International Journal of Difference Equations.
ISSN 0973-6069 Volume 1 Number 1 (2006), pp. 19–29

http://www.ripublication.com/ijde.htm

Trigonometric Recurrence Relations and Tridiagonal Trigonometric Matrices*

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Abstract

We show that every tridiagonal symmetric matrix can be transformed by a special transformation into the so-called tridiagonal trigonometric matrix. The relationship of this transformation to 2×2 trigonometric symplectic system and to three-term trigonometric recurrence relations is discussed as well.

AMS subject classification: 39A10.
Keywords: Three-term recurrence relation, symplectic difference system, trigonometric transformation, trigonometric system, Sturm-Liouville difference equation.

1. Introduction

In this paper, we consider the three-term symmetric recurrence relation

$$r_{k+1}x_{k+2} + \beta_k x_{k+1} + r_k x_k = 0, \quad r_k \neq 0, \quad k \in \{0, \dots, N-1\} \tag{1.1}$$

*Research supported by the Grant 201/04/0580 of the Czech Grant Agency and the Research Project MSM0022162409 of the Czech Ministry of Education
Received April 11, 2006; Accepted April 23, 2006

which is closely related to the 2×2 symplectic difference system

$$x_{k+1} = a_k x_k + b_k u_k, \quad u_{k+1} = c_k x_k + d_k u_k \tag{1.2}$$

with $a_k = 1$, $b_k = \frac{1}{r_k}$, $c_k = -r_{k+1} - r_k - \beta_k$, and $d_k = 1 + \frac{c_k}{r_k}$, and to the tridiagonal symmetric matrix

$$\mathcal{L} = -\begin{pmatrix} \beta_0 & r_1 & 0 & 0 & \dots & 0 & 0 & 0 \\ r_1 & \beta_1 & r_2 & 0 & \dots & 0 & 0 & 0 \\ 0 & r_2 & \beta_2 & r_3 & & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & & & \vdots & \\ & & & & & & & \\ 0 & 0 & & 0 & r_{N-3} & \beta_{N-3} & r_{N-2} & 0 \\ 0 & \dots & & & & r_{N-2} & \beta_{N-2} & r_{N-1} \\ 0 & \dots & & & & 0 & r_{N-1} & \beta_{N-1} \end{pmatrix}. \tag{1.3}$$

More precisely, under the "closely related" we mean that a sequence $x = \{x_k\}_{k=0}^{N+1}$ is a solution of (1.1) if and only if there exists $u = \{u_k\}_{k=0}^{N}$ such that x, u solves (1.2), and this happens if and only if $x = \{x_k\}_{k=0}^{N+1}$ with $x_0 = 0 = x_{N+1}$ solves the linear system $\mathcal{L}x = 0$. The integer N in (1.1) and (1.3) can be taken arbitrarily large and can be understood $N = \infty$ when dealing with asymptotic properties of solutions of (1.1), (1.2) or with Jacobi matrices which are (1.3) with $N = \infty$. For a general background of the qualitative theory of equation (1.1), system (1.2) and the tridiagonal symmetric matrices (Jacobi matrices) (1.3) we refer to the books [1, 2, 9, 10, 13].

Our principal concern are transformations and oscillatory properties of (1.1) and (1.2), and their relationship to positive definiteness of (1.3). Recall that a 2×2 difference system (1.2) is said to be *symplectic*, if the 2×2 matrix $\mathcal{S} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is symplectic, i.e.,

$$\mathcal{S}^T \mathcal{J} \mathcal{S} = \mathcal{J}, \quad \mathcal{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{1.4}$$

It is not difficult to verify that the 2×2 matrix $\mathcal{S}$ is symplectic if and only if $\det \mathcal{S} = 1$, i.e., $ad - bc = 1$. In the special case $a = d =: p$ and $b = -c =: q$, i.e., $p^2 + q^2 = 1$, the system

$$s_{k+1} = p_k s_k + q_k c_k, \quad c_{k+1} = -q_k s_k + p_k c_k \tag{1.5}$$

is called *trigonometric system*. This terminology is motivated by the fact that there exists $\varphi_k \in [0, 2\pi)$ such that $\cos \varphi_k = p_k$, $\sin \varphi_k = q_k$ and a solution of (1.5) is

$$s_k = \sin\left(\sum^{k-1} \varphi_j\right), \quad c_k = \cos\left(\sum^{k-1} \varphi_j\right). \tag{1.6}$$

