International Journal of Computational and Applied Mathematics.
ISSN 1819-4966 Volume 12, Number 3 (2017), pp. 935-942

http://www.ripublication.com

Positive and Associative Implicative Filters of Lattice Wajsberg Algebras

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Abstract

In this paper, we introduce the notions of positive implicative filter and an associative implicative filter in lattice Wajsberg algebras. We find a characterization of the positive implicative filter and investigate some properties of the associative implicative filter.

Keywords: Wajsberg algebra, Implicative filter, Positive implicative filter, Associative implicative filter.

Mathematical Subject Classification 2010: 03G10, 03G25.

1. INTRODUCTION

Mordchaj Wajsberg[6] introduced the concept of Wajsberg algebra in 1935 and analyzed by Font, Rodriguez and Torrens[1] in 1984. Further, they[1] introduced notion of implicative filters and family of implicative filter in lattice Wajsberg algebra and investigated some of their properties. In this paper, we introduce the definitions of the positive implicative filters and the associative implicative filter. Also, we discuss some of their properties with illustrations. We provide equivalent conditions for both the positive and the associative implicative filters.

2. PRELIMINARIES

In this section, we recall some basic definitions and related properties those are supporting for our main results.

Definition 2.1[1] Let $(A, \rightarrow, *, \ 1)$ be an algebra with complement "$*$" and a binary operation "$\rightarrow$" is called a Wajsberg algebra if and only if it satisfies the following axioms for all $x, \ y, \ z \in A$.

(i) $1 \rightarrow x = x$

(ii) $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = 1$

(iii) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$

(iv) $(x^* \rightarrow y^*) \rightarrow (y \rightarrow x) = 1$

Proposition 2.2[1] The Wajsberg algebra $(A, \rightarrow, *, \ 1)$ satisfies the following axioms for all $x, \ y, \ z \in A$.

(i) $x \rightarrow x = 1$

(ii) If $x \rightarrow y = y \rightarrow x = 1$ then $x = y$

(iii) $x \rightarrow 1 = 1$

(iv) $x \rightarrow (y \rightarrow x) = 1$

(v) If $x \rightarrow y = y \rightarrow z = 1$ then $x \rightarrow z = 1$

(vi) $(x \rightarrow y) \rightarrow ((z \rightarrow x) \rightarrow (z \rightarrow y)) = 1$

(vii) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$

(viii) $x \rightarrow 0 = x \rightarrow 1^* = x^*$

(ix) $(x^*)^* = x$

(x) $x^* \rightarrow y^* = y \rightarrow x$

Proposition 2.3[1] The Wajsberg algebra $(A, \rightarrow, *, \ 1)$ satisfies the following axioms for all $x, \ y, \ z \in A$.

(i) If $x \leq y$ then $x \rightarrow z \geq y \rightarrow z$

(ii) If $x \leq y$ then $z \to x \leq z \to y$

(iii) $x \leq y \to z$ if and only if $y \leq x \to z$

(iv) $(x \vee y)^* = (x^* \wedge y^*)$

(v) $(x \wedge y)^* = (x^* \vee y^*)$

(vi) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$

(vii) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$

(viii) $(x \to y) \vee (y \to x) = 1$

(ix) $x \to (y \vee z) = (x \to y) \vee (x \to z)$

(x) $(x \wedge y) \to z = (x \to y) \vee (x \to z)$

(xi) $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$

(xii) $(x \wedge y) \to z = (x \to y) \to (x \to z)$

Definition 2.4[1] The Wajsberg algebra A is a called lattice Wajsberg algebra if it satisfies the following conditions for all $x, y \in A$.

(i) A partial ordering $\leq$ on a lattice Wajsberg algebra A such that $x \leq y$ if and only if $x \to y = 1$

(ii) $(x \vee y) = (x \to y) \to y$

(iii) $(x \wedge y) = ((x^* \to y^*) \to y^*)^*$. Thus, we have $(A, \vee, \wedge, *, 0, 1)$ is a lattice Wajsberg algebra with lower bound 0 and upper bound 1.

Definition 2.5[1] Let A be a lattice Wajsberg algebra. A subset F of A is called an implicative filter of A if it satisfies the following axioms for all $x, y \in A$.

(i) $1 \in F$

(ii) $x \in F$ and $x \to y \in F$ imply $y \in F$

3. MAIN RESULTS

In this section, we introduce a positive implicative filter and an associative implicative filter of lattice Wajsberg algebras and investigate some properties with illustrations.

3.1 Positive implicative filters

Definition 3.1.1. A subset F of A is called a positive implicative filter of A if it satisfies

(i) $1 \in F$

(ii) $(x \to y) \to y \in F$ and $x \in F$ imply $y \in F$ for all $x, y \in A$.

