International Journal of Computational and Applied Mathematics.
ISSN 1819-4966 Volume 12, Number 3 (2017), pp. 887-898

http://www.ripublication.com

A Note on Signed Semigraphs

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Abstract

The concept of Semigraphs introduced by E.Sampathkumar [2] generalizes many properties of graphs. In this paper we try to generalize the concept of signed graphs to semigraphs. The end vertices and middle vertices in a semigraph give rise to the definition of e-signed semigraph, v-signed semigraph and ve–signed semigraph. Also, we discuss the conditions for e-signed semigraph, v-signed semigraph and ve–signed semigraphs to be balanced.

Keywords: Semigraphs, e-signed semigraph, v-signed semigraph, ve–signed semigraph

AMS Subject Classification: 05C22

1. INTRODUCTION

A signed graph [6], or briefly an S-graph, is a generalization of linear graphs which consists of a set E of n points $P_1, P_2, ..., P_n$ together with two disjoint subsets L^+, L^- of the set of all unordered pairs of distinct points. Signed graphs concept was originated from the Heider's [7] conception to analyze the cognitive units such as human behaviour, likes or dislikes, hate or love, extra. The structural representation of human behaviour by using a graph was first introduced by Cartwright and Harary [1] in the Psychological Review "Structural Balance : A Generalization of Heider's Theory". The points (or vertices) of a graph indicate the individual person and the lines (or edges) represent the relationship among the people. Further to denote the positive and negative relationship between two people by the signs +1 and -1 respectively are assigned. Further, E. Sampathkumar [3] defined pl-signed graph where the signs +1 and -1 are assigned together for the vertices and edges and also obtained the characterizations of p-balanced , l-balanced and pl-signed graphs

2. REVIEW OF SEMIGRAPHS

Definition 2.1: [3] A semigraph G is a pair (V, X) where V is a nonempty set whose elements are called vertices of G, and X is a set of n-tuples, called edges of G, of distinct vertices, for various $n \geq 2$, satisfying the following conditions:

S.G.1. Any two edges have at most one vertex in common.

S.G.2. Two edges $(u_1, u_2, ..., u_n)$ and $(v_1, v_2, ..., v_m)$ are considered to be equal if, and

only if, (i) $m = n$ and (ii) either $u_i = v_i$ for $1 \leq i \leq n$, or $u_i = v_{n-i+1}$ for $1 \leq i \leq n$.

Thus the edge $E = (u_1, u_2, ..., u_n)$ is same as the edge $(u_n, u_{n-1}, ..., u_1)$.

Definition 2.2: [3] Let $G = (\mathrm{V}, \mathrm{X})$ be a semigraph and $E = (v_1, v_2, ..., v_n)$ be an edge of G. Then v_1 and v_n are the end vertices of E which are identified as dark points and v_i , $2 \leq i \leq n-1$ are the middle vertices (or m-vertices) of E which are denoted by hollow circles, while the (m,e)-vertex is denoted by a hollow circle with a small tangent drawn to it marking the end of its adjacent edge. The vertices in a semigraph are adjacent to each other if they belong to an edge of a semigraph.

Example 2.3:

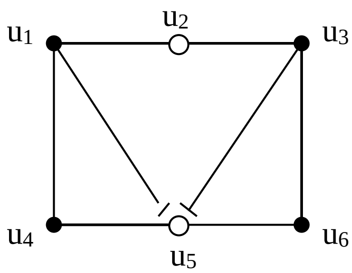

Figure 1

Let $G = (\mathrm{V}, \mathrm{X})$ be a semigraph with vertex set $V = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ and edge set $X = \{(u_1, u_2, u_3), (u_1, u_4), (u_4, u_5, u_6), (u_3, u_6), (u_1, u_5), (u_3, u_5)\}$ is shown in figure 1.

Definition 2.4: [3] For a vertex v in a semigraph $G = (\mathrm{V}, \mathrm{X})$, various types of degrees are defined as follows:

(i) Degree: deg v is the number of edges having v as an end vertex. (ii) Edge Degree: $\deg_e v$ is the number of edges containing v.

(iii) Adjacent Degree: $\deg_a v$ is the number of vertices adjacent to v.

(iv) Consecutive Adjacent Degree: $\deg_{ca} v$ is the number of vertices which are consecutively adjacent to v.

