, y) \qquad \ldots (2.4)$$

**Proof:** (i) consider

$$(1-yt)^{-1-\alpha} \exp\left(\frac{-xt}{1-yt}\right) = (1-yt)^{-(\alpha-\beta)}\ (1-yt)^{1-\beta} \exp\left(\frac{-xt}{1-yt}\right).$$

Now using (1.7) and some series rearrangements

$$\sum_{n=0}^{\infty} \mathcal{L}_n^{(\alpha)}(x, y)\, t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(\alpha-\beta)_k\ (yt)^k}{k!}\ \mathcal{L}_{n-k}^{(\beta)}(x, y)\, t^{n-k}$$

Equating the coefficient of $t^n$ on both side, we get required result (2.1)

(ii) Consider an equation

$$(1-yt)^{-1-\alpha} \exp\left(\frac{-xt}{1-yt}\right)(1-yt)^{-1-\beta} \exp\left(\frac{-zt}{1-yt}\right)$$

$$= (1-yt)^{-1-(\alpha+\beta+1)} \exp\left(\frac{-(x+z)\,t}{1-yt}\right)$$

Now using (1.7) and using some series rearrangement

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} \mathcal{L}_k^{(\alpha)}(x, y)\, t^k\, \mathcal{L}_{n-k}^{(\beta)}(z, y) t^{n-k} = \sum_{n=0}^{\infty} \mathcal{L}_n^{(\alpha+\beta+1)}(x+z, y) t^n$$

Equation the coefficient of $t^n$, we get required result (2.2)

(iii) Since linear generating function of Laguerre polynomial defined by

$$\sum_{n=0}^{\infty} (c)_n\, \mathcal{L}_n^{(\alpha)}(x, y)\, t^n /(1+\alpha)_n = (1-yt)^{-c}\ {}_1F_1\left[\begin{matrix} c; & \dfrac{-xt}{1-yt} \\ 1+\alpha; & \end{matrix}\right]$$

Using Kummer's first formula (1.10) in R.H.S. and after some calculation

$$\sum_{n=0}^{\infty}(c)_n\, \mathcal{L}_n^{(\alpha)}(x, y)\, t^n /(1+\alpha)_n$$

$$= (1-yt)^{-1-(2c-\alpha-2)} \exp\left(\frac{-xt}{1-yt}\right)(1-yt)^{-(1+\alpha-c)}\ {}_1F_1\left[\begin{matrix} 1+\alpha-c; & \dfrac{-xt}{1-yt} \\ 1+\alpha; & \end{matrix}\right]$$

using (1.7) and (1.8), we get

$$\sum_{n=0}^{\infty}(c)_n\, \mathcal{L}_n^{(\alpha)}(x, y)\, t^n /(1+\alpha)_n$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \mathcal{L}_{n-k}^{(2c-\alpha-2)}(x, y)\, \frac{(1+\alpha-c)_k}{(1+\alpha)_k}\, \mathcal{L}_k^{(\alpha)}(-x, y)\, t^n$$

Equating the coefficient of $t^n$ on both side, we get required result (2.3)

Put c = 1 + $\alpha$/2+ m/2 in (2.3), we get

$$\mathcal{L}_n^{(\alpha)}(x, y) = \frac{(1+\alpha)_n}{(1+\alpha/2+m/2)_n} \sum_{k=0}^{n} \frac{\left(\frac{\alpha}{2}-\frac{m}{2}\right)_k}{(1+\alpha)_k}\, \mathcal{L}_k^{(\alpha)}(-x, y)\, \mathcal{L}_{n-k}^{(m)}(x, y) \qquad \ldots (2.5)$$

Again put c = 1 + $\alpha$ + m in (2.3) and using (1.8) we get

$$\mathcal{L}_n^{(\alpha)}(x, y) = (1+\alpha)_n (1+\alpha)_m /(1+\alpha)_{m+n}$$

$$\times \sum_{k=0}^{n}(-m)_k\, \mathcal{L}_k^{(\alpha)}(-x, y)\, \mathcal{L}_{n-k}^{(\alpha+2m)}(x, y)/(1+\alpha)_k \qquad \ldots (2.6)$$

(iv) Consider the series

$$\sum_{n=0}^{\infty} \mathcal{L}_n^{(\alpha)}(xy, z)\, t^n/(1+\alpha)_n = \sum_{n=0}^{\infty} L_n^{(\alpha)}\left(\frac{xy}{z}\right)(tz)^n(1+\alpha)_n$$

using [6; P.113(1)], we get

$$\sum_{n=0}^{\infty} \mathcal{L}_n^{(\alpha)}(xy, z)\, t^n/(1+\alpha)_n = e^{tz}\ {}_0F_1\left[\begin{matrix} -; & \frac{-x}{z}(tzy) \\ 1+\alpha; & \end{matrix}\right]$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} [tz(1-y)]^{n-k}/(n-k)!\ L_k^{(\alpha)}\left(\frac{x}{z}\right)(tzy)^k/(1+\alpha)_k$$

using (1.5) then equating the coefficient of $t^n$ on both side, we get required result (2.4)