Another important property of (1.5) is that it is *self-reciprocal,* i.e., the so-called *reciprocity transformation* (see, e.g., [8])

$$\begin{pmatrix} x \\ u \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}$$

transforms (1.5) into itself. This means that the matrix (with s, c given by (1.6))

$$Z_k = \begin{pmatrix} c_k & s_k \\ -s_k & c_k \end{pmatrix}$$

forms the fundamental matrix of (1.5) which is *symplectic* and *orthogonal*.

The aim of this paper is to deal with various aspects of the transformation theory of (1.2), in particular, with the so-called *trigonometric transformation* of (1.2). First we establish an alternative proof of the main result of [4] which deals with this transformation. Then we study (1.2) under the assumption $b_k \neq 0$. We show that then (1.2) and an associated trigonometric system (1.5) can be written in the form of three-term relation (1.1). We introduce the concepts of the *trigonometric three-term symmetric recurrence relation* and the *tridiagonal trigonometric symmetric matrix*, and using these concepts we present a geometric proof of the relationship between positivity of the quadratic functional associated with (1.5) and nonexistence of generalized zeros of a certain solution of this system. We also compare our results with their continuous counterparts which are treated in details in the books [5, 11].

2. Preliminaries

Symplectic system (1.2) is closely related to the discrete quadratic functional

$$\mathcal{F}(x, u; 0, N) = \sum_{k=0}^{N} \left\{ a_k c_k x_k^2 + 2 b_k c_k x_k u_k + b_k d_k u_k^2 \right\} \tag{2.1}$$

considered over the class of $\{x_k, u_k\}_{k=0}^{N+1}$ satisfying the boundary condition $x_0 = 0 = x_{N+1}$ and the first equation of (1.2) $x_{k+1} = a_k x_k + b_k u_k$ (the so-called *equation of motion*). This class of $\{x, u\}$ will be referred to as the class of *admissible* sequences for $\mathcal{F}$. It is known that the functional $\mathcal{F}(x, u; 0, N)$ is positive over the class of admissible $\{x, u\}$ with $x \not\equiv 0$ if and only if the solution $\begin{pmatrix} x \\ u \end{pmatrix}$ given by the initial condition $x_0 = 0$, $u_0 = 1$ has no *generalized zero* in $(0, N+1]$, i.e., $x_{k+1} \neq 0$ whenever $x_k \neq 0$, and $x_k x_{k+1} b_k \geq 0$, $k = 1, \ldots, N$, see, e.g., [3].

We will also need some results of the transformation theory of symplectic difference systems (see e.g., [8] for details). Let

$$\mathcal{R}_k = \begin{pmatrix} h_k & 0 \\ g_k & \dfrac{1}{h_k} \end{pmatrix},$$

where h, g are real-valued sequences. The transformation

$$\begin{pmatrix} x_k \\ u_k \end{pmatrix} = \mathcal{R}_k \begin{pmatrix} y_k \\ z_k \end{pmatrix} \tag{2.2}$$

transforms system (1.2) into a system (which is again symplectic)

$$y_{k+1} = \tilde{a}_k y_k + \tilde{b}_k z_k, \quad z_{k+1} = \tilde{c}_k y_k + \tilde{d}_k z_k$$

with the sequences $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, $\tilde{d}$ given by the formulas

$$\begin{aligned} \tilde{a}_k &= \frac{a_k h_k + b_k g_k}{h_{k+1}}, \\ \tilde{b}_k &= \frac{b_k}{h_k h_{k+1}}, \\ \tilde{c}_k &= -g_{k+1}(a_k h_k + b_k g_k) + h_{k+1}(c_k h_k + d_k g_k), \\ \tilde{d}_k &= \frac{-g_{k+1} b_k + h_{k+1} d_k}{h_k}. \end{aligned} \tag{2.3}$$

3. Trigonometric Recurrence Relations and Matrices

We start with an alternative proof of the statement that every 2×2 symplectic system can be transformed by a transformation preserving oscillatory behavior into a trigonometric system. This result is formulated in [4] for *general* $2n \times 2n$ symplectic systems. The fact that we consider here a scalar 2×2 symplectic system enables to present a proof which is based on the direct computation and which is more transparent than that given in [4].