Proposition 2.5: [3] Let $G = (\mathrm{V}, \mathrm{X})$ be a semigraph where $V = (v_1, v_2, \ldots\ldots, v_p)$ and $X = (E_1, E_2, \ldots., E_q)$. Then

1. $\sum_{i=1}^{p} \deg v_i = 2q$

2. $\sum_{i=1}^{p} \deg v_i = \sum_{i=1}^{q} |E_i|$

3. $\sum_{i=1}^{p} (\deg_a v_i + \deg_e v_i) = \sum_{i=1}^{q} |E_i|^2$

Definition 2.6: [3] A subedge of an edge $E = (v_1, v_2, ..., v_n)$ is a *k*-tuple $E' = (v_{i_1}, v_{i_2}, ..., v_{i_k})$, where $1 \le i_1 < i_2 < .... < i_k \le n$ or $1 \le i_k < i_{k-1} < .... < i_1 \le n$.

Definition 2.7: [3] A *partial edge* or fp edge E is a $j - i + 1$-tuple $E(v_i, v_j) = (v_i, v_{i+1}, ..., v_j)$, where $1 \le i \le n$. Thus a subedge E' of an edge E is a partial edge if, and only if, any two consecutive vertices in E' are also consecutive vertices of E.

Definition 2.8: [3] An fs-edge in a semigraph G is an edge or a subedge. An fp -edge is an edge or a partial edge. Two subedges E_1 and E_2 are disjoint if $|E_1 \cap E_2| < 1$.

Definition 2.9: [3] A walk in a semigraph G is an alternating sequence of vertices and fs-edges $v_0 E_1 v_1 E_2 v_2 .... v_{n-1} E_n v_n$ beginning and ending with the vertices, such that v_{i-1} and v_i are the end vertices of the fs-edges E_i, $1 \le i \le n$. It is called a $v_0 - v_n$ walk. It is closed if $v_0 - v_n$ and open otherwise.

A $v_0 - v_n$ walk is a trial if any two fs-edges in it are disjoint. Note that in a trial vertices may be repeated. A $v_0 - v_n$ path is a $v_0 - v_n$ trial in which all the vertices are distinct. A cycle is a closed path with atleast three vertices.

3. SIGNED SEMIGRAPHS

Definition 3.1: Let $S = (\mathrm{V}, \mathrm{E})$ be a signed semigraph. In S, the edge with odd number of middle vertices (or m-vertices) is assigned negative sign and the edge with even number of m-vertices or without m-vertices is assigned positive sign. Then S is called an e-signed semigraph. In other words, a e-signed semigraph $S = (\mathrm{V}, \mathrm{E})$ is a set V of vertices, together with two disjoint subsets E^+ and E^- of the set E, where $E = E^+ \cup E^-$ and E^+ is the set of all edges with positive sign E^- is the set of all edges with negative sign. The elements of E^+ and E^- are called positive edges and negative edges respectively.

Example 3.2:

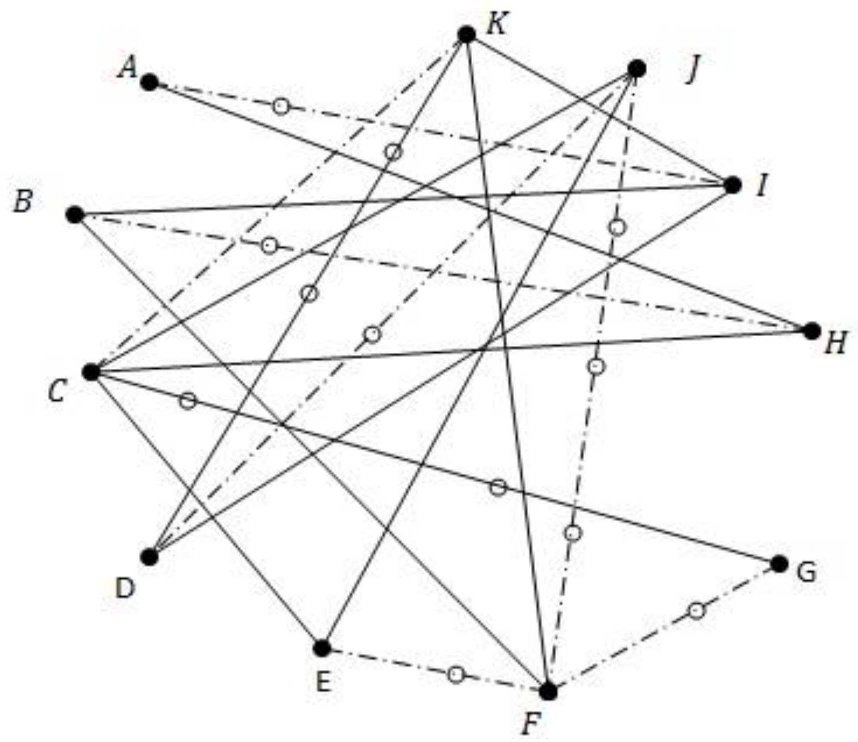

Figure 2

Definition 3.3: A path or a cycle in a e-signed semigraph $S = (\mathrm{V}, \mathrm{E})$ is called positive if it contains an even number of negative edges (or no negative edges) and odd if it contains an odd number of negative edges.