Example 3.1.2. Let A = $\{0, a, b, c, 1\}$ be a set with Figure (1) as partial ordering. Define unary operation "$*$" and a binary operation "$\to$" on A as in the Table (1) and Table (2).

x	x^*
0	1
a	b
b	a
c	c
1	0

Table (1)

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	c	1	1
b	a	c	1	1	1
c	c	c	c	1	1
1	0	a	b	c	1

Table (2)

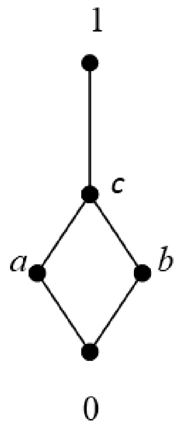

Figure (1)

Define $\vee$ and $\wedge$ an operation on A as follows:

$(x \vee y) = (x \to y) \to y$,

$(x \wedge y) = ((x^* \to y^*) \to y^*)^*$ for all $x, y \in A$. Then, we have $(A, \vee, \wedge, 0, 1)$ is a lattice Wajsberg algebra. It is easy verify that $F = \{a, c, 1\}$ is a positive implicative filter of A. But, $G=\{a,b,1\}$ is not a positive implicative filter of A. Since $(a \vee b) = c \notin G$ and $(a \wedge b) = c \notin G$.

Proposition 3.1.3. Every positive implicative filter of A is an implicative filter.

Proof. Let F be a positive implicative filter of A and $x \to y \in F$ and $x \in F$ for all $x, y \in A$. Putting $x = y$ in (ii) of Definition 3.1.1, we have

$(y \to y) \to y = 1 \to y = y \in F$ and $x \in F$. It follows from (ii) of Definition 3.1.1. that $y \in F$. Hence F is a implicative filter of A. ■

Remark 3.1.4. Converse of Proposition 3.1.3. need not be true. Consider a lattice W-algebra A as in Example 3.1.2. We know that {1} is a implicative filter of A, but it is not a positive implicative filter, since $(1 \to b) \to b = b \to b = 1 \in F$ and $b \notin \{1\}$. Also, a subset F_1 = {a, 1} is a implicative, but not a positive implicative filter of A, since $(1 \to b) \to b = b \to b = 1 \in F_1$ for $b \notin F_1$.

Next we show an equivalent condition that every implicative filter is a positive implicative filter.

Proposition 3.1.5. Let F be a implicative filter of A. Then F is a positive implicative filter of A if and only if for all $x, y \in A$, $(x \to y) \to x \in F$ implies $x \in F$.

Proof. Let F be a positive implicative filter of A and let $(x \to y) \to x$ for all $x, y \in A$. Then we have, $1 \to ((x \to y) \to x) = (x \to y) \to x \in F$. Since $1 \in F$ it follows from (ii) of Definition 3.1.1. that $x \in F$. Thus, $(x \to y) \to x \in F$ implies $x \in F$.

Conversely, suppose that F satisfies $(x \to y) \to x \in F$ implies $x \in F$. Let $(x \to y) \to y \in F$ and $x \in F$ for all $x, y \in A$. Then, $y \in F$ by (ii) of Definition 3.1.1. Hence F is a positive implicative filter of A. ■

3.2 Associative implicative filters

Definition 3.2.1 Let x be a fixed element of A. A subset F of A is called an associative implicative filter of A with respect to x if it satisfies

(i) $x \in F$

(ii) $x \to (y \to z) \in F$ and $x \to y \in F$ imply $z \in F$ for all $y, z \in A$.

An associative implicative filter of A with respect to all $x \neq 0$ is called an associative implicative filter of A.

Note. An associative implicative filter with respect to 0 is the whole algebra A. An associative implicative filter with respect to 1 is coincident with a implicative filter.

Example 3.2.2 Let A = {0, a, b, c, d, 1} be a set with Figure (2) as partial ordering. Define unary operation "$*$" and a binary operation "$\rightarrow$" on A as in the Tables (3) and (4)

x	x^*
0	1
a	d
b	c
c	b
d	a
1	0

Table (3)

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	a	c	c	1
b	c	1	1	c	c	1
c	b	a	b	1	a	1
d	a	1	a	1	1	1
1	0	a	b	c	d	1

Table (4)

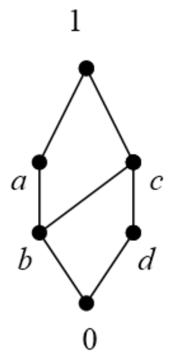

Figure (2)

Define $\vee$ and $\wedge$ an operation on A as follows:

$$(x \vee y) = (x \rightarrow y) \rightarrow y,$$

$(x \wedge y) = ((x^* \rightarrow y^*) \rightarrow y^*)^*$ for all $x, y \in A$. Then, we have $(A, \vee, \wedge, 0, 1)$ is a lattice Wajsberg algebra.