**Expansions of polynomials**

**Theorem – 2:** If $\alpha$ is a positive and arbitrary integer, then

(i) $$H_n\ (x, y) = 2^n\ (1+\alpha)_n \sum_{k=0}^{\infty}\ {}_2F_2\left[\begin{matrix} -(n-k)/2, -(n-k-1)/2; \\ -(\alpha+n)/2, -(\alpha+n-1)/2; \end{matrix}\ -\frac{1}{4y}\right]$$

$$\times(-n)_k\ y^{n-k}\mathcal{L}_k^{(\alpha)}\ (x, y)/(1+\alpha)_k \qquad \ldots(3.1)$$

(ii) $$P_n(x) = 2^n(\tfrac{1}{2})_n\ (1+\alpha)_n/n!\sum_{k=0}^{n}\ (-n)_k\ y^{n-k}\ \mathcal{L}_k^{(\alpha)}\ (x, y)/(1+\alpha)_k$$

$$\times\ {}_2F_3\left[\begin{matrix} -(n-k)/2, -(n-k-1)/2; \\ (\tfrac{1}{2}-n), -\tfrac{1}{2}(\alpha+n), -\tfrac{1}{2}(\alpha+n-1); \end{matrix}\ \frac{1}{4y^2}\right] \qquad \ldots(3.2)$$

(iii) $$\mathcal{L}_n^{(\alpha)}\ (x, y) = \sum_{k=0}^{n} \frac{(1+\alpha)_n\ (-1)^k\ y^{n-k}}{(n-k)!\,k!\,2^k\ (1+\alpha)_k} H_k\ (x, y)$$

$$\times\ {}_2F_2\left[\begin{matrix} -(n-k)/2, -(n-k-1)/2; \\ (1+\alpha+k)/2, 2+\alpha+k/2; \end{matrix}\ \frac{1}{4y}\right] \qquad \ldots(3.3)$$

(iv) $$\mathcal{L}_n^{(\alpha)}\ (x, y) = \sum_{k=0}^{n}\ {}_2F_3\left[\begin{matrix} \frac{-(n-k)}{2}, \frac{-(n-k-1)}{2}; \\ \frac{(1+\alpha+k)}{2}, \frac{2+\alpha+k}{2}, \frac{3}{2}+k; \end{matrix}\ \frac{1}{4y^2}\right]$$

$$\times \frac{(-1)^k \; y^{n-k} \; (1+\alpha)_n \; (2k+1) \, P_n \; (x)}{2^k \; (n-k)! \, (3/2)_k \; (1+\alpha)_k} \qquad \ldots (3.4)$$

**Proof:** (i) Consider the following series

$$S = \sum_{n=0}^{\infty} \; H_n \; (x, y) \, \frac{t^n}{n!} \qquad \ldots (3.5)$$

$$S = \sum_{n=0}^{\infty} \; \sum_{s=0}^{[n/2]} \frac{(-y)^s \; (2x)^{n-2s} \; t^n}{s! \, (n-2s)!}$$

Replacing n by n + 2s and using (1.9), and again replacing n by n + k, we get

$$S = \sum_{n=0}^{\infty} \; \sum_{s=0}^{\infty} \; \sum_{k=0}^{\infty} \frac{(-1)^{k+s} \; (1+\alpha)_{n+k}}{s! \, n! \, (1+\alpha)_k} \, 2^{n+k} \; y^{n+s} \; \mathcal{L}_k^{(\alpha)} \; (x, y) \, t^{n+2s+k}$$

Now replacing n by n – 2s and by some calculations we get

$$S = \sum_{n=0}^{\infty} \; \sum_{k=0}^{\infty} \; \sum_{s=0}^{[n/2]} \frac{(-1)^s \; (4y)^{-s} \; (-n)_{2s}}{s! \, (-\alpha-n-k)_{2s}} \, \frac{(-1)^k \; (1+\alpha)_{n+k}}{n! \, (1+\alpha)_k} \, 2^{n+k} \; y^n$$

$$\times \; \mathcal{L}_k^{(\alpha)} \; (x, y) \, t^{n+k}$$

using Legendre's duplication formula and replacing n by n – k we get

$$S = \sum_{n=0}^{\infty} \; \sum_{k=0}^{n} \; {}_2F_2 \left[ \begin{matrix} -\tfrac{1}{2}(n-k), -\tfrac{1}{2}(n-k-1); \\ -\left(\frac{\alpha+n}{2}\right), -\tfrac{1}{2}(\alpha+n-1); \end{matrix} \; \frac{-1}{4y} \right]$$

$$\times (-1)^k \, 2^n \, y^{n-k} \; (1+\alpha)_n \; \mathcal{L}_n^{(\alpha)} \; (x, y) \, t^n (n-k)! \, (1+\alpha)_k \qquad \ldots (3.6)$$

Equating the coefficient of $t^n$ of equations (3.5) and (3.6), we get required result (3.1)

