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q-Bernoulli and q-Euler Polynomials, an Umbral Approach

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Abstract

We present a variety of q-formulas linked together by the q-umbral calculus introduced here, equivalent to the approach to q-hypergeometric functions and q-Appell functions given earlier, influenced by Rota and Cigler. This q-umbral calculus is connected to formal power series; three q-Taylor formulas occur; one of these forms the basis of an umbral formula influenced by Nörlund, which enables q-Euler-Maclaurin- and q-Euler-Boole formulas. q-Appell polynomials in the spirit of Milne-Thomson are also treated before specialization to q-Bernoulli and q-Euler polynomials. A brief survey of the theory and history of finite differences and umbral calculus is given.

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1. Introduction

The aim of this paper is to describe how different q-difference operators combine with q-Bernoulli and q-Euler numbers and polynomials to form various q-formulas.

The Bernoulli numbers were first used by Jacob Bernoulli (1654–1705) [9], who calculated the sum

$$s_m(n) \equiv \sum_{k=0}^{n-1} k^m = \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k} n^{m+1-k} B_k. \tag{1.1}$$

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In 1738 Euler used the generating functions to study the Bernoulli polynomials. The Bernoulli polynomials were also studied by J.-L. Raabe (1801–1859) [69] and Schlömilch. Raabe found two important formulas for these polynomials.

It was Raabe who in 1851 first used the name Euler numbers for a multiple of the secant numbers. It was then used by Sylvester, Catalan, Glaisher, Lucas, and from 1877 the name was used in Germany. The Bernoulli and Euler polynomials were later systematically studied by Nörlund [66].

Gould [38] remarks that many sums involving binomial coefficients greatly benefit from the use of Bernoulli numbers. Bernoulli and Stirling numbers have wideranging applications in computer technology [41] and in numerical analysis [31]. One reason is that computers use difference operators rather than derivatives, and these numbers are used in the transformation process.

We will now describe the q-umbral method invented by the author [21–24, 26], which also involves the Nalli–Ward–Alsalam q-addition and the Jackson–Hahn–Cigler q-addition. This method is a mixture of Heine 1846 [44] and Gasper–Rahman [32]. The advantages of this method have been summarized in [24, p. 495].

Definition 1.1. The power function is defined by $q^a \equiv e^{a\log(q)}$. We always use the principal branch of the logarithm. The variables

$$a, b, c, a_1, a_2, \ldots, b_1, b_2, \ldots \in \mathbb{C}$$

denote certain parameters. The variables i, j, k, l, m, n, p, r will denote natural numbers except for certain cases where it will be clear from the context that i will denote the imaginary unit.

The q-analogues of a complex number a and of the factorial function are defined by:

$$\{a\}_q \equiv \frac{1-q^a}{1-q},\ q \in \mathbb{C}\backslash\{1\},$$

$$\{n\}_q! \equiv \prod_{k=1}^{n} \{k\}_q,\ \{0\}_q! \equiv 1,\ q \in \mathbb{C}.$$

Definition 1.2. Let the q-shifted factorial (compare [33, p. 38]) be defined by

$$\langle a;q\rangle_n \equiv \begin{cases} 1, & n=0; \\ \displaystyle\prod_{m=0}^{n-1}(1-q^{a+m}), & n \in \mathbb{N}. \end{cases} \tag{1.2}$$

The Watson notation [32] will also be used

$$(a;q)_n \equiv \begin{cases} 1, & n=0; \\ \displaystyle\prod_{m=0}^{n-1}(1-aq^{m}), & n \in \mathbb{N}. \end{cases} \tag{1.3}$$

Since products of q-shifted factorials occur so often, to simplify them, we shall frequently use the following more compact notation:

$$\langle a, b; q\rangle_n \equiv \langle a; q\rangle_n \langle b; q\rangle_n. \tag{1.4}$$

Furthermore,

$$(a; q)_\infty \equiv \prod_{m=0}^{\infty}(1 - aq^m),\ 0 < |q| < 1, \tag{1.5}$$

$$(a; q)_\alpha \equiv \frac{(a; q)_\infty}{(aq^\alpha; q)_\infty},\ a \neq q^{-m-\alpha},\ m \in \mathbb{N}_0. \tag{1.6}$$

Let the Gauss q-binomial coefficient be defined by

$$\binom{n}{k}_q \equiv \frac{\langle 1; q\rangle_n}{\langle 1; q\rangle_k \langle 1; q\rangle_{n-k}}, \tag{1.7}$$

for $k = 0, 1, \ldots, n$, and by

$$\binom{\alpha}{\beta}_q \equiv \frac{\langle \beta + 1, \alpha - \beta + 1; q\rangle_\infty}{\langle 1, \alpha + 1; q\rangle_\infty}, \tag{1.8}$$

for complex α and β when $0 < |q| < 1$.

The q-multinomial coefficient, a q-analogue of [81, p. 10], is defined by

$$\binom{n}{k_1, \ldots, k_l}_q \equiv \frac{\langle 1; q\rangle_n}{\prod_{i=1}^{l} \langle 1; q\rangle_{k_i}}, \tag{1.9}$$

for $\{k_i\}_{i=1}^{l} = 0, 1, \ldots, n$ and $\sum_{i=1}^{l} k_i = n$.

If the number of k_i is unspecified, we denote the q-multinomial coefficient by

$$\binom{n}{\vec{k}}_q,\ \sum_{i=1}^{\infty} k_i = n.$$

We give some examples of q-multinomial coefficients.

Example 1.3.

$$\binom{3}{1,1,1}_q = 1 + 2q + 2q^2 + q^3,$$

$$\binom{4}{1,1,1,1}_q = 1 + 3q + 5q^2 + 6q^3 + 5q^4 + 3q^5 + q^6,$$

$$\binom{4}{2,1,1}_q = 1 + 2q + 3q^2 + 3q^3 + 2q^4 + q^5,$$

$$\binom{4}{2,2}_q = 1 + q + 2q^2 + q^3 + q^4.$$

Definition 1.4. If $0 < |q| < 1$ and $|z| < |1-q|^{-1}$, the q-exponential function $E_q(z)$ was defined by Jackson [49] in 1904, and by Exton [30].

$$\mathrm{E}_q(z) \equiv \sum_{k=0}^{\infty} \frac{1}{\{k\}_q!} z^k. \tag{1.10}$$

Let the q-Pochhammer symbol $\{a\}_{n,q}$ be defined by

$$\{a\}_{n,q} \equiv \prod_{m=0}^{n-1} \{a+m\}_q.$$

The following notation will be convenient:

$$\mathrm{QE}(x) \equiv q^x, \quad q^{\binom{\vec{k}}{2}} \equiv \prod_{j=1}^{n} q^{\binom{k_j}{2}},$$

where $\vec{k}$ is a vector of length n.

Definition 1.5. The Nalli–Ward–Alsalam q-addition (NWA), compare [2, p. 240], [61, p. 345], [92, p. 256] is given by

$$(a \oplus_q b)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q a^k b^{n-k},\ n \in \mathbb{N}_0. \tag{1.11}$$

Furthermore,we put

$$(a \ominus_q b)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q a^k (-b)^{n-k},\ n \in \mathbb{N}_0. \tag{1.12}$$

There is a q-addition dual to the NWA, which will be presented here for reasons to be given shortly. The following polynomial in 3 variables x, y, q originates from Gauss.

Definition 1.6. The Jackson–Hahn–Cigler q-addition (JHC), compare [13, p. 91], [43, p. 362], [51, p. 78] is the function

$$\begin{aligned}(x \boxplus_q y)^n \equiv& [x+y]_q^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q q^{\binom{k}{2}}\, y^k x^{n-k} \\ =& x^n \left(-\frac{y}{x}; q\right)_n \equiv P_{n,q}(x,y),\ n \in \mathbb{N}_0,\end{aligned} \tag{1.13}$$

$$(x \boxminus_q y)^n \equiv P_{n,q}(x,-y),\ n \in \mathbb{N}_0. \tag{1.14}$$

Remark 1.7. The notation $[x + y]_q^n$ is due to Hahn [43, p. 362], and the notation $P_{n,q}(x, y)$ is due to Cigler [13, p. 91]. We will only use $(x \boxplus_q y)^n$ as it resembles the notation for NWA.

For symbolic purposes, we will define a general q-addition.

Definition 1.8. Let $f(k, n)$ be a given function. The general q-addition is defined by

$$(a \oplus_{g,q} b)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q q^{f(k,n)} a^k b^{n-k},\ n \in \mathbb{N}_0,$$

and similar for $(a \ominus_{g,q} b)$.

