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Optimality and Convexity Theorems for Linear Fractional Programming Problem

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Abstract

Linear programming is a mathematical programming technique to optimize performance under a set of resource constraints as specified by organization. Linear fractional programming is a generalization of linear programming. The objective functions in linear programs are linear functions while the objective function in a linear fractional program is a ratio of two linear functions. In this paper an attempt is made to prove the optimality and convexity theorems in linear fractional programming problem.

Keywords: Linear Fractional programming, Optimality condition, Convexity.

I. INTRODUCTION

In this paper we formulate and prove the main theorems of linear fractional programming (LFP). Problems of LFP arise when there appears a necessity to optimize the efficiency of some activity, profit gained by company per unit of expenditure labor, cost of production per unit of produced goods etc.

Linear fraction problems (i.e. ratio objective that have numerator and denominator) have attracted considerable research and interest, since they are useful in production planning, financial and corporate planning, health care and hospital planning. Several methods to solve these problems are proposed in (1962), Charnes and Cooper have proposed their method depends on transforming this (LFP) to an equivalent linear program.

II. LINEAR FRACTIONAL PROGRAMMING

In mathematical optimization, linear-fractional programming (LFP) is a generalization of linear programming linear programming (LP). Whereas the objective functions in linear programs are linear functions, the objective function in a linear-fractional program is a ratio of two linear functions. A linear program can be regarded as a special case of a linear-fractional program in which the denominator is the constant function one.

Formally, a linear-fractional program is defined as the problem of maximizing (or minimizing) a ratio of affine functions over a polyhedron,

$$maximize\ \frac{c^T x+\alpha}{d^T x+\beta}$$
$$subject\ to\ \ Ax \leq b$$

where $x \in R^n$ represents the vector of variables to be determined, $c, d \in R^n$ and $b \in R^m$ are vectors of (known) coefficients, $A \in R^{m\times n}$ is a (known) matrix of coefficients and $\alpha, \beta \in R$ are constants. The constraints have to restrict the feasible region to $\{x|d^T x + \beta > 0\}$, i.e. the region on which the denominator is positive. Alternatively, the denominator of the objective function has to be strictly negative.

III. OPTIMAL SOLUTION FOR LINEAR FRACTIONAL PROGRAMMING

Convex Polyhedron: The set of all convex combinations of a finite number of points is called the convex polyhedron spanned by these points.

Theorem 1: If the convex set of feasible solution of $AX \leq b, X \geq 0$ is a convex polyhedron, then at least one of the extreme points gives an optimal solution.

Proof: Let $X^{(1)}, X^{(2)}, X^{(3)} \dots\dots\dots\dots\dots. X^{(k)}$ be the extreme points of the feasible region F of linear fractional problem.

$$Max\ z = \frac{u(x)}{v(x)} X$$
$$subject\ to\ constraints\ AX \leq b,$$
$$and\ non-negative\ restriction\ \ X \geq 0$$

Suppose $X^{(m)}$ is the extreme point among $X^{(1)}, X^{(2)}, X^{(3)} \dots\dots\dots\dots\dots. X^{(k)}$ at which the value of objective function is max (say z*).

$$z^* = \max\left\{\frac{u(x)}{v(x)}\right\} X^{(k)}$$

$$z^* = \left\{\frac{u(x)}{v(x)}\right\} X^{(m)}$$

We now consider a point $X^{(0)}$, in the feasible region F which is not an extreme point and let $z^{(0)}$ be the corresponding value of objective function. Then

$$z^{(0)} = \frac{u(x)}{v(x)} X^{(0)}$$

Since $X^{(0)}$ is not an extreme point, it can be expressed as a convex combination of the extreme points $X^{(1)}, X^{(2)}, X^{(3)} \dots\dots\dots\dots\dots . X^{(k)}$ of the feasible region F, where F is assumed to be a bounded set. Then

$$X^{(0)} = \lambda_1 X^{(1)} + \lambda_2 X^{(2)} + \cdots \dots\dots\dots\dots\dots\dots .. + \lambda_k X^{(k)}$$

Where $\lambda_1, \lambda_2 \dots\dots\dots\dots\dots\dots\dots .. \lambda_k \geq 0, \quad \sum_{t=1}^{k} \lambda_k = 1$

Now substitute the value of $X^{(0)}$ in the eq (2), we get

$$z^{(0)} = \frac{u(x)}{v(x)} \left[\lambda_1 X^{(1)} + \lambda_2 X^{(2)} + \cdots \dots\dots\dots\dots\dots\dots .. + \lambda_k X^{(k)}\right]$$

$$z^{(0)} \leq \frac{u(x)}{v(x)} X^{(m)}$$

$$z^{(0)} \leq z^* \qquad \text{since } cX^{(m)} = z^*$$

Which implies that at optimal solution, the extreme point solution is at least as good as any other feasible solution.

Theorem 2: If the optimal solutions occur at more than one extreme point, the value of the objective function will be the same for all convex combinations of this extreme point.

Proof: Let $x^{(1)}, x^{(2)}, \dots\dots\dots, x^{®}$ $(r \leq k)$ be the extreme points of the feasible region F at which the objective function assumes the same optimum value. This means

$$z^* = \frac{u_x}{v_x} x^{(1)} = \frac{u_x}{v_x} x^{(2)} = \cdots \dots\dots\dots\dots\dots .. = \frac{u_x}{v_x} x^{(r)}$$

Let $x = \lambda_1 x^{(1)} + \lambda_2 x^{(2)} + \cdots \dots\dots\dots\dots .. + \lambda_r x^{(r)}$

$\lambda_{1,} \lambda_{2,} \dots\dots\dots \lambda_{r,} > 0$

$\sum_{j=1}^{r} \lambda_j = 1$

Be the convex combination of $x^{(1)}, x^{(2)}, \dots\dots\dots, x^{®}$.