(ii) Now consider following the series

$$S = \sum_{n, s=0}^{\infty} \; P_n \; (x) \, t^n \qquad \ldots (3.7)$$

$$= \sum_{n, s=0}^{\infty} \frac{(-1)^s \; (1/2)_{n+s} \; (2x)^n \; t^{n+2s}}{s! n!}$$

using (1.9) and the same procedure as [6; P.208 (4)], we get then n by n – k, we get

$$S = \sum_{n=0}^{\infty} \sum_{k=0}^{n} {}_2F_3 \left[ \begin{matrix} -(n-k)/2, -(n-k-1)/2\,; \\ \tfrac{1}{2} - n, -(\alpha+n)/2, -(\alpha+n-1)/2\,; \end{matrix} \frac{1}{4y^2} \right]$$

$$\times (-1)^k\, 2^n\, (\tfrac{1}{2})_n\, (1+\alpha)_n\, y^{n-k}\, \mathcal{L}_k^{(\alpha)}(x, y)\, t^n / (n-k)!\, (1+\alpha)_k \qquad \ldots (3.8)$$

Equating the coefficient of $t^n$ of equations (3.7) and (3.8) and using (1,12)then, we get required result (3.2)

(iii) Consider the following series

$$S = \sum_{n=0}^{\infty} \mathcal{L}_n^{(\alpha)}(x, y)\, t^n / (1+\alpha)_n \qquad \ldots (3.9)$$

$$S = \sum_{n,k=0}^{\infty} (-1)^k\, y^n\, x^k\, t^{n+k} / k!\, n!\, (1+\alpha)_k$$

$$= \sum_{n,k=0}^{\infty} \sum_{r=0}^{[k/2]} \frac{(-1)^k\, y^n\, H_{k-2r}(x, y)\, t^{n+k}}{n!\, (1+\alpha)_k\, 2^k\, (k-2r)!\, r!}$$

Now using the same procedure as in part (i) and (ii), we get

$$S = \sum_{n=0}^{\infty} \sum_{k=0}^{n} {}_2F_2 \left[ \begin{matrix} \dfrac{-(n-k)}{2}, \dfrac{-(n-k-1)}{2}\,; \\ \left(\dfrac{1+\alpha+k}{2}\right), \left(\dfrac{2+\alpha+k}{2}\right); \end{matrix} \frac{1}{4y} \right]$$

$$\times \frac{(-1)^k\, y^{n-k}\, H_k(x, y)}{(n-k)!\, k!\, (1+\alpha)_k} \frac{t^n}{2^k} \qquad \ldots (3.10)$$

Equating the coefficient of $t^n$ of equation (3.9) and (3.10), we get required result (3.3).

(iv) For proof of (3.4) consider the following series

$$\sum_{n=0}^{\infty} \mathcal{L}_n^{(\alpha)}(x, y)\, t^n (1+\alpha)_n = \sum_{n,k=0}^{\infty} (-1)^k\, y^n\, x^k\, t^{n+k} / k!\, n!\, (1+\alpha)_k$$

Now using [6; P.181 (4)], we get

$$\sum_{n=0}^{\infty} \frac{\mathcal{L}_n^{(\alpha)}(x, y)\, t^n}{(1+\alpha)_n} = \sum_{n,k=0}^{\infty} \sum_{r=0}^{[k/2]} \frac{(-1)^k\, y^n\, (2k-4r+1)}{n!\, (1+\alpha)_k\, 2^k\, r!} \frac{P_{k-2r}(x)\, t^{n+k}}{(\tfrac{3}{2})_{k-r}},$$

using same procedure in part (i) and (ii) then we get

$$\sum_{n=0}^{\infty} \frac{{}_{n}^{(\alpha)}(x, y)\, t^n}{(1+\alpha)_n} = \sum_{n,k=0}^{\infty} {}_2F_3\left[\begin{matrix} -n/2, -(n-1)/2; \\ (1+\alpha+k)/2, (2+\alpha+k)/2, 3/2+k; \end{matrix} \quad \frac{1}{4y^2}\right]$$

$$x \, \frac{(-1)^k \; y^n \; (2k+1)\, P_k \; (x)\, t^n}{n!\, 2^k \left(\frac{3}{2}\right)_k \; (1+\alpha)_k}$$

Equating the coefficient of $t^n$ on both side, we get required result (3.4)

**Special cases:**

I. For y = 1, then (2.1) and (2.2) reduces to a known result [6; P. 209 (283)]

For y = 1, then (2.3) reduces to a known result [6; P.210 (8)]

For z =1, then (2.4) reduces to a known result [6; P. 209 (15)].

For y = 1, then (2.5) and (2.6) reduces to a known result [6; P. 216 (6 & 7)].

II For y = 1, then (3.1) reduces to a known result [6; P.207 (3)]

For y = 1, then (3.2) reduces to a known result [6; P. 208 (4)].

For y = 1, then (3.3), (3.4) reduces to a known result [6; P. 216 (2&3)].

Special cases I and II are known formulae for some properties and expansions of polynomials for ordinary Laguerre Polynomials $L_n(x)$

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