We will discuss how the NWA q-addition enables q-analogues of many results from Nörlund's investigations of difference analysis [65, 66]. We will use Milne-Thomson [60] as a basis for the notation of the various polynomials. The generating functions will play a major role here.

The main topic is the set of almost parallel formulas, anticipated by Ward [92], for the NWA q-Bernoulli, q-Euler, q-Lucas and q-G numbers and polynomials. These numbers will belong to $\mathbb{C}(q)$. In number theory there is also a Lucas number, which is not to be confused with this one. The reason for the name G-number is that E.T. Bell, and after him his graduate student Ward, for some reason called these numbers Genocchi numbers.

The reason for introducing the second or JHC polynomials is that they are needed in the q-analogues of complementary argument formulas. The notation second or JHC polynomials will be used throughout.

These equations are more systematically presented here than in Nörlund [66], which makes this paper an amplification and a complement to [66] even for the case $q = 1$. Despite the "telegraphic style" of the proofs, which assume that the reader knows the basic technical tools and various q-identities, it is likely that readers with a taste for q-analogues will find much to enjoy. For example, the q-analogues of the Euler–Maclaurin summation formula might even be of some general interest.

We have added q-analogues of some formulas from Szegö's review [87] of [65]. The operational q-additions makes the formulas remarkably pretty.

It is well known that there are at least two types of q-Bernoulli numbers, now let us consider the first one, i.e., NWA. Its complement is JHC. The different Carlitz' 1948 q-Bernoulli numbers are not considered here.

In the spirit of Milne-Thomson [60] and Rainville [71], we replace the $=$ by $\doteqdot$ to indicate the symbolic nature. Lucas [56] used a different symbol.

Example 1.9. [92, p. 265], [2, p. 245, 4.3]

$$B_{\text{NWA},0,q} = 1,\ (B_{\text{NWA},q} \oplus_q 1)^k - B_{\text{NWA},k,q} \doteqdot \delta_{1,k}, \tag{1.15}$$

where $B_{\text{NWA},q}^n$ is replaced by $B_{\text{NWA},n,q}$ on expansion.

Another improvement in the present paper is that the operational umbral formulas for q-Bernoulli and q-Euler polynomials are extended from polynomials to formal power series. The formulas are adaptable to formal power series with corresponding q-Taylor formulas whereas the formulas in another paper about q-Stirling numbers [27] are adaptable to functions of q^x or equivalently $\binom{x}{k}_q$, with corresponding q-Taylor formulas.

In the year 1706 Johann Bernoulli (1667–1748) invented the difference symbol $\triangle$. Fifty years latter, 1755, Leonhard Euler used its inverse, the $\sum$ operator [29, Chapter 1]. Euler was Johann Bernoulli's student together with Bernoulli's two sons, Nicolaus II and Daniel. Even though Johann Bernoulli used the symbol $\triangle$ already in 1706, he did not imply finite differences thereby but differential quotients. Euler stands out as the one who devised the designation that has remained in use. Euler's proofs were however from a modern point of view not entirely satisfactory according to J. Herschel [46, p. 87].

As Sharma and Chak [80, p. 326] remarked, the operator D_q, defined by

$$\left(D_q\varphi\right)(x) \equiv \begin{cases} \dfrac{\varphi(x)-\varphi(qx)}{(1-q)x}, & \text{if } q \in \mathbb{C}\backslash\{1\},\ x \neq 0; \\ \dfrac{d\varphi}{dx}(x) & \text{if } q = 1; \\ \dfrac{d\varphi}{dx}(0) & \text{if } x = 0 \end{cases} \tag{1.16}$$

plays the same role for polynomials in x as the difference operator

$$\triangle_q f(x) \equiv f(x+1) - f(x),\ \triangle_q^{n+1} f(x) \equiv \triangle_q^n f(x+1) - q^n \triangle_q^n f(x)$$

does for polynomials in q^x.

If we want to indicate the variable which the q-difference operator is applied to, we write $\left(D_{q,x}\varphi\right)(x,y)$ for the operator. The same notation will also be used for a general operator.

We will find q-analogues of Leibniz-type formulas from Jordan [54] for the $\triangle_q$ and ∇_qoperators.

All the next three equations were found by Euler. They have the following form, where E is the forward shift operator and $\triangle = E - I$.

Theorem 1.10. [28, p. 200], [15, p. 26], [82, p. 9]

$$\triangle^n f(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} E^{n-k} f(x). \tag{1.17}$$

This formula can be inverted.

Theorem 1.11. [79, p. 15, 3.1]

$$E^n f(x) = \sum_{i=0}^{n} \binom{n}{i} \triangle^i f(x). \tag{1.18}$$

The Leibniz rule is as follows.

Theorem 1.12. [54, p. 97, 10], [15, p. 27, 2.13], [60, p. 35, 2], [67, p. 19]

$$\triangle^n (fg) = \sum_{i=0}^{n} \binom{n}{i} \triangle^i f \, (\triangle^{n-i} E^i))g. \tag{1.19}$$

Curiously, in this paper we keep the binomial coefficients in the corresponding q-formulas, whereas in the q-Stirling paper [27], the q-binomial coefficients are used for the corresponding formulas.

We now give a short review of the history of umbral calculus and finite differences. Interpolation theory, often used by nineteenth century astronomers (Gauss, Bessel, W. Herschel [1738–1822], J. Herschel), is essentially equivalent to theory of finite differences. Gudermann (1786–1852), the teacher of Weierstrass, was one of the first to use this technique. Calculations on elliptic functions by finite differences were made by Jacobi, Weierstrass and Milne-Thomson (1891–1974). The mathematician and astronomer Johann August Grünert (1797–1872), editor of the journal *Archiv der Mathematik und Physik*, which started in 1841, used this technique to publish some of the first tables of Stirling numbers. The umbral calculus was initiated by Euler [29], who used operator equations like (3.43), and Lagrange (1736–1813). Later Arbogast (1759–1803) suggested to substitute a capital D for the little d of Leibniz to simplify the computations. Textbooks on the subject were written by Ettingshausen (1796–1878), J. Herschel (1792–1871), Pearson 1850 and De Morgan (1806–1871). Robert Murphy (1806–1843) was a forerunner to Boole and Heaviside, who among other things found beautiful operator formulas for derivatives in the spirit of Carlitz. J.J. Sylvester (1814–1897) edited *Quarterly Journal of Mathematics* from 1855 to 1878, where attempts at umbral calculus were made by Horner 1861, Blissard (1803–1875) 1861–68, and Glaisher (1848–1928). It was Sylvester who coined the name umbral calculus. By 1860 two textbooks on finite differences were in print in England, one of them by Boole (1815–1864), which covered almost all the theorems that we know now. Oliver Heaviside (1850–1925) was able to greatly simplify Maxwell's 20 equations in 20 variables to four equations in two variables. This and other articles about electrical problems, which appeared in 1892–98, were severely criticized for their lack of rigour by the contemporary mathematicians. It seems that Heaviside's contribution to mathematics was underestimated by his contemporaries, since he both discussed formal power series and the rudiments of the umbral calculus that we give in this paper. Genocchi (1817–1889) and Pincherle (1853–1936) contributed to the early Italian development of the subject. Clebsch (1833–1872) and Gordan (1837–1912) continued the theory of invariants that had started with Sylvester and Cayley.

In the same way as the Γ function plays a basic role in complex analysis, the Γ_q function is fundamental for q-calculus. The Γ_q function is defined in the unit disk $0 < |q| < 1$ by

Definition 1.13.

$$\Gamma_q(x) \equiv \frac{\langle 1; q\rangle_\infty}{\langle x; q\rangle_\infty}(1-q)^{1-x}. \tag{1.20}$$

Heine, P. Appell 1879, and Daum [19] used another function without the factor $(1-q)^{1-x}$, which they called the Heine Ω-function. Ashton [7] in his thesis supervised by Lindemann, showed its connection to the Jacobi–Neville elliptic functions. The main difference between the two functions is that Ω has zeros, in contrast to the Γ_q function which has no zeros, and therefore $\frac{1}{\Gamma_q}$ is entire. Sonine [84] wrote a book about the Heine Ω-function in Russian.

The Heine q-umbral calculus reached its peak in the thesis by Smith [83] 1911, supervised by Pringsheim. The Austrian school of q-analysis started in the sixties when Wolfgang Hahn (1911–1998) moved to Graz in 1964 after visits to India 1959–1961 and America 1962. Cigler (1937–) [15] wrote an excellent book on finite differences with a view to umbral calculus.