Theorem 3.1. Let $\begin{pmatrix} x^{[1]} \\ u^{[1]} \end{pmatrix}$, $\begin{pmatrix} x^{[2]} \\ u^{[2]} \end{pmatrix}$ be solutions of (1.2) such that the matrix $\begin{pmatrix} x^{[1]} & u^{[1]} \\ x^{[2]} & u^{[2]} \end{pmatrix}$ is symplectic, and let

$$h_k^2 = \left(x_k^{[1]}\right)^2 + \left(x_k^{[2]}\right)^2, \quad g_k = \frac{x_k^{[1]} u_k^{[1]} + x_k^{[2]} u_k^{[2]}}{h_k}. \tag{3.1}$$

Then transformation (2.2) transforms (1.2) into trigonometric system (1.5).

Proof. According to transformation formulas (2.3) we need to prove the identities $\tilde{a} = \tilde{d}$ and $\tilde{b} = -\tilde{c}$, i.e., the identities

$$\begin{aligned} \frac{a_k h_k + b_k g_k}{h_{k+1}} &= \frac{-g_{k+1} b_k + h_{k+1} d_k}{h_k}, \\ \frac{b_k}{h_k h_{k+1}} &= g_{k+1}(a_k h_k - b_k g_k) - h_{k+1}(c_k h_k + d_k g_k). \end{aligned}$$

We have

$$\tilde{a}_k = \frac{a_k h_k + b_k g_k}{h_{k+1}} = \frac{a_k h_k^2 + b_k \left(x_k^{[1]} u_k^{[1]} + x_k^{[2]} u_k^{[2]} \right)}{h_k h_{k+1}}$$

and

$$\begin{aligned}
\tilde{d}_k &= \frac{-b_k x_{k+1}^{[1]} u_{k+1}^{[1]} - b_k x_{k+1}^{[2]} u_{k+1}^{[2]} + h_{k+1}^2 d_k}{h_k h_{k+1}} \\
&= \frac{1}{h_k h_{k+1}} \Big[-b_k \left(a_k x_k^{[1]} + b_k u_k^{[1]} \right) \left(c_k x_k^{[1]} + d_k u_k^{[1]} \right) \\
&\quad - b_k \left(a_k x_k^{[2]} + b_k u_k^{[2]} \right) \left(c_k x_k^{[2]} + d_k u_k^{[2]} \right) + \left[\left(x_{k+1}^{[1]} \right)^2 + \left(x_{k+1}^{[2]} \right)^2 \right] d_k \Big] \\
&= \frac{1}{h_k h_{k+1}} \Big[a_k \left(x_k^{[1]} \right)^2 (-b_k c_k + a_k d_k) + a_k \left(x_k^{[2]} \right)^2 (-b_k c_k + a_k d_k) \\
&\quad + b_k x_k^{[1]} u_k^{[1]} (a_k d_k - b_k c_k) + b_k x_k^{[2]} u_k^{[2]} (a_k d_k - b_k c_k) \Big] \\
&= \frac{a_k h_k^2 + b_k \left(x_k^{[1]} u_k^{[1]} + x_k^{[2]} u_k^{[2]} \right)}{h_k h_{k+1}},
\end{aligned}$$

so the first identity is proved. To prove the second identity $\tilde{b} = -\tilde{c}$ we proceed as follows:

$$\begin{aligned}
-\tilde{c} h_k h_{k+1} &= h_k h_{k+1} g_{k+1} (a_k h_k + b_k g_k) - h_k h_{k+1}^2 (c_k h_k + d_k g_k) \\
&= \left(x_{k+1}^{[1]} u_{k+1}^{[1]} + x_{k+1}^{[2]} u_{k+1}^{[2]} \right) \left[a_k h_k^2 + b_k \left(x_k^{[1]} u_k^{[1]} + x_k^{[2]} u_k^{[2]} \right) \right] \\
&\quad - h_{k+1}^2 \left[c_k h_k^2 + d_k \left(x_k^{[1]} u_k^{[1]} + x_k^{[2]} u_k^{[2]} \right) \right] \\
&= \left(x_k^{[1]} u_k^{[1]} + x_k^{[2]} u_k^{[2]} \right) \left[\left(x_{k+1}^{[1]} u_{k+1}^{[1]} + x_{k+1}^{[2]} u_{k+1}^{[2]} \right) b_k - h_{k+1}^2 d_k \right] \\
&\quad + h_k^2 \left[a_k \left(x_{k+1}^{[1]} u_{k+1}^{[1]} + x_{k+1}^{[2]} u_{k+1}^{[2]} \right) - c_k h_{k+1}^2 \right] \\
&= h_k^2 \left[a_k (b_k c_k - a_k d_k) \left(x_k^{[1]} u_k^{[1]} + x_k^{[2]} u_k^{[2]} \right) \right] \\
&\quad + \left[\left(u_k^{[1]} \right)^2 + \left(u_k^{[2]} \right)^2 \right] h_k^2 b_k (a_k d_k - b_k c_k) \\
&\quad + \left(x_k^{[1]} u_k^{[1]} + x_k^{[2]} u_k^{[2]} \right) \{ [b_k (a_k d_k + b_k c_k) - 2 a_k b_k d_k] \left(x_k^{[1]} u_k^{[1]} + x_k^{[2]} u_k^{[2]} \right) \\
&\quad + h_k^2 a_k (a_k d_k - b_k c_k) \} \\
&= -h_k^2 a_k \left(x_k^{[1]} u_k^{[1]} + x_k^{[2]} u_k^{[2]} \right) + \left[\left(u_k^{[1]} \right)^2 + \left(u_k^{[2]} \right)^2 \right] h_k^2 b_k \\
&\quad + \left(x_k^{[1]} u_k^{[1]} + x_k^{[2]} u_k^{[2]} \right) \left[-b_k \left(x_k^{[1]} u_k^{[1]} + x_k^{[2]} u_k^{[2]} \right) + h_k^2 a_k \right]
\end{aligned}$$

$$= b_k \left\{ \left[\left(x_k^{[1]}\right)^2 + \left(x_k^{[2]}\right)^2 \right] \left[\left(u_k^{[1]}\right)^2 + \left(u_k^{[2]}\right)^2 \right] - \left(x_k^{[1]} u_k^{[1]} + x_k^{[2]} u_k^{[2]} \right)^2 \right\}$$
$$= b_k \left(x_k^{[1]} u_k^{[2]} - x_k^{[2]} u_k^{[1]} \right)^2 = b_k = \tilde{b}_k h_k h_{k+1},$$

which proves that $\tilde{b} = -\tilde{c}$ since $h \neq 0$. ■

Definition 3.2. A three-term recurrence relation (1.1) is said to be *trigonometric*, if $|r_k| \geq 1$, $k = 1, \dots, N-1$, and there exists a sequence $e = \{e_k\}_{k=0}^N$, $e_k \in \{-1, 1\}$, such that

$$\beta_k = -e_{k+1} \operatorname{sgn} r_{k+1} \sqrt{r_{k+1}^2 - 1} - e_k \operatorname{sgn} r_k \sqrt{r_k^2 - 1}. \tag{3.2}$$

A tridiagonal matrix $\mathcal{L}$ of the form (1.3) is said to be *trigonometric*, if there exist $\varphi_0, \dots \varphi_{N-1} \in (0, 2\pi)$, $\varphi_j \neq \pi$, $j = 0, \dots, N-1$, such that

$$\beta_k = -\operatorname{cotg} \varphi_{k+1} - \operatorname{cotg} \varphi_k \quad \text{and} \quad r_k = \frac{1}{\sin \varphi_k}. \tag{3.3}$$

Now we relate the concepts from the previous definition.

Lemma 3.3. The following statements are equivalent:

(i) Recurrence relation (1.1) is trigonometric;

(ii) The associated symplectic difference system (1.2) is trigonometric;

(iii) The related symmetric tridiagonal matrix (1.3) is trigonometric.