Let F = {1, a, b} is an associative implicative filter of A with respect to a and b but not with respect to c and d, since $c \rightarrow (b \rightarrow d) = c \rightarrow c = 1 \in F$, $c \rightarrow b = b \in F$ but $d \notin F$, and $d \rightarrow (b \rightarrow c) = d \rightarrow c = 1 \in F$, $d \rightarrow b = a \in F$ but $c \notin F$.

Proposition 3.2.3 Every associative implicative filter with respect to x contains x itself.

Proof. If $x = 0$ and $x = 1$ then it is trivial. If $x \neq 0$. Let F be an associative implicative filter of A with respect to x. we have $x \rightarrow (1 \rightarrow x) = x \rightarrow x = 1 \in F$ and $x \rightarrow 1 = 1 \in F$. It follows from (ii) of definition 3.2.1, and we have $x \in F$. ■

Proposition 3.2.4 Every associative implicative filter is an implicative filter.

Proof. Let F be an associative implicative filter of A and let $x \rightarrow y \in F$ and $x \in F$ for all $x, y \in A$. Then $1 \rightarrow x = x \in F$ and $1 \rightarrow (x \rightarrow y) = x \rightarrow y \in F$. It follows from (ii) of definition 3.2.1, we have $y \in F$. Therefore, F is an implicative filter of A. ■

Remark 3.2.5 The converse of proposition 3.2.4 need not be true. From example 3.2.2, {1} is an implicative filter of A. But it not an associative implicative filter of A, since $a \to (b \to c) = a \to 1 = 1 \in \{1\}$ and $a \to b = 1 \in \{1\}$, but $c \notin \{1\}$.

Next we show equivalent conditions that every implicative filter is an associative implicative filter.

Proposition 3.2.6 Let F be an implicative filter of A. Then F is an associative implicative filter if and only if it satisfies

$$x \to (y \to z) \in F \text{ implies } (x \to y) \to z \in F \text{ for all } x, y, z \in A. \qquad \text{(I)}$$

Proof. If an implicative filter F of A satisfies $x \to (y \to z) \in F$ implies $(x \to y) \to z \in F$ for all $x, y, z \in A$. Then, we get F is an associative implicative filter.

Conversely, let F be an associative implicative filter of A and let $x \to (y \to z) \in F$ for all $x, y, z \in A$. Then, we have

$$x \to ((y \to z) \to ((x \to y) \to z))$$

$$= (y \to z) \to (x \to ((x \to y) \to z)) \text{ from (vii) of propostion2.2}$$

$$= (y \to z) \to ((x \to y) \to (x \to z)) \text{ from (vii) of proposition 2.2}$$

$$= 1 \in F, \qquad \text{from (vii) of proposition 2.2 and (ii) definition 2.1}$$

Which implies from (ii) of definition 3.2.1, we have $(x \to y) \to z \in F$. ■

Proposition 3.2.7 Let F be an implicative filter of A. Then F is an associative implicative filter if and only if it satisfies $x \to (x \to y) \in F$ implies $y \in F$ for all $x, y \in A$. (II)

Proof. Let F be an implicative filter of A. It is enough to prove that (I) and (II) are equivalent. Putting $x = y$ in (I) and using (i) of definition 2.1 and (i), (ii) of proposition 2.2, we get (II). If (II) holds and let $x \to (y \to z) \in F$ for all $x, y, z \in A$. Using (i) and (ii) of definition 2.1, (i) and (vii) of proposition 2.2 and (i) and (ii) of proposition 2.3, we have

$$(x \to (y \to z)) \to (x \to (x \to ((x \to y) \to z)))$$

$$= 1 \to ((x \to (y \to z)) \to (x \to (x \to ((x \to y) \to z)))$$

$$= ((y \to z) \to (x \to ((x \to y) \to z))) \to ((x \to (y \to z)) \to (x \to (x \to ((x \to y) \to z)))$$

$$= (x \to (y \to z)) \to (((y \to z) \to (x \to ((x \to y) \to z))) \to (x \to (x \to ((x \to y) \to z)))$$

$$\geq (x \to (y \to z)) \to (x \to (y \to z)) = 1.$$

This implies that $(x \to (y \to z)) \to (x \to (x \to ((x \to y) \to z))) = 1 \in F$.

Since $x \to (y \to z) \in F$ and since F is an implicative filter, it follows that

$$x \to (x \to ((x \to y) \to z)) \in F.$$

By using (II), we get $(x \to y) \to z \in F$. This completes the proof. ■

4. CONCLUSION

We have proposed the notions of a positive implicative filter and an associative implicative filter in lattice Wajsberg algebra, discussed an equivalent condition that every implicative filter is a positive implicative filter. We have shown that every associative implicative filter is an implicative filter. Also, we have given some illustrations for positive and associative implicative filters. Finally, we have concluded that an equivalent condition for an implicative filter is an associative implicative filter. In future, we can extend these concepts in fuzzy environment.

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