In 1880 Appell (1855–1930) [5] characterized certain polynomial sequences $F_\nu(x)$ including Bernoulli and Euler by the property

$$DF_\nu(x) = \nu F_{\nu-1}(x).$$

This was equivalent to Euler's generating function. Chapter 3 is devoted to q-analogues of these so-called Appell polynomials. Another French contribution was made by E. Lucas (1842–1891), who invented a modern notation for umbral calculus, which we will follow closely. F.H. Jackson (1870–1960) followed this path in the early twentieth century, and fully understood the symbolic nature of the subject in his first investigations of q-functions. Like Blissard, Jackson worked as a priest his whole life; both of them had studied in Cambridge. To commemorate Jackson, we will use his notation for $E_q(x)$.

Pia Nalli (1886–1964) was the first to use the Ward q-addition in her only paper on q-calculus. Letterio Toscano published interesting papers involving Bernoulli, Euler and Stirling numbers in connection with the operator xD. Geronimus (1898–1984) wrote about certain Appell polynomials. E.T. Bell (1883–1960) tried to write about umbral calculus, but he is best remembered for his books about the history of mathematics. Morgan Ward (1901–1963), became doctor at Caltech 1928 supervised by E. T. Bell.

Thorvald N. Thiele (1838–1910) was a prolific Danish actuary, astronomer, and mathematician. Thiele's book "Interpolationsrechnung", which contains a table of Stirling numbers, was published 1909.

Niels Erik Nörlund (1885–1981) was a Danish/Swedish mathematician, astronomer, and geodeticist. The remarkable work [66] presented the first rigorous treatment of finite differences, written from the point of view of the mathematician. According to

Grigoriew [42, p. 147], the generalized Bernoulli numbers that Nörlund uses in [66] had previously also been used by Blissard [10] and Imchenetsky [48]. Steffensen [86], Jordan [54], and Milne-Thomson [60] wrote books about finite differences intended both for mathematicians and statisticians.

Other famous people are L. Carlitz (1907–1999), J. Riordan, and Rota (1932–1999).

2. The Ward–Alsalam–Rota–Cigler q-umbral Calculus

Cigler [14] used a special case of the following q-umbral calculus, the case $q = 1$ was treated in [15].

Definition 2.1. A q-analogue of [76, p. 696]. A q-umbral calculus contains a set A, called the alphabet, with elements called letters or umbrae.

Assume that $\alpha,\ \beta,$ are distinct umbrae. Then a new umbra is obtained by $\alpha * \beta$, where $*$ is $\oplus_q,\ \boxplus_q,\ \ominus_q,\ \boxminus_q,$ or any general q-addition.

There is a certain linear functional $eval,\ \mathbb{C}[[x]] \times A \to \mathbb{C}$, called the evaluation, such that $eval(1, \alpha) = 1, \alpha \in A$. In the following, an arbitrary $f \in \mathbb{C}[[x]]$ will be used.

If $\alpha,\ \beta,\ \ldots,\ \gamma$ are distinct umbrae, and $i, j,\ \ldots,\ k$ positive integers,

$$eval(f, \alpha^i\beta^j \ldots \gamma^k) = eval(f, \alpha^i)eval(f, \beta^j) \ldots eval(f, \gamma^k).$$

Two umbrae α and β are called equivalent, denoted $\alpha \sim \beta$ if $eval(f, \alpha) = eval(f, \beta)$. The set of equivalent umbrae form an equivalence class.

There is a distinguished element ϵ of the alphabet called the zero, such that

$$eval(f, \epsilon^n) = \delta_{n,0} \text{ and } x \boxminus_q x \sim \epsilon.$$

Elements α and $\beta \in A$ are said to be inverse to each other if $\alpha \boxplus_q \beta \sim \epsilon$.

There is a Ward number $\overline{n}_q$

$$\overline{n}_q \sim 1 \oplus_q 1 \oplus_q \ldots \oplus_q 1,$$

where the number of 1 in the RHS is n.

There is a Jackson number $\tilde{n}_q$

$$\tilde{n}_q \sim 1 \boxplus_q 1 \boxplus_q \ldots \boxplus_q 1,$$

where the number of 1 in the RHS is n.

There is a multiplication with Ward and Jackson numbers in the following sense: Assume that $a \in \mathbb{C}$ or $a \in \mathbb{C}[D_q]$. Then we define

$$a\overline{n}_q \sim a \oplus_q a \oplus_q \ldots \oplus_q a,$$

where the number of a in the RHS is n. In the same way,

$$a\tilde{n}_q \sim a \boxplus_q a \boxplus_q \ldots \boxplus_q a,$$

where the number of a in the RHS is n.

If $\alpha_1, \dots, \alpha_n \ \in A,\ \alpha \sim \alpha_i,\ i = 1, \dots, n$, then $\alpha_1 \oplus_q \dots \oplus_q \alpha_n \sim \overline{n}_q \alpha$. The last condition is a q-analogue of [65, p. 125, (13)], [66, p. 132, (49)]:

$$F(x \oplus_q B_{\text{NWA},q}(y)) \doteqdot \sum_{n=0}^{\infty} \frac{B_{\text{NWA},n,q}(y)}{\{n\}_q!} D_q^n F(x). \tag{2.1}$$

Here B can be changed to any q-polynomial sequence.

Three examples of $eval$ are the NWA, JHC and the general q-addition.

Theorem 2.2. [61, p. 345] The q-addition (1.11) has the following properties, where $a, b \in A;\ c \in \mathbb{C}$:

$$\begin{aligned} (a \oplus_q b) \oplus_q c \sim a \oplus_q (b \oplus_q c), \quad a \oplus_q b \sim b \oplus_q a, \\ a \oplus_q 0 \sim 0 \oplus_q a \sim a, \quad ca \oplus_q cb \sim c(a \oplus_q b). \end{aligned} \tag{2.2}$$

The first three conditions mean that the umbrae of Nalli–Ward–Alsalam q-addition form a commutative monoid.

Definition 2.3. For $\alpha \in \mathbb{C}$, NWA is extended to

$$(a \oplus_q b)^\alpha \equiv a^\alpha \sum_{k=0}^{\infty} \binom{\alpha}{k}_q \left(\frac{b}{a}\right)^k, \ \left|\frac{b}{a}\right| < 1.$$

Remark 2.4. The associative law does not hold here.

Definition 2.5. For $\alpha \in \mathbb{C}$, JHC is extended to

$$(a \boxplus_q b)^\alpha \equiv a^\alpha \sum_{k=0}^{\infty} \binom{\alpha}{k}_q (\frac{b}{a})^k q^{\binom{k}{2}},\ |\frac{b}{a}| < 1.$$

We will give three examples of other scientists who have used these q-additions in other contexts.

In 1954 Sharma A. & Chak A. M. [80] constructed q-Appell sequences for the JHC. The JHC has also been used by Goulden & D.M. Jackson [40], who used the notation

$$Q_n(-y, x) \equiv P_{n,q}(x, y).$$

In 1994 [16] Chung K. S. & Chung W. S. & Nam S. T. & Kang H. J. rediscovered the NWA together with a new form of the q-derivative.

Definition 2.6. A generalization of [16]. Let $*$ be $\oplus_q$ or $\boxplus_q$. Then

$$D_* f(x) = \lim_{\delta x \to 0} \frac{f(x * \delta x) - f(x)}{\delta x}. \tag{2.3}$$

Theorem 2.7. A generalization of [24]. Let $*$ be $\oplus_q$ or $\boxplus_q$. Then the two operators D_q and D_* are identical when operating on functions which can be expressed as $x^\alpha \sum_{k=0}^{\infty} a_k x^k,\ \alpha \in \mathbb{C}$.

Unlike the NWA, the JHC is neither commutative nor associative, but on the other hand, it can be written as a finite product.

Jackson and Exton have presented several addition theorems for q-exponential and q-trigonometric functions. These are presented in a more lucid style using the JHC q-addition in [23].

Ward explained how the NWA can be used as the function argument in a formal power series. The theory of formal power series is outlined in Niven [64], see also Hofbauer [47].

The formal power series form a vector space with respect to term-wise addition and multiplication by complex scalars. In the rest of this chapter, as in [92, p. 258], unless otherwise stated, we assume that functions $f(x), g(x), F(x), G(x) \in \mathbb{C}[[x]]$.