Proof. (i) $\Longrightarrow$ (ii): Suppose that (1.1) is a trigonometric recurrence relation and put

$$q_k = \frac{1}{r_k}, \quad p_k = \frac{e_k \sqrt{r_k^2 - 1}}{|r_k|}. \tag{3.4}$$

Then obviously $p_k^2 + q_k^2 = 1$ and

$$\frac{p_{k+1}}{q_{k+1}} + \frac{p_k}{q_k} = \frac{e_{k+1} r_{k+1} \sqrt{r_{k+1}^2 - 1}}{|r_{k+1}|} + \frac{e_k r_k \sqrt{r_k^2 - 1}}{|r_k|} = -\beta_k,$$

i.e., (1.1) can be written in the form

$$\frac{1}{q_{k+1}} x_{k+2} - \left(\frac{p_{k+1}}{q_{k+1}} + \frac{p_k}{q_k} \right) x_{k+1} + \frac{1}{q_k} x_k = 0. \tag{3.5}$$

Put $u_k = \dfrac{1}{q_k}(x_{k+1} - p_k x_k)$, i.e., $x_{k+1} = p_k x_k + q_k u_k$, and using (3.5)

$$\begin{aligned} u_{k+1} &= \frac{1}{q_{k+1}}(x_{k+2} - p_{k+1}x_{k+1}) \\ &= \frac{p_k}{q_k}x_{k+1} - \frac{1}{q_k}x_k \\ &= \frac{p_k}{q_k}(p_k x_k + q_k u_k) - \frac{1}{q_k}x_k \\ &= \frac{p_k^2 - 1}{q_k}x_k + p_k u_k \\ &= -q_k x_k + p_k u_k, \end{aligned}$$

so, x, u is a solution of (1.5) with p, q given by (3.4).

(ii) $\Longrightarrow$ (iii): System (1.5) with $q_k \neq 0$ is trigonometric if and only if $p_k^2 + q_k^2 = 1$, i.e., there exists $\varphi_k \in (0, 2\pi)$, $\varphi_k \neq \pi$, such that $\sin\varphi_k = q_k$, $\cos\varphi_k = p_k$. In the previous part of the proof we have shown that if x, u is a solution of (1.5) with $q_k \neq 0$, then $x = \{x_k\}_{k=0}^{N+1}$ solves (3.5) and hence, if $x_0 = 0 = x_{N+1}$, it solves the linear system $\mathcal{L}x = 0$, where

$$r_k = \frac{1}{q_k} = \frac{1}{\sin\varphi_k}, \quad \beta_k = -\frac{p_{k+1}}{q_{k+1}} - \frac{p_k}{q_k} = -\operatorname{cotg}\varphi_{k+1} - \operatorname{cotg}\varphi_k,$$

i.e., the matrix $\mathcal{L}$ in (1.3) is trigonometric.

(iii) $\Longrightarrow$ (i): This implication is an immediate consequence of the relationship between Jacobi matrices and three-term symmetric recurrence relations, see e.g., [10] or [13]. ■

In the next statement we use the previous lemma and Theorem 3.1 to show that any tridiagonal symmetric matrix can be reduced via a diagonal transformation matrix to a tridiagonal *trigonometric* matrix.

Theorem 3.4. Given a tridiagonal symmetric $N \times N$ matrix $\mathcal{L}$, there exists a sequence $\{f_k\}_{k=0}^{N-1}$ such that $\mathcal{L}$ can be expressed in the form

$$\mathcal{L} = \operatorname{diag}\{f_0, \dots, f_{N-1}\}\tilde{\mathcal{L}}\operatorname{diag}\{f_0, \dots, f_{N-1}\}, \tag{3.6}$$

where $\tilde{\mathcal{L}}$ is a tridiagonal trigonometric matrix.