Definition 3.4: A e-signed semigraph $S = (\mathrm{V}, \mathrm{E})$ is e-balanced if its edges can be partitioned into two sets V_1 and V_2 so that all the edges within each subset are positive and the only edges joining a vertex of V_1 to a vertex of V_2 are negative. The sets V_1 and V_2 are called semi clusters.

Example 3.5:

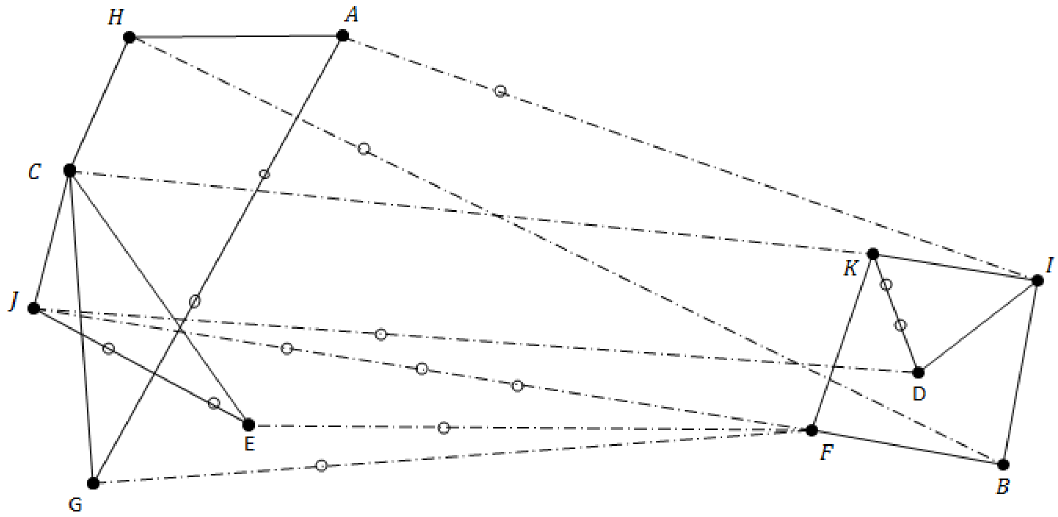

Figure 3

Definition 3.6: A e-signed semigraph $S = (V, E)$ is called semi clusterable if its edges can be partitioned into subsets called semiclusters, so that every positive edge joins edges in the same semicluster and every negative edge joins edges in different semiclusters.

Definition 3.7: A semigraph $S = (V, E)$ is called v-signed semigraph if every end vertex of S is assigned either positive or negative sign.

Definition 3.8: A v-signed semigraph is v-balanced if no component of S contain an odd number of negative end vertices. In other words, let S be a connected semigraph, S is v-balanced if S contains an even number of negative end vertices.

Definition 3.9: A semigraph $S = (V, E)$ is called ve-signed semigraph if every end vertex and every edge of S is assigned either positive or negative sign.

Definition 3.10: A ve-signed semigraph $S = (\mathrm{V}, \mathrm{E})$ is ve-balanced if

(1) it is both e-balanced and v-balanced
and

(2) the sign of any vertex u in S is equal to the product of the signs of the

edges incident with u.

Theorem 3.11: A connected e-signed semigraph S is e-balanced if, and only if, every pair of distinct end vertices u and v of S, all $u-v$ paths have the same parity.

Proof: Suppose S is e-balanced semigraph with semiclusters V and W.

1. Let u and v be two end vertices of S. If u and v are in the same semicluster, then all $u-v$ paths are even, because if they go to the other cluster they must also come back.
2. If u and v are in opposite semiclusters, all $u-v$ paths must be odd, because they must cross the semicluster-divide an odd number of times to finish in the other semicluster.