Definition 2.8. If

$$F(x) = \sum_{k=0}^{\infty} a_k x^k,$$

then [92, p. 258]

$$F(x \oplus_q y) \equiv \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} \binom{n}{k}_q x^k y^{n-k}, \tag{2.4}$$

[51, p. 78]

$$F(x \boxplus_q y) \equiv F[x+y]_q \equiv \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} \binom{n}{k}_q q^{\binom{k}{2}} y^k x^{n-k}. \tag{2.5}$$

In 1936 Ward [92, p. 256] proved the following equations for q-subtraction (the original paper seems to contain a misprint of (2.6)):

$$(x \ominus_q y)^{2n+1} = \sum_{k=0}^{n} (-1)^k \binom{2n+1}{k}_q x^k y^k (x^{2n+1-2k} - y^{2n+1-2k}), \tag{2.6}$$

$$(x \ominus_q y)^{2n} = (-1)^n \binom{2n}{n}_q x^n y^n + \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k}_q x^k y^k (x^{2n-2k} + y^{2n-2k}). \tag{2.7}$$

We could also use norms for q-additions. To this aim we put

Definition 2.9. The norm for NWA is defined by

$$|a \oplus_q b|_n \equiv |(a \oplus_q b)^n|^{\frac{1}{n}}.$$

Assume that we would like to use the q-binomial theorem for

$$\frac{1}{((a \oplus_q b); q)_\alpha},$$

where $(a \oplus_q b)^n$ is defined by (1.11) and $a, b \in A$. To use the q-binomial theorem we should require $|a \oplus_q b|_n < 1$, for $n > 0$.

Before we embark on q-Taylor theorems, the following remark of Pearson [67] might be of interest. The differential calculus is a particular case of the direct method of finite differences, and the integral calculus is a particular case of the inverse method of finite differences. In fact Taylor published his formula in terms of finite differences.

There are at least three q-analogues of the Taylor formula for formal power series known from the literature, which we will list here. Compare [64, p. 877] for reference on formal power series.

Theorem 2.10. The Nalli–Ward q-Taylor formula [61, p. 345], [92, p. 259]:

$$F(x \oplus_q y) = \sum_{n=0}^{\infty} \frac{y^n}{\{n\}_q!} D_q^n F(x). \tag{2.8}$$

Theorem 2.11. The first Jackson q-Taylor formula [50, p. 63]:

$$F(x) = \sum_{n=0}^{\infty} \frac{(x \boxminus_q y)^n}{\{n\}_q!} D_q^n F(y). \tag{2.9}$$

Theorem 2.12. The second Jackson q-Taylor formula [51, (51, p. 77)]:

$$F(x \boxplus_q y) = \sum_{n=0}^{\infty} \frac{y^n}{\{n\}_q!} q^{\binom{n}{2}} D_q^n F(x). \tag{2.10}$$

Remark 2.13. Wallisser [91] has found a criterion for an entire function to be expanded in the q-Taylor series (2.9) for the special case $y = 1$ and $q < 1$. Put $M_{\mathrm{E}_{\frac{1}{q}}}(r) = \max_{|x|=r} |\mathrm{E}_{\frac{1}{q}}(x)|$.

If the maximum of the absolute value of an entire function F on $|x| = r$ satisfies the inequality

$$M_F(r) \leq C M_{\mathrm{E}_{\frac{1}{q}}}(r\tau),\ q \in \mathbb{R},\ q < 1,\ \tau < \left(\frac{1}{q} - 1\right)^{-1},$$

then $F(x)$ can be expanded in the q-Taylor series (2.9) for the special case $y = 1$.

Remark 2.14. Schendel [78] first proved (2.9), possibly influenced by Gauss.

The following general inversion formula will prove useful in the sequel.

Theorem 2.15. Gauss inversion [1, p. 96], a corrected version of [36, p. 244]. A q-analogue of [72, p. 4]. The following two equations for arbitrary sequences $a_n,\ b_n$ are equivalent:

$$a_n = q^{-f(n)} \sum_{l=0}^{n} (-1)^l q^{\binom{l}{2}} \binom{n}{l}_q b_{n-l}, \tag{2.11}$$

$$b_n = \sum_{i=0}^{n} q^{f(i)} \binom{n}{i}_q a_i. \tag{2.12}$$

Proof. It will suffice to prove that

$$a_n = q^{-f(n)} \sum_{l=0}^{n} (-1)^l \binom{n}{l}_q q^{\binom{l}{2}} \sum_{i=0}^{n-l} q^{f(i)} \binom{n-l}{i}_q a_i.$$

The first sum is zero except for $i = n$ and $l = 0$. ∎

Definition 2.16. The Ward–Alsalam q-shift operator [2, p. 242, 3.1], a q-analogue of [11, p. 16], [15, p. 18], [66, p. 3], is given by

$$E(\oplus_q)^{\omega}(x^n) \equiv (x \oplus_q \omega)^n.$$

We denote the corresponding operator for the JHC by $E(\boxplus_q)$, i.e.,

$$E(\boxplus_q)^{\omega}(x^n) \equiv (x \boxplus_q \omega)^n.$$

When $\omega = 1$, we denote these operators $E(\oplus_q)$ and $E(\boxplus_q)$.

To save space in the following, except for certain special cases, we only write formulas for NWA (Ward) according to the following two lists. The corresponding formulas for JHC (Jackson) follow from the next conversion table.

NWA	E_q	$\oplus_q$	$\overline{n}_q$	η	β	first	$S_{\mathrm{B,N},q}$	$J_{\mathrm{B,N},q}$
JHC	$\mathrm{E}_{\frac{1}{q}}$	$\boxplus_q$	$\tilde{n}_q$	θ	γ	second	$S_{\mathrm{B,J},q}$	$J_{\mathrm{B,J},q}$

(2.13)

The function $E_q(xt)$ in formulas is not changed because of the generating function.

The following equations for NWA have JHC equivalents according to (2.13). (2.14) to (2.16), (2.21) to (2.22), (2.25) to (2.26), (2.29) to (2.37), (3.7) to (3.8), (3.9) to (3.18), (3.19) to (3.26), (3.29) to (3.30), (3.33) to (3.42), (3.44) to (3.48), (3.51) to (3.61), (3.69) to (3.74), (3.77) to (3.80).

In formulas (3.62) to (3.68), (3.75) to (3.76), and (3.82), just replace NWA by JHC and keep $\oplus_q$. Observe that $\oplus_q$ is only changed to $\boxplus_q$ in an operator expression to the right preceding a $\tilde{n}_q$. In a pure expression for a q-Appell polynomial, $\oplus_q$ remains unchanged.

Definition 2.17. The invertible linear difference operator for the NWA, a q-analogue of [66, p. 3], is defined by

$$\underset{\omega}{\triangle_{\text{NWA},q}} \equiv \frac{E(\oplus_q)^\omega - I}{\omega},\ \omega \in \mathbb{C}, \tag{2.14}$$

where I is the identity operator. When $\omega = 1$, we denote this operator $\triangle_{\text{NWA},q}$ [2, p. 243, 3.5], compare [92, p. 264, 15.1].

Remark 2.18. In contrast to [60], $\lim_{\omega\to 0} \underset{\omega}{\triangle_{\text{NWA},q}}$ does not correspond to a q-difference operator.

Definition 2.19. If ω is a Ward number $\overline{n}_q$, the difference operator for the NWA is defined by

$$\underset{\overline{n}_q}{\triangle_{\text{NWA},q}} \equiv \frac{E(\oplus_q)^{\overline{n}_q} - I}{n}.$$

Remark 2.20. The formulas (2.21) and (2.32) show that the minus between $E(\oplus_q)$ and I is not a q-subtraction.

We are now going to present an operational equation, which was first found by Lagrange 1772 for $q = 1$. It played a major role in the theory of finite differences, for example in Lacroix's treatise on differences from 1800, [6], [11, p. 18], [45, p. 26], [46, p. 66]; and later in the first umbral calculus by Blissard [10]. The following dual q-analogues of [15, p. 28], see [2, p. 242, 3.3, p. 243, 3.9], [92, p. 264] hold:

$$E(\oplus_q)^\omega = \mathrm{E}_q(\omega D_q). \tag{2.15}$$

The difference operator $\triangle$ was used by Boole [11, p. 16], who showed that $\triangle$ is distributive, commutative with respect to any constant coefficients in the terms of the object to which it is applied, and obeys the index law for exponents. The same laws hold for $\triangle_{\text{NWA},q}$.

Definition 2.21. A q-analogue of the mean value operator of Jordan [54, p. 6] ($\omega = 1$), Nörlund [66, p. 3], and [60, p. 30]:

$$\underset{\omega}{\nabla_{\text{NWA},q}} \equiv \frac{E(\oplus_q)^\omega + I}{2}. \tag{2.16}$$

When $\omega = 1$, we denote this operator $\nabla_{\text{NWA},q}$.

The following definition is reminiscent of [66, p. 6, (12)].