Proof. Consider a symplectic difference system (1.2) with $b_k \neq 0$. Then from the first equation in this system $u_k = \dfrac{1}{b_k}(x_{k+1} - a_k x_k)$ and substituting into the second equation we get the recurrence relation

$$\frac{x_{k+2}}{b_{k+1}} - \left(\frac{a_{k+1}}{b_{k+1}} + \frac{d_k}{b_k}\right)x_{k+1} + \frac{x_k}{b_k} = 0,$$

i.e., (1.1) with

$$r_k = \frac{1}{b_k} \quad \text{and} \quad \beta_k = -\frac{a_{k+1}}{b_{k+1}} - \frac{d_k}{b_k}.$$

The same idea applied to the symplectic system which results from (1.2) upon the transformation (2.2) gives the recurrence relation (1.1) with

$$\begin{aligned}
\tilde r_k &= \frac{1}{\tilde b_k} = \frac{h_k h_{k+1}}{b_k},\\
\tilde\beta_k &= -\frac{\tilde a_{k+1}}{\tilde b_{k+1}} - \frac{\tilde d_k}{\tilde b_k}\\
&= -\frac{\frac{1}{h_{k+2}}(ah+bg)_{k+1}}{\frac{b_{k+1}}{h_{k+1}h_{k+2}}} - \frac{\frac{1}{h_k}(-g_{k+1}b_k + h_{k+1}d_k)}{\frac{b_k}{h_k h_{k+1}}}\\
&= -h_{k+1}^2\frac{a_{k+1}}{b_{k+1}} - h_{k+1}^2\frac{d_k}{b_k}\\
&= h_{k+1}^2\beta_k.
\end{aligned}$$

Now, by Theorem 3.1, there are sequences h, g such that (2.2) transforms (1.2) into (1.5) and this fact, coupled with the statement of Lemma 3.3, gives the required result where $f_k := h_k^{-1}, k = 0, \dots, N-1$. ■

In the last statement of this section we give a discrete analogue of the result that the trigonometric quadratic functional (with a positive function q)

$$\mathcal{F}(x; a, b) = \int_a^b \left[\frac{1}{q(t)}x'^2 - q(t)x^2\right] dt$$

is positive over the class of nontrivial differentiable functions satisfying $x(a) = 0 = x(b)$ if and only if $\int_a^b q(t)\, dt < \pi$, see e.g., [5].

Theorem 3.5. The tridiagonal trigonometric matrix (1.3), where β_k, r_k are given by (3.3) with $\varphi_k \in (0, \pi)$, is positive definite if and only if $\sum_{k=0}^{N-1} \varphi_k < \pi$. In particular, the m-th principal minor D_m of $\mathcal{L}$, $m = 1, \dots, N-1$, is given by the formula

$$D_m = \frac{\sin\left(\sum_{j=0}^m \varphi_j\right)}{q_0 \cdots q_m}. \tag{3.7}$$

Proof. We prove the statement by induction. Denote $q_k = \sin\varphi_k$, $p_k = \cos\varphi_k$. Then we have

$$D_1 = \frac{p_0}{q_0} + \frac{p_1}{q_1} = \operatorname{cotg}\varphi_0 + \operatorname{cotg}\varphi_1 = \frac{\sin(\varphi_0+\varphi_1)}{q_0 q_1}.$$

Consider the determinant of the $(k+1)$-th order D_{k+1}. Expanding D_{k+1} by the $(k+1)$-th row (using the Laplace rule), we get