Then

$$\frac{u_x}{v_x} x = \frac{u_x}{v_x} \left[\lambda_1 x^{(1)} + \lambda_2 x^{(2)} + \cdots \dots\dots\dots\dots .. + \lambda_r x^{(r)}\right]$$

$$= \lambda_1 \left(\frac{u_x}{v_x} x^{(1)}\right) + \lambda_2 \left(\frac{u_x}{v_x} x^{(2)}\right) + \cdots \dots\dots\dots\dots .. + \lambda_r \left(\frac{u_x}{v_x} x^{(r)}\right)$$

$$= \lambda_1 z^* + \lambda_2 z^* + \cdots \dots\dots\dots .. + \lambda_r z^*$$

$$= (\lambda_1 + \lambda_2 + \cdots \dots\dots\dots + \lambda_r) z^*$$

$$\frac{u_x}{v_x} x = z^* \qquad \text{since } \sum_{j=1}^{r} \lambda_j = 1$$

IV: CONVEXITY FOR LINEAR FRACTIONAL PROGRAMMING

Convex set: A set C in n- dimensional space is said to be convex if for any points $x^{(1)}$, $x^{(2)}$ in set C, the line segment joining these points is also in the set C.

Mathematically, this definition implies that $x^{(1)}$ and $x^{(2)}$ are two distinct points in C, then every point $x = \lambda x^{(2)} + (1-\lambda)x^{(1)}$, $0 \leq \lambda \leq 1$ must also be in the set C.

Feasible Solution: Feasible solution is any element of the feasible region of an optimization problem. The feasible region is the set of all possible solution of an optimization problem.

Basic feasible solution: It is one that occurs at the corner point of the feasible region in a graph.

Theorem: The collection of all feasible solution to linear fractional programming model constitutes of a convex set whose extreme points correspond to the basic feasible solutions.

Proof: Let F be a set of all feasible solution of the system

AX=1, x≥0

If the set F of solutions has only one element, then F is convex set. Hence the theorem is true in this case.

Now assume that there are at least two distinct points $x^{(1)}$ and $x^{(2)}$ in F. then we have

A$x^{(1)}$= 1 for $x^{(1)} \geq 0$

A$x^{(2)}$ = 1 for $x^{(2)} \geq 0$

We only need to show that every convex combination of any two feasible solution is also a feasible solution, we define a point $x^{(0)}$ as the convex combination of $x^{(1)}$ and $x^{(2)}$. This implies that

$$x^{(0)} = \lambda x^{(2)} + (1-\lambda)x^{(1)}, \qquad 0 \leq \lambda \leq 1$$

By definition F is convex if $x^{(0)}$ also belongs to F. To shoe this is true we must show that $x^{(0)}$ satisfies the system of constraints AX = 1, x≥0

Thus A $x^{(0)}$= A $\{\lambda x^{(2)} + (1-\lambda)x^{(1)}\}$

$= \lambda Ax^{(2)} + (1-\lambda) Ax^{(1)}$

$= \lambda .1 + (1 - \lambda) .1$

$= \lambda + 1 - \lambda$

$= 1$

Also since $0 \leq \lambda \leq 1$ and $x^{(1)} \geq 0, x^{(2)} \geq 0$, then $x^{(0)} \geq 0$. This means that $x^{(0)} \epsilon F$ and consequently F is convex.

Extreme point correspondence:

Suppose that X = $[X_B, 0]$ is a basic feasible solution where X_B is an mX1 vector s.t. for a non-singular sub matrix B of A.

$$\mathrm{B}X_B = 1$$

If possible let us suppose that x be a point of F. Such that there exist $x^{(1)}, x^{(2)} \epsilon F$, so that

$$x = \lambda x^{(2)} + (1 - \lambda) x^{(1)}, \quad 0 < \lambda < 1$$

Let $x^{(1)} = [u_1, v_1]$ and $x^{(2)} = [u_2, v_2]$ where u_1, u_2 are mX1 vectors and v_1, v_2 are (n-m)X1 vectors then

$[X_B, 0] = \lambda [u_1, v_1]$ + (1- λ)$[u_2, v_2]$

$X_B = \lambda u_1 + (1 - \lambda) u_2$

$0 = \lambda v_1 + (1 - \lambda) v_2, \quad 0 < \lambda < 1$

Since $x^{(1)}, x^{(2)}$ are feasible solutions therefore $u_1, v_1, u_2, v_2 \geq 0$. Now $0 < \lambda < 1$ and $0 = \lambda v_1 + (1 - \lambda) v_2$. Therefore we must have $v_1 = v_2 = 0$. Thus $x^{(1)} = [u_1, 0]$ and $x^{(2)} = [u_2, 0]$. Again since $x^{(1)}$ *and* $x^{(2)}$ satisfy AX = 1, we have $Bu_1 = 1$ and $\mathrm{B}u_2$=1. Also since $\mathrm{B}X_B = 1$ and since expression of 1 as linear combination of basis vectors must be unique, therefore $u_1 = u_2 = X_B$

Hence $x^{(1)} = x^{(2)} = x$. This is contradiction for $x^{(1)} \neq x^{(2)}$. Hence u is an extreme point of F.

V. CONCLUSION

In this paper, we have discussed linear fractional programming. We have proved the optimality condition of linear fractional programming problem. For proving the optimality we have considered linear fractional programming model which states that if the convex set of feasible solution of $AX \leq b, X \geq 0$ is a convex polyhedron, then at least one of the extreme points gives an optimal solution. We have also proved the convexity of linear fractional programming problem. For proving the convexity we have shown that the collection of all feasible solution to linear fractional programming model constitutes of a convex set whose extreme points correspond to the basic feasible solutions

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