Definition 2.22.

$$\triangle_{\text{NWA},2,q} \equiv 2\triangle_{\text{NWA},q}\nabla_{\text{NWA},q} \equiv E(\oplus_q)^{\overline{2}_q} - I, \tag{2.17}$$

$$\nabla_{\mathrm{NWA},2,q} \equiv \frac{E(\oplus_q)^{\overline{2}_q} + I}{2}.$$

This can be generalized to

Definition 2.23.

$$\begin{aligned} \underset{\omega}{\triangle_{\mathrm{NWA},2,q}} &\equiv \frac{E(\oplus_q)^{\overline{2}_q\omega} - I}{2\omega},\ \omega \in \mathbb{C}, \\ \underset{\omega}{\nabla_{\mathrm{NWA},2,q}} &\equiv \frac{E(\oplus_q)^{\overline{2}_q\omega} + I}{2},\ \omega \in \mathbb{C}. \end{aligned} \tag{2.18}$$

In the following definition, the q-additions are written first in additive, then in multiplicative form. In the first case, we assume that the function argument operate from left to right when using the two q-additions. In the second case, we assume that the function argument operate from right to left in accordance with (2.15). So do not forget that the following two equations are not associative.

Definition 2.24. If

$$f(x) = \sum_{k=0}^{\infty} a_k x^k,$$

then

$$f(x \oplus_q y \boxminus_q z) \equiv \sum_{k=0}^{\infty} a_k \sum_{l=0}^{k} \binom{k}{l}_q y^{k-l} \sum_{m=0}^{l} (-1)^m \binom{l}{m}_q q^{\binom{m}{2}} z^m x^{l-m}, \tag{2.19}$$

$$f(E(\boxplus_q)^{-z} E(\oplus_q)^y x)) \equiv \sum_{k=0}^{\infty} a_k \sum_{l=0}^{k} \binom{k}{l}_q y^{k-l} \sum_{m=0}^{l} (-1)^m \binom{l}{m}_q q^{\binom{m}{2}} z^m x^{l-m}. \tag{2.20}$$

We will now give a number of theorems for arbitrary letters which illustrate certain symmetry properties of this umbral calculus.

Theorem 2.25. The NWA and the JHC are dual operators:

$$f(x \oplus_q a \boxminus_q a) \equiv f(E(\boxplus_q)^{-a} E(\oplus_q)^a x) = f(x).$$

Proof. Use (2.15). ∎

By Goulden & D.M. Jackson [40] we obtain two further formulas of this type.

Theorem 2.26. [40, p. 228]

$$(\alpha \boxplus_q \beta) \oplus_q (\gamma \boxplus_q \delta) \sim (\alpha \boxplus_q \delta) \oplus_q (\gamma \boxplus_q \beta),\ \alpha, \beta, \gamma, \delta \in A.$$

Theorem 2.27. Goulden & D.M. Jackson [40, p. 228]:

$$(\alpha \boxminus_q \gamma) \sim (\alpha \boxminus_q \beta) \oplus_q (\beta \boxminus_q \gamma),\ \alpha, \beta, \gamma \in A.$$

The two Leibniz theorems go as follows. Notice the binomial coefficient on the right. Remember that we only write one equation to save space.

Theorem 2.28. A q-analogue of [15, p. 27, 2.13], [54, p. 97, 10], [60, p. 35, 2]. Let $f(x)$ and $g(x)$ be formal power series. Then

$$\triangle_{\text{NWA},q}^{n}(fg) = \sum_{i=0}^{n} \binom{n}{i} \triangle_{\text{NWA},q}^{i} f(\triangle_{\text{NWA},q}^{n-i} E(\oplus_q)^i) g. \tag{2.21}$$

Proof. Same as [54, p. 96 f]. ∎

The difference of a quotient of functions can be computed as

Theorem 2.29. A q-analogue of [67, p. 2], [11, p. 29]:

$$\triangle_{\text{NWA},q} \frac{f(x)}{g(x)} = \frac{g(x)\triangle_{\text{NWA},q} f(x) - f(x)\triangle_{\text{NWA},q} g(x)}{g(x)g(x \oplus_q 1)}. \tag{2.22}$$

The following theorem reminding of [92, p. 258] shows how Ward numbers usually appear in applications. Compare with [2, p. 244, 3.16], where the notation $P_k(n)$ was used.

Theorem 2.30.

$$(\overline{n}_q)^k = \sum_{m_1+\ldots+m_n=k} \binom{k}{m_1, \ldots, m_n}_q, \tag{2.23}$$

where each partition of k is multiplied with its number of permutations. We have the following special cases:

$$(\overline{0}_q)^k = \delta_{k,0};\ (\overline{n}_q)^0 = 1;\ (\overline{n}_q)^1 = n.$$

The following theorem shows how Jackson numbers usually appear in applications.

Theorem 2.31.

$$(\tilde{n}_q)^k = \sum_{m_1+\ldots+m_n=k} \binom{k}{m_1, \ldots, m_n}_q q^{\binom{\vec{m}}{2}},\ \vec{m} = (m_2, \ldots, m_n), \tag{2.24}$$

where for each partition of k, all permutations are counted. We have the following special cases:

$$(\tilde{0}_q)^k = \delta_{k,0};\ (\tilde{n}_q)^0 = 1;\ (\tilde{n}_q)^1 = n.$$

The following table lists some of the first $(\overline{n}_q)^k$. Compare [17, p. 309], where a long list of multinomial coefficients is given. The reader can check that the results agree with the definition (2.23) for the case $q = 1$.

	$k = 2$	$k = 3$	$k = 4$
$n = 1$	1	1	1
$n = 2$	$3 + q$	$4 + 2q + 2q^2$	$5 + 3q + 4q^2 + 3q^3 + q^4$
$n = 3$	$6 + 3q$	$10 + 8q + 8q^2 + q^3$	$3\left(5 + 5q + 7q^2 + 6q^3 + 3q^4 + q^5\right)$

and for $n = 4$

$k=2$	$k=3$	$k=4$
$10+6q$	$4\left(5+5q+5q^2+q^3\right)$	$5\left(7+9q+13q^2+12q^3+7q^4+3q^5\right)+q^6$

The following table lists some of the first $(\tilde{n}_q)^k$.

	$k=2$	$k=3$	$k=4$
$n=1$	1	1	1
$n=2$	$2+2q$	$2(1+q+q^2+q^3)$	$2+2q+2q^2+3q^3+2q^4+2q^5+q^6$
$n=3$	$4+5q$	$4+8q+8q^2+7q^3$	$\left(2+2q+2q^2+3q^3\right)^2$

and for $n = 4$

$k=2$	$k=3$	$k=4$
$7+9q$	$4\left(2+5q+5q^2+4q^3\right)$	$8+24q+41q^2+63q^3+56q^4+39q^5+25q^6$

According to Netto [62], the so-called multinomial expansion theorem was first mentioned in a letter 1695 from Leibniz to Johann Bernoulli, who proved it. In 1698 De Moivre first published a paper about multinomial coefficients in England [20, p. 114].

Two natural q-analogues are given by

Definition 2.32. If $f(x)$ is the formal power series $\sum_{l=0}^{\infty} a_l x^l$, it's k'th NWA-power is given by

$$(\oplus_{q,l=0}^{\infty} a_l x^l)^k \equiv (a_0 \oplus_q a_1 x \oplus_q \ldots)^k \equiv \sum_{|\vec{m}|=k} \prod_{l=0}^{\infty} (a_l x^l)^{m_l} \binom{k}{\vec{m}}_q .$$

Definition 2.33. If $f(x)$ is the formal power series $\sum_{l=0}^{\infty} a_l x^l$, it's k'th JHC-power is given by

$$(\boxplus_{q,l=0}^{\infty} a_l x^l)^k \equiv (a_0 \boxplus_q a_1 x \boxplus_q \ldots)^k \equiv \sum_{|\vec{m}|=k} \prod_{l=0}^{\infty} (a_l x^l)^{m_l} \binom{k}{\vec{m}}_q q^{\binom{\vec{n}}{2}},$$

where $\vec{n} = (m_2, \ldots, m_n)$.

Definition 2.34. If $f(x)$ is the formal power series $\sum_{k=0}^{\infty} a_k x^k$, the (Ward) q-sum is defined by

$$\sum_{k=n}^{m} f(\overline{k}_q) \equiv \sum_{k=n}^{m} \sum_{l=0}^{\infty} a_l (\overline{k}_q)^l,\ n, m \in \mathbb{N},\ n \leq m, \tag{2.25}$$

where for each k the function value for the corresponding Ward number is computed. If $n > m$, the sum $= 0$. Similarly, we define

$$(-1)^{\overline{n}_q} \equiv (-1)^n. \tag{2.26}$$

Definition 2.35. Let the q-extended real numbers $\mathbb{R}_q$ be the set generated by $\mathbb{R}$ together with the operation $\oplus_q$, where $0 < q < 1$.