$$
\begin{aligned}
D_{k+1} &= \left(\frac{p_k}{q_k} + \frac{p_{k+1}}{q_{k+1}}\right) D_k - \frac{1}{q_k^2} D_{k-1} \\
&= \frac{(\sin\varphi_k \cos\varphi_{k+1} + \cos\varphi_k \sin\varphi_{k+1})}{q_0 \cdots q_{k-1} q_k^2 q_{k+1}} \\
&\quad \times \frac{\sin\left(\sum_{j=0}^{k-1}\varphi_j\right)\cos\varphi_k + \cos\left(\sum_{j=0}^{k-1}\varphi_j\right)\sin\varphi_k}{q_0 \cdots q_{k-1} q_k^2 q_{k+1}} - \frac{\sin\left(\sum_{j=0}^{k-1}\varphi_j\right)}{q_k^2} \\
&= \frac{\sin\varphi_k \cos\varphi_{k+1} \sin\left(\sum_{j=0}^{k-1}\varphi_j\right)\cos\varphi_k}{q_0 \cdots q_{k-1} q_k^2 q_{k+1}} \\
&\quad + \frac{\cos\left(\sum_{j=0}^{k-1}\varphi_j\right)\sin\varphi_k \sin\varphi_k \cos\varphi_{k+1} + \cos\varphi_k \sin\varphi_{k+1} \sin\left(\sum_{j=0}^{k-1}\varphi_j\right)\cos\varphi_k}{q_0 \ldots q_{k-1} q_k^2 q_{k+1}} \\
&\quad + \frac{\cos\varphi_k \sin\varphi_{k+1} \cos\left(\sum_{j=0}^{k-1}\varphi_j\right)\sin\varphi_k - \sin\varphi_{k+1}\sin\left(\sum_{j=0}^{k-1}\varphi_j\right)}{q_0 \ldots q_{k-1} q_k^2 q_{k+1}} \\
&= \frac{\sin\left(\sum_{j=0}^{k-1}\varphi_j\right)\sin\varphi_{k+1}(\cos^2\varphi_k - 1) + \cos\left(\sum_{j=0}^{k-1}\varphi_j\right)\sin^2\varphi_k \cos\varphi_{k+1}}{q_0 \ldots q_{k-1} q_k^2 q_{k+1}} \\
&\quad + \frac{\sin\varphi_k \cos\varphi_{k+1} \sin\left(\sum_{j=0}^{k-1}\varphi_j\right)\cos\varphi_k + \cos\varphi_k \sin\varphi_{k+1}\cos\left(\sum_{j=0}^{k-1}\varphi_j\right)\sin\varphi_k}{q_0 \ldots q_{k-1} q_k^2 q_{k+1}} \\
&= \frac{\sin^2\varphi_k\left[-\sin\left(\sum_{j=0}^{k-1}\varphi_j\right)\sin\varphi_{k+1} + \cos\left(\sum_{j=0}^{k-1}\varphi_j\right)\cos\varphi_{k+1}\right]}{q_0 \ldots q_{k-1} q_k^2 q_{k+1}} \\
&\quad + \frac{\sin\varphi_k \cos\varphi_k\left[\sin\left(\sum_{j=0}^{k-1}\varphi_j\right)\cos\varphi_{k+1} + \cos\left(\sum_{j=0}^{k-1}\varphi_j\right)\sin\varphi_{k+1}\right]}{q_0 \ldots q_{k-1} q_k^2 q_{k+1}} \\
&= \frac{\sin^2\varphi_k \cos\left(\sum_{j=0}^{k-1}\varphi_j + \varphi_{k+1}\right) + \sin\varphi_k \cos\varphi_k \sin\left(\sum_{j=0}^{k-1}\varphi_j + \varphi_{k+1}\right)}{q_0 \ldots q_{k-1} q_k^2 q_{k+1}} \\
&= \frac{\sin\varphi_k \sin\left(\sum_{j=0}^{k+1}\varphi_j\right)}{q_0 \ldots q_{k-1} q_k^2 q_{k+1}} = \frac{\sin\left(\sum_{j=0}^{k+1}\varphi_j\right)}{q_0 \ldots q_{k+1}}
\end{aligned}
$$

which proves (3.7) and also the statement of theorem. ∎

4. Remarks

(i) As we have already mentioned earlier, the transformation theory of the second order differential equation

$$(r(t)x')' + c(t)x = 0 \tag{4.1}$$

was studied in details in the monographs [5, 11]. It was shown there (among others) that (4.1) can be transformed via the transformation $x = h(t)y$ into the equation

$$\left(\frac{1}{q(t)}y'\right)' + q(t)y = 0, \quad q = \frac{1}{rh^2}. \tag{4.2}$$

Theorem 3.1 can be regarded as a discrete analogue of this statement. Theorem 3.1 (which is not a new result in our paper, only its proof is new) has been used (following the "continuous pattern") to study oscillatory properties of the second order difference equation

$$\Delta(r_k \Delta x_k) + c_k x_{k+1} = 0, \quad r_k \neq 0, \tag{4.3}$$

in the recent papers [6, 7, 12]. However, the relationship of (4.3) to three-term recurrence relations (1.1) and Jacobi matrices (1.3) is not considered in those papers.