Conversely, now suppose S is a connected e-signed semigraph and for every pair of distinct end vertices u and v of S, all $u-v$ paths have the same parity. Select one end vertex u. Let V be the set of all vertices v such that there exists an even $u-v$ path (If there is one even $u-v$ path, then all $u-v$ paths are even, by our assumption) and let W be the set of all end vertices not in V. Suppose ab is a edge joining two end vertices of V, then ab must be a positive edge. If it were negative, then there would be an odd path from a to u, formed by joining to any of the even paths from a to u. Similarly, if ab is an edge joining vertices of W, it must be positive. Finally, if $a \in V$ and $b \in W$, then $u-a$ paths are even and $u-b$ paths are odd, which implies that ab must be negative.

Corollary 3.12: A e-signed semigraph S is e-balanced if, and only if, every cycle of S is even.

Proof: Let $S = (\mathrm{V}, \mathrm{E})$ be a e-signed semigraph which is e-balanced. Then every pair of distinct end vertices u and v of S, all $u-v$ paths have the same parity. Therefore, any two distinct paths will form a cycle. If $u-v$ paths are odd, then two

$u-v$ paths which form a cycle is even. If $u-v$ paths are even, then also two $u-v$ paths which form a cycle is even.

Conversely, suppose every cycle of a signed semigraph S is even, let $u,v \in C_1$ where C_1 is a even cycle. Then there are two $u-v$ paths belong to C_1, which should have the same parity. Therefore S is e-balanced.

This corollary can be stated as follows

Corollary 3.13: A e-signed semigraph S is e-balanced if, and only if, no cycle of S contains an odd number of negative edges.

Corollary 3.14: A e-signed semigraph S is e-balanced if the number of unbalanced paths incident on a common end vertex is even.

Corollary 3.15: Every subsemigraph of a e-balanced, e-signed semigraph is e-balanced.

Proof: Each cycle of the subsemigraph is a cycle of the e-balanced semigraph and is therefore positive.

Theorem 3.16: A e-signed semigraph is clusterable if, and only if, there is no cycle having exactly one negative edge.

Proof: Suppose the semigraph $S = (\mathrm{V}, \mathrm{E})$ is semiclusterable and C is a cycle in S. If all the end vertices are in the same semicluster, then C has zero negative edges. If C contains end vertices in at least two semiclusters, then it must contain at least two negative edges. Hence no cycle can contain only one negative edge.

Conversely, suppose S is a semigraph in which no cycle has exactly one negative edge. Define a relation R on the end vertices of S. uRv if, and only if, $u=v$ or there is a $u-v$ path containing only positive edges. This relation is reflexive, symmetric and transitive and so it is an equivalence relation. We claim the equivalence classes are semiclusters in the semigraph. If u and v are in different classes, then there is no positive edge whose end vertices are u and v. So, only negative edges can join vertices in different equivalence classes. If u and v are in the same class and uv is a negative edge with end vertices u and v, then since there is $u-v$ path containing only positive edges, there is a cycle containing exactly one negative edge. This is a contradiction, so only positive edges can join vertices in the same class. This proves that the classes are the semiclusters in a clusterable graph.

Theorem 3.17: A connected e-signed semigraph S is e-balanced if, and only if, it is possible to assign signs to its end vertices such that the sign of each edge uv is equal to the product of the signs of u and v (which are end vertices of uv).

Proof: Let S be e-balanced. Then its end vertices can be partitioned into two disjoint sets V_1 and V_2 in such a way that each positive edge of S joins two end vertices of the same set and each negative edge joins two end vertices of different sets.

Therefore, all the edges in V_1 and V_2 are positive. Without loss of generality, let all the vertices of V_1 be assigned positive. Therefore, all the vertices of V_2 must be assigned negative, so that the edges in V_2 have positive sign and the edges which join vertices of V_1 and V_2 have negative sign. Conversely, suppose it is possible to assign signs to its end vertices such that the sign of each edge uv is equal to the product of the signs of u and v. Collect all the vertices with positive signs in one set V_1 and the vertices with negative sign in another set V_2. Then in V_1 and V_2 all edges are positive and the edges which join the vertices of V_1 and V_2 are negative. Therefore, V_1 and V_2 are disjoint sets. Therefore S is e-balanced.

Theorem 3.18: A v-signed semigraph S is v-balanced if, and only if, it is possible to assign signs to the edges of S such that the sign of any end vertex u is equal to the product of the signs of the edges incident with u.