Definition 2.36. The real q-integral is defined by

$$\int_0^a f(t,q)\, d_q(t) \equiv a(1-q) \sum_{n=0}^{\infty} f(aq^n, q) q^n,\ 0 < q < 1,\ a \in \mathbb{R}_q. \tag{2.27}$$

Definition 2.37. Let $a = a(q) \in \mathbb{R}_q$ and $\lim_{q \to 1} a(q) > 0$. Then we can define a q-analogue of a closed interval as usual as [0,a].

Definition 2.38. Let $a \in \mathbb{R}_q$ and let I be the product interval $[0, a] \times (0, 1)$. Then $\mathrm{L}_q^1(I)$ is the space of all functions $f(x, q) \in \mathbb{R}[[x]]$ on I such that

$$\int_0^a f(t,q)\, d_q(t) \text{ converges.} \tag{2.28}$$

Theorem 2.39. $\mathrm{L}_q^1(I)$ is a vector space with respect to term-wise addition and multiplication by complex scalars.

We will now find several q-analogues of formulas by Nörlund et al. for difference operators. Some of them have been published before.

Theorem 2.40. A q-analogue of the Newton–Gregory series [15, p. 21, 2.7], [54, p. 26], [60, 2.5.1], [56, p. 243]:

$$f(\overline{n}_q) = \sum_{k=0}^{n} \binom{n}{k} \triangle_{\mathrm{NWA},q}^k f(0). \tag{2.29}$$

This can be generalized to

Theorem 2.41. A q-analogue of [66, p. 4, (7)]:

$$f(\omega\overline{n}_q) = \sum_{k=0}^{n} \binom{n}{k} \omega^k \left(\frac{\triangle_{\mathrm{NWA},q}}{\omega} \right)^k f(0). \tag{2.30}$$

Theorem 2.42. A q-analogue of [66, p. 4, (8)]:

$$f(\omega\overline{n}_q) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} 2^k \left(\frac{\nabla_{\mathrm{NWA},q}}{\omega} \right)^k f(0). \tag{2.31}$$

The formula (2.29) can be inverted as follows.

Theorem 2.43. A corrected form of [92, p. 264, (iii)], and a q-analogue of [41, p. 188, (5.50)], [56, p. 136, (3)], [60, 2.5.2]:

$$\triangle_{\mathrm{NWA},q}^{n} f(x)=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k} f(x\oplus_q \overline{k}_q). \tag{2.32}$$

We will use the following abbreviations, $k_l \in \mathbb{N}$:

$$\Omega \equiv (k_1\omega_1 \oplus_q k_2\omega_2 \oplus_q \ldots \oplus_q k_n\omega_n),\; k\equiv\sum_{l=1}^{n} k_l,$$

$$\Phi \equiv (\overline{k_1 m_1}_q \oplus_q \overline{k_2 m_2}_q \oplus_q \ldots \oplus_q \overline{k_n m_n}_q).$$

The notation $\sum_{\vec{k}}$ denotes a multiple summation with each of the indices $k_1,\ldots,k_n$ running between $0,1$. The formula (2.32) can be generalized to

Definition 2.44. Two q-analogues of [66, p. 4]:

$$\underset{\omega_1,\ldots,\omega_n}{\triangle_{\mathrm{NWA},q}^{n}} f(x)\equiv(\omega_1\ldots\omega_n)^{-1}\sum_{\vec{k}}(-1)^{n-k} f(x\oplus_q\Omega), \tag{2.33}$$

$$\underset{\overline{m_1}_q,\ldots,\overline{m_n}_q}{\triangle_{\mathrm{NWA},q}^{n}} f(x)\equiv(m_1\ldots m_n)^{-1}\sum_{\vec{k}}(-1)^{n-k} f(x\oplus_q\Phi). \tag{2.34}$$

There is a similar formula for $\nabla_{\mathrm{NWA},q}$:

$$\nabla_{\mathrm{NWA},q}^{n} f(x)\equiv 2^{-n}\sum_{k=0}^{n}\binom{n}{k} f(x\oplus_q \overline{k}_q). \tag{2.35}$$

In a similar way, the formula (2.35) can be generalized to

Definition 2.45. A q-analogue of [66, p. 4]:

$$\underset{\omega_1,\ldots,\omega_n}{\nabla_{\mathrm{NWA},q}^{n}} f(x)\equiv 2^{-n}\sum_{\vec{k}} f(x\oplus_q\Omega). \tag{2.36}$$

Theorem 2.46. A q-analogue of [54, (12), p. 114]:

$$\nabla_{\mathrm{NWA},q}^{-1}=\sum_{m=0}^{\infty}\frac{(-1)^m}{2^m}\triangle_{\mathrm{NWA},q}^{m}. \tag{2.37}$$

3. q-Appell Polynomials

We will now describe the q-Appell polynomials, which already have been characterized by Al-Salam [3], who described its algebraic structure. In the spirit of Milne-Thomson [60, p. 125–147], which we will follow closely, we will call these q-polynomials Φ_q polynomials, and express them by a certain generating function. Some examples of q-Appell polynomials or Φ_q polynomials are $B^{(n)}_{\text{NWA},\nu,q}(x)$, $E^{(n)}_{\text{NWA},\nu,q}(x)$, $L^{(n)}_{\text{NWA},\nu,q}(x)$, and $G^{(n)}_{\text{NWA},\nu,q}(x)$. We will see that these polynomials have many similar properties. Now back to q-Appell polynomials.

Definition 3.1. A q-analogue of [60, p. 124]. For every power series $f_n(t)$, the Φ_q polynomials of degree ν and order n have the following generating function

$$f_n(t)\mathrm{E}_q(xt) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!}\Phi^{(n)}_{\nu,q}(x). \tag{3.1}$$

By putting $x = 0$, we have

$$f_n(t) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!}\Phi^{(n)}_{\nu,q},$$

where $\Phi^{(n)}_{\nu,q}$ is called a Φ_q number of degree ν and order n.

It will be convenient to fix the value for $n = 0$ and $n = 1$:

$$\Phi^{(0)}_{\nu,q}(x) \equiv x^\nu;\ \Phi^{(1)}_{\nu,q}(x) \equiv \Phi_{\nu,q}(x). \tag{3.2}$$

The special case $\Phi^{(n)}_{\nu,q}(x)$ independent of x in (3.1) is called Eulerian generating function in [34, p. 69], [58, p. 116].

By (3.1) we obtain

Theorem 3.2. A q-analogue of [5], [60, p. 125 (4), (5)]:

$$D_q\Phi^{(n)}_{\nu,q}(x) = \{\nu\}_q\Phi^{(n)}_{\nu-1,q}(x), \tag{3.3}$$

$$\int_a^x \Phi^{(n)}_{\nu,q}(t)\, d_q(t) = \frac{\Phi^{(n)}_{\nu+1,q}(x) - \Phi^{(n)}_{\nu+1,q}(a)}{\{\nu+1\}_q}.$$

By (2.8), (2.10) we obtain the two q-Taylor formulas

Theorem 3.3.

$$\Phi^{(n)}_{\nu,q}(x \oplus_q y) = \sum_{k=0}^{\nu} \binom{\nu}{k}_q \Phi^{(n)}_{\nu-k,q}(x)y^k, \tag{3.4}$$

$$\Phi^{(n)}_{\nu,q}(x \boxplus_q y) = \sum_{k=0}^{\nu} \binom{\nu}{k}_q q^{\binom{k}{2}}\Phi^{(n)}_{\nu-k,q}(x)y^k. \tag{3.5}$$

Note the slight difference to polynomials of q-binomial type in (3.4).

The first formula (or [3, p. 33 2.5]) gives the symbolic equality

Theorem 3.4. A q-analogue of [60, p. 125 (3)]:

$$\Phi_{\nu,q}^{(n)}(x) \ddot{=} (\Phi_q^{(n)} \oplus_q x)^\nu. \tag{3.6}$$

Theorem 3.5. A q-analogue of [60, p. 125]:

$$(\mathrm{E}_q(t)-1)f_n(t)\mathrm{E}_q(xt) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} \triangle_{\mathrm{NWA},q} \Phi_{\nu,q}^{(n)}(x). \tag{3.7}$$

Proof. Operate on (3.1) with $\triangle_{\mathrm{NWA},q}$. ∎

Theorem 3.6. A q-analogue of [60, p. 125]:

$$\frac{(\mathrm{E}_q(t)+1)}{2} f_n(t)\mathrm{E}_q(xt) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} \nabla_{\mathrm{NWA},q} \Phi_{\nu,q}^{(n)}(x). \tag{3.8}$$

Proof. Operate on (3.1) with $\nabla_{\mathrm{NWA},q}$. ∎

The simplest example of a Φ_q polynomial is the Rogers–Szegö polynomials [73], [88], [13, p. 90 (11)]

$$H_{n,q}(x,a) \equiv (x \oplus_q a)^n.$$

A special case of the Φ_q polynomials are the β_q polynomials of degree ν and order n, which are obtained by putting $f_n(t) = \dfrac{t^n g(t)}{(\mathrm{E}_q(t)-1)^n}$ in (3.1).