(ii) It was shown in [4] that the sequence h in Theorem 3.1 can be taken in such a way that if $b_k \neq 0$ in (1.2), then $q_k > 0$ in (1.5), see [4]. Consequently, one can suppose, without loss of generality that $\varphi_k \in (0, \pi)$ in trigonometric matrix (1.3) (see Definition 3.2).

(iii) Assume that the recurrence relation (1.1) is trigonometric, i.e., (in view of Lemma 3.3) there exist $\varphi_k \in (0, 2\pi) \setminus \{\pi\}$ such that

$$\frac{x_{k+2}}{\sin \varphi_{k+1}} - (\operatorname{cotg} \varphi_{k+1} + \operatorname{cotg} \varphi_k)\, x_{k+1} + \frac{x_k}{\sin \varphi_k} = 0. \tag{4.4}$$

By a direct computation (or again using Lemma 3.3) one can verify that then

$$x_k^{[1]} = \sin\left(\sum^{k-1} \varphi_j\right), \quad x_k^{[2]} = \cos\left(\sum^{k-1} \varphi_j\right)$$

is the fundamental system of solutions of (4.4). This is a discrete analogue of the fact that

$$x_1(t) = \sin\left(\int^t q(s)\, ds\right), \quad x_2(t) = \cos\left(\int^t q(s)\, ds\right)$$

is the fundamental system of solution of (4.2).

(iv) Using the previous remark, (4.4) with $\varphi_k \in (0, \pi)$ is nonoscillatory if and only if $\sum^{\infty} \varphi_k < \infty$. An important concept of the theory of three-term symmetric recurrences is the so-called *recessive* solution which is a solution $\tilde{x}$ with the property

$$\lim_{k\to\infty} \frac{\tilde{x}_k}{x_k} = 0$$

for any solution x linearly independent of $\tilde{x}$. The recessive solution of (4.4) can be computed explicitly, it is the solution given by the formula

$$\tilde{x}_k = \sin\left(\sum_{j=k}^{\infty} \varphi_j\right).$$

We hope to use this fact in investigating asymptotic properties of (1.1) in a subsequent paper.

References

[1] R.P. Agarwal, Difference Equations and Inequalities. Theory, Methods, and Applications. Second edition. Monographs and Textbooks in Pure and Applied Mathematics, 228. Marcel Dekker, Inc., New York, 2000.

[2] R.P. Agarwal, M. Bohner, S.R. Grace, and D. O'Regan, Discrete Oscillation Theory, Hindawi Publishing Corporation, New York, 2005.

[3] M. Bohner, Linear Hamiltonian difference systems: disconjugacy and Jacobi-type conditions, *J. Math. Anal. Appl.*, 199:804–826, 1996.

[4] M. Bohner and O. Došlý, Trigonometric transformations of symplectic difference systems, *J. Differential Equations*, 163(1):113–129, 2000.

[5] O. Borůvka, Linear differential transformations of the second order, The English Universities Press, Ltd., London, 1971.

[6] Z. Došlá and Š. Pechancová, Conjugacy and phases of second order difference equations, submitted.

[7] Z. Došlá and D. Škrabáková, Phases of linear difference equations and symplectic systems, *Math. Bohememica*, 128:293–308, 2003.

[8] O. Došlý, R. Hilscher, Linear Hamiltonian difference systems: transformations, recessive solutions, generalized reciprocity, *Dynam. Systems Appl.*, 8:401–420, 1999.

[9] S. Elaydi, An Introduction to Difference Equations. Third edition. Undergraduate Texts in Mathematics. Springer, New York, 2005.

[10] W.G. Kelley and A.C. Peterson, Difference Equations, An Introduction with Applications. Second edition. Harcourt/Academic Press, San Diego, 2001.

[11] F. Neuman, Global Properties of Linear Ordinary Differential Equations, Mathematics and its Applications, Kluwer Academic Publishers Group, Dordrecht, 1991.

[12] Š. Ryzí, On the first and second phases of 2×2 symplectic difference systems, *Stud. Univ. Žilina, Math. Ser.*, 17:129–136, 2003.

[13] G. Teschl, Jacobi Operators and Completely Integrable Nonlinear Lattices, American Mathematical Society, New York, 2000.