Proof: Let S be connected. If S is v-balanced, then it has an even number of negative end vertices. Since, S is connected, the vertices can be partitioned into two end vertex subsets $\{u_i, v_i\}$ such that for each i, there is a $u_i - v_i$ path P_i and for $i = j$ the path P_i and P_j are edge disjoint. Give the negative signs to all the edges in each of the above paths P_i and positive signs to all the other edges in S. To prove that the semigraph thus obtained has the property (1), consider a vertex in S. Let u be positive. If all the edges incident with u are positive, then (1) is true at u. Suppose some negative lines are incident with u. Then since each negative edge lies on some path P_i and the paths P_i are edge disjoint, it follows that there are an even number of negative edges at u and (1) is true in this case also. Suppose u is negative. Then there is atleast one negative edge x at u, the one on some path P_r having u as an end vertex. If x is the only negative edge at u, then (1) is true at u. Suppose there are negative edges at u other than x. Then each such edge is on some path P_i, between two negative vertices. Therefore, u is an end vertex of exactly one path P_r, and therefore, the paths P_i are edge disjoint, it follows that there are an odd number of negative edges at u and hence (1) is true at u.

Conversely, suppose it is possible to assign signs to the edges of a v-signed semigraph S, satisfying (1), let v_i be the end vertex set of S such that $\deg v_i = d_i$. Then $\sum \deg v_i = 2q$, where q is the number of edges of S. Let s_i and t_i denote respectively the number of positive and negative edges incident with v_i and let $d_i = d_i - (s_i + c_i)$ where $c_i = t_i$ if t_i is even and $c_i = t_i - 1$ if t_i is odd. Then $d_i = 0$ if v_i is positive and $d_i = 1$ if v_i is negative. Clearly, $\sum d_i$ and $\sum c_i$ are even. Also, since each positive edge contributes exactly two to the sum $\sum s_i$, it follows that $\sum s_i$ is even. Hence d_i is even and the number of negative vertices in S is even. Hence S is v-balanced. Hence the theorem.

Let S be an e-signed and e-balanced semigraph. Suppose we allot signs to each vertex of S, which is equal to the product of the signs of edges incident with it, Then by the above theorem 2.4, S is v-balanced and hence ve-balanced.

Corollary 3.19: One can always obtain a ve-balanced graph from a given e-signed and e-balanced semigraph.

Corollary 3.20: A connected ve-signed semigraph S is ve-balanced if, and only if,

1. S is e-balanced
2. The sign of any vertex v of S is equal to the product of the signs of the edges incident with v.

Theorem 3.21: In a ve-balanced connected semigraph S, one can partition the set of negative end vertices into 2-vertex subsets $\{u_i, v_i\}$ such that

1. For each i there exists a trail P_i between u_i and v_i consisting of negative edges only

and

2. For $i \neq j$, the trials P_i and P_j are edge disjoint.

Proof: (1) Let u_i be a negative end vertex of S. Since, the sign of u_i is the product of signs of the edges incident with u_i, it follows that there exists at least one negative edge, say, $u_i v_1$ incident with u_i. If v_1 is positive, then as before, there exists a negative edge $v_1 v_2$ where $v_2 = u_i$. If v_2 is positive, there exists a negative edge $v_2 v_3$ where $v_3 \neq v_1$. Also $v_3 \neq u_i$, for otherwise we have a triangle $u_i v_1 v_2$, whose edges are all negative and since S is e-balanced, this cannot happen. If v_3 is positive, then there exists a negative edge $v_3 v_4$, $v_4 \neq v_2$. If $v_4 = u_i$ then there exists a negative edge

$u_i v_5 = v_4 v_5$ at u_i different from $u_i v_1$ and $u_i v_3$. If $v_4 \neq u_i$ and v_4 is positive, there exists a negative edge $v_4 v_5$ where $v_5 \neq v_3$. Considering this way along $v_4 v_5$, we ultimately arrive at a negative end vertex v_i and the $u_i - v_i$ trail, say P_i which we have traversed consisting of negative edges only.

(2) Let u_j be a negative end vertex different from u_i and v_i. As before, we proceed along a trail from u_j consisting of negative edges which were not traversed earlier. If at some stage u_i or v_i, say v_i occurs, then since u_i is negative and the sign of u_i is equal to the product of the signs of the edges incident with u_i, it follows that there is a negative edge x incident with u_i which does not lie on the $u_i - v_i$ trial P_i. At u_i we proceed along this edge x until we arrive at a negative vertex v_j different from u_i, v_i and v_j. This process ultimately leads to the required partition of the negative vertices satisfying the given conditions (1) and (2).

ACKNOWLEDGMENT

The author MMK is thankful to Dr. Mukti Acharya and Dr. E.Sampathkumar for their valuable suggestions.

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