Definition 3.7.

$$\frac{t^n g(t)}{(\mathrm{E}_q(t)-1)^n} \mathrm{E}_q(xt) \equiv \sum_{\nu=0}^{\infty} \frac{t^\nu \beta_{\nu,q}^{(n)}(x)}{\{\nu\}_q!}. \tag{3.9}$$

Theorem 3.8. [2, p. 255, 10.8], a q-analogue of [60, (2), p. 126], [57, p. 21], [76, p. 704], [56, p. 240]:

$$\triangle_{\mathrm{NWA},q} \beta_{\nu,q}^{(n)}(x) = \{\nu\}_q \beta_{\nu-1,q}^{(n-1)}(x) = D_q \beta_{\nu,q}^{(n-1)}(x). \tag{3.10}$$

Proof. Use (3.7). ∎

By (3.6) the following symbolic relations are obtained.

Theorem 3.9. A q-analogue of [60, p. 126]. The second equation implies (1.15):

$$(\beta_q^{(n)} \oplus_q x \oplus_q 1)^\nu - (\beta_q^{(n)} \oplus_q x)^\nu \ddot{=} \{\nu\}_q (\beta_q^{(n-1)} \oplus_q x)^{\nu-1}, \tag{3.11}$$

$$(\beta_q^{(n)} \oplus_q 1)^\nu - \beta_{\nu,q}^{(n)} \ddot{=} \{\nu\}_q \beta_{\nu-1,q}^{(n-1)}.$$

Theorem 3.10. A q-analogue of [65, (20), p. 163]:

$$\triangle_{\mathrm{NWA},q} f(\beta_{\nu,q}^{(n)}(x)) \equiv f(\beta_{\nu,q}^{(n)}(x) \oplus_q 1) - f(\beta_{\nu,q}^{(n)}(x)) \ddot{=} D_q f(\beta_{\nu,q}^{(n-1)}(x)). \tag{3.12}$$

Theorem 3.11. Almost a q-analogue of [85, p. 378, (26)]:

$$\sum_{k=1}^{\nu} \binom{\nu}{k}_q \beta_{\nu-k,q}^{(n)}(x) = \{\nu\}_q \beta_{\nu-1,q}^{(n-1)}(x). \tag{3.13}$$

Proof. Use (3.4) and (3.11). ■

A particular case of β_q polynomials are the generalized q-Bernoulli polynomials $B_{\mathrm{NWA},\nu,q}^{(n)}(x)$ of degree ν and order n, which were defined for $q = 1$ in [60, p. 127], [65] and for complex order in [2, p. 254, 10.3].

Definition 3.12. [2, p. 254, 10.3], [66, (36) p. 132], [85]. The generating function for $B_{\mathrm{NWA},\nu,q}^{(n)}(x)$ is a q-analogue of [76, p. 704], [68, p. 1225, ii]:

$$\frac{t^n}{(\mathrm{E}_q(t) - 1)^n} \mathrm{E}_q(xt) = \sum_{\nu=0}^{\infty} \frac{t^\nu B_{NWA,\nu,q}^{(n)}(x)}{\{\nu\}_q!}, \ |t| < 2\pi. \tag{3.14}$$

This can be generalized to

Definition 3.13. The generating function for $B_{\mathrm{NWA},\nu,q}^{(n)}(x|\omega_1, \ldots, \omega_n)$ is the following q-analogue of [66, (77) p. 143]:

$$\begin{gathered} \frac{t^n \omega_1 \ldots \omega_n}{\prod_{k=1}^n (\mathrm{E}_q(\omega_k t) - 1)} \mathrm{E}_q(xt) = \sum_{\nu=0}^{\infty} \frac{t^\nu B_{NWA,\nu,q}^{(n)}(x|\omega_1, \ldots, \omega_n)}{\{\nu\}_q!}, \\ |t| < \min\left(\left| \frac{2\pi}{\omega_1} \right|, \ldots, \left| \frac{2\pi}{\omega_n} \right| \right). \end{gathered} \tag{3.15}$$

Remark 3.14. The values for t above are for $q = 1$. For general t and q, the convergence area can be different. The above definitions are mostly formal.

Definition 3.15. For the q-Lucas polynomials $L_{\mathrm{NWA},\nu,q}^{(n)}(x|\omega_1, \ldots, \omega_n)$, the generating function is

$$\begin{gathered} \frac{(2t)^n \omega_1 \ldots \omega_n}{\prod_{k=1}^n (\mathrm{E}_q(\omega_k t \overline{2}_q) - 1)} \mathrm{E}_q(xt) = \sum_{\nu=0}^{\infty} \frac{t^\nu L_{NWA,\nu,q}^{(n)}(x|\omega_1, \ldots, \omega_n)}{\{\nu\}_q!}, \\ |t| < \min\left(\left| \frac{\pi}{\omega_1} \right|, \ldots, \left| \frac{\pi}{\omega_n} \right| \right). \end{gathered} \tag{3.16}$$

Theorem 3.16. Obviously, $B^{(n)}_{\text{NWA},\nu,q}(x|\omega_1,\ldots,\omega_n)$ is symmetric in $\omega_1,\ldots,\omega_n$, and in particular

$$B^{(1)}_{\text{NWA},\nu,q}(x|\omega) = \omega^\nu B_{\text{NWA},\nu,q}\left(\frac{x}{\omega}\right). \tag{3.17}$$

And the same for $L^{(n)}_{\text{NWA},\nu,q}(x|\omega_1,\ldots,\omega_n)$:

$$\mathop{\triangle^n_{\text{NWA},q}}_{\omega_1,\ldots,\omega_n} B^{(n)}_{\text{NWA},\nu,q}(x|\omega_1,\ldots,\omega_n) = \{\nu\}_q x^{\nu-1},$$

$$\mathop{\triangle^n_{\text{NWA},2,q}}_{\omega_1,\ldots,\omega_n} L^{(n)}_{\text{NWA},\nu,q}(x|\omega_1,\ldots,\omega_n) = \{\nu\}_q x^{\nu-1}.$$

Theorem 3.17. The successive differences of q-Bernoulli polynomials can be expressed as q-Bernoulli polynomials. A q-analogue of [66, (46) p. 131]:

$$\mathop{\triangle^p_{\text{NWA},q}}_{\omega_1,\ldots,\omega_p} B^{(n)}_{\text{NWA},\nu,q}(x|\omega_1,\ldots,\omega_n) = \frac{\{\nu\}_q!}{\{\nu-p\}_q!} B^{(n-p)}_{\text{NWA},\nu-p,q}(x|\omega_{p+1},\ldots,\omega_n).$$

We will use the following notation:

$$B^{(n)}_{\text{NWA},\nu,q}(x|\omega_1,\ldots,\omega_n) = (B^{(n)}_{\text{NWA},\nu,\omega_1,\ldots,\omega_n,q} \oplus_q x)^\nu.$$

The following special case is often used.

Definition 3.18. The Ward q-Bernoulli numbers [92, p. 265, 16.4], [2, p. 244, 4.1] are given by

$$B_{\text{NWA},n,q} \equiv B^{(1)}_{\text{NWA},n,q}. \tag{3.18}$$

The following table lists some of the first Ward q-Bernoulli numbers:

$n=0$	$n=1$	$n=2$	$n=3$
1	$-(1+q)^{-1}$	$q^2(\{3\}_q!)^{-1}$	$(1-q)q^3(\{2\}_q)^{-1}(\{4\}_q)^{-1}$

$n=4$
$q^4(1-q^2-2q^3-q^4+q^6)(\{2\}^2_q\{3\}_q\{5\}_q)^{-1}$

Theorem 3.19. We have the following operational representations, a q-analogue of [87]:

$$B^{(n)}_{\text{NWA},\nu,q}(\omega_1,\ldots,\omega_n) \doteq (\oplus^n_{q,l=1}\omega_l B_{\text{NWA},l,q})^\nu.$$

And the same for L-polynomial:

$$B^{(n)}_{\text{JHC},\nu,q}(\omega_1,\ldots,\omega_n) \doteq (\oplus^n_{q,l=1}\omega_l B_{\text{JHC},l,q})^\nu.$$

Corollary 3.20. A q-analogue of [55, p. 639]:

$$\text{E}_q(tB_{\text{NWA},q}) \doteq \frac{t}{\text{E}_q(t)-1},$$

$$\mathrm{E}_q(tB_{\text{JHC},q}) \ddot{=} \frac{t}{\mathrm{E}_{\frac{1}{q}}(t) - 1},$$

$$\mathrm{E}_q(tL_{\text{NWA},q}) \ddot{=} \frac{2t}{\mathrm{E}_q(t\overline{2}_q) - 1}.$$

The following operator will be useful in connection with $B^{(n)}_{\text{NWA},\nu,q}(x)$.

Definition 3.21. Compare [15, p. 32] $(n = 1)$. The invertible operator $S^n_{\text{B,N},q} \in \mathbb{C}(D_q)$ is given by

$$S^n_{\text{B,N},q} \equiv \frac{(\mathrm{E}_q(D_q) - I)^n}{D^n_q}. \tag{3.19}$$

This implies

Theorem 3.22.

$$\triangle^n_{\text{NWA},q} = D^n_q S^n_{\text{B,N},q}.$$

Theorem 3.23. A q-analogue of [68, p. 1225, i]. The q-Bernoulli polynomials of degree ν and order n can be expressed as

$$B^{(n)}_{\text{NWA},\nu,q}(t) = S^{-n}_{\text{B,N},q} t^\nu. \tag{3.20}$$

Proof.

$$LHS = \sum_{k=0}^{\nu} \binom{\nu}{k}_q B^{(n)}_{\text{NWA},k,q} t^{\nu-k} = \sum_{k=0}^{\infty} \frac{B^{(n)}_{\text{NWA},k,q}}{\{k\}_q!} D^k_q t^\nu \overset{\text{by}(3.14))}{=} RHS.$$

∎

Theorem 3.24. A q-analogue of a generalization of [15, p. 43, 3.3]:

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} B^{(n)}_{\text{NWA},\nu,q}(x \oplus_q \overline{k}_q) = \{\nu - n + 1\}_{n,q} x^{\nu-n}. \tag{3.21}$$

Proof.

$$\begin{aligned}\triangle^n_{\text{NWA},q} B^{(n)}_{\text{NWA},\nu,q}(x) =& D^n_q S^n_{\text{B,N},q} B^{(n)}_{\text{NWA},\nu,q}(x) = D^n_q S^n_{\text{B,N},q} S^{-n}_{\text{B,N},q} x^\nu \\ =& D^n_q x^\nu = \{\nu - n + 1\}_{n,q} x^{\nu-n}.\end{aligned} \tag{3.22}$$

∎

Theorem 3.25. [2, p. 253, 9.5], a q-analogue of [56, (1), p. 240], [76, p. 699]:

$$f(x \oplus_q B_{\text{NWA},q} \oplus_q 1) - f(x \oplus_q B_{\text{NWA},q}) \ddot{=} D_q f(x),$$

where here and in the sequel, we have abbreviated the umbral symbol by $B_{\text{NWA},q}$.

We will also state the corresponding equation for $B^{(n)}_{\mathrm{NWA},\nu,q}$ written in two different forms.

Theorem 3.26. A q-analogue of [65, (11) p. 124], [66, (36) p. 132]:

$$f(x \oplus_q B^{(n)}_{\mathrm{NWA},q} \oplus_q 1) - f(x \oplus_q B^{(n)}_{\mathrm{NWA},q}) \ddot{=} D_q f(x \oplus_q B^{(n-1)}_{\mathrm{NWA},q}), \tag{3.23}$$

$$f(B^{(n)}_{\mathrm{NWA},q}(x) \oplus_q 1) - f(B^{(n)}_{\mathrm{NWA},q}(x)) \ddot{=} D_q f(B^{(n-1)}_{\mathrm{NWA},q}(x)). \tag{3.24}$$

Theorem 3.27. Compare [15, 3.15 p. 51], where the corresponding formula for Euler polynomials was given:

$$B_{\mathrm{NWA},\nu,q}(x) \equiv \frac{\{\nu\}_q}{\mathrm{E}_q(D_q) - I} x^{\nu-1} = \frac{\{\nu\}_q}{E(\oplus_q) - I} x^{\nu-1} \ddot{=} (x \oplus_q B_{\mathrm{NWA},q})^{\nu}.$$

We now follow Cigler [15] and give some q-analogues of equations for Bernoulli polynomials. The first two of these equations are well known in the literature ($q = 1$).

Definition 3.28. A q-analogue of [46, p. 87], [15, p. 13], [90, p. 575]:

$$s_{\mathrm{NWA},m,q}(n) \equiv \sum_{k=0}^{n-1} (\overline{k}_q)^m, \ s_{\mathrm{NWA},0,q}(1) \equiv 1.$$

Theorem 3.29. [2, p. 248, 5.13], [92, p. 265, 16.5], a q-analogue of [15, p. 13, p. 17: 1.11, p. 36], [56, p. 237]:

$$\begin{aligned} s_{\mathrm{NWA},m,q}(n) =& \frac{B_{\mathrm{NWA},m+1,q}(\overline{(n)}_q) - B_{\mathrm{NWA},m+1,q}}{\{m+1\}_q} \\ \equiv& \frac{1}{\{m+1\}_q} \sum_{k=1}^{m+1} \binom{m+1}{k}_q (\overline{n}_q)^k B_{\mathrm{NWA},m+1-k,q} \\ \equiv& \frac{1}{\{m+1\}_q} \sum_{k=0}^{m} \binom{m+1}{k}_q (\overline{n}_q)^{m+1-k} B_{\mathrm{NWA},k,q}. \end{aligned} \tag{3.25}$$

Theorem 3.30. A q-analogue of [15, p. 45], [65, p. 127, (17)].

$$x^n = \int_x^{x \oplus_q 1} B_{\mathrm{NWA},n,q}(t) \, d_q(t) = \frac{B_{\mathrm{NWA},n+1,q}(x \oplus_q 1) - B_{\mathrm{NWA},n+1,q}(x)}{\{n+1\}_q}. \tag{3.26}$$

Proof. q-Integrate (3.3) for $n = 1$ and use (3.10). ∎

This can be rewritten as a q-analogue of the well-known identity [39, p. 496, 8.2].

$$x^n = \frac{1}{\{n+1\}_q} \sum_{k=0}^{n} \binom{n+1}{k}_q B_{\mathrm{NWA},k,q}(x). \tag{3.27}$$

The JHC-version is

$$x^n = \frac{1}{\{n+1\}_q} \sum_{k=0}^{n} \binom{n+1}{k}_q B_{\text{JHC},k,q}(x) q^{\binom{n+1-k}{2}}. \tag{3.28}$$

Cigler has given some examples of translation invariant operators. One of them is the Bernoulli operator.

Definition 3.31. The first q-Bernoulli operator is given by the following q-integral, a q-analogue of [15, p. 91], [18, p. 154], [75, p. 59], [76, p. 701, 703], [68, p. 1217]:

$$J_{\text{B,N},q} f(x) \equiv \int_{x}^{x \oplus_q 1} f(t)\, d_q(t). \tag{3.29}$$

Theorem 3.32. A q-analogue of [15, p. 44–45], [68, p. 1217]. The first q-Bernoulli operator can be expressed in the form

$$J_{\text{B,N},q} f(x) = \frac{\triangle_{\text{NWA},q}}{D_q} f(x).$$

Proof. Use (3.20) and (3.26). ∎

Theorem 3.33. A q-analogue of [15, p. 44–45]. We can expand a given formal power series in terms of the $B_{\text{NWA},k,q}(x)$ as follows:

$$f(x) = \sum_{k=0}^{\infty} \int_{\bar{0}_q}^{\bar{1}_q} D_q^k f(t)\, d_q(t) \frac{B_{\text{NWA},k,q}(x)}{\{k\}_q!}. \tag{3.30}$$

Proof. Assume that

$$f(x) = \sum_{k=0}^{\infty} \frac{a_k}{\{k\}_q!} B_{\text{NWA},k,q}(x).$$

As we have

$$\begin{aligned} x^k &= S_{\text{B,N},q} B_{\text{NWA},k,q}(x), \\ f(x) &= \sum_{k=0}^{\infty} \frac{a_k}{\{k\}_q!} S_{\text{B,N},q}^{-1} x^k, \qquad (3.31) \\ S_{\text{B,N},q} f(x) &= \sum_{k=0}^{\infty} \frac{a_k}{\{k\}_q!} x^k. \end{aligned}$$

This implies

$$a_k = D_q^k S_{\text{B,N},q} f(x)|_{x=0} = D_q^k \frac{\triangle_{\text{NWA},q}}{D_q} f(x)|_{x=0} = \int_{\bar{0}_q}^{\bar{1}_q} D_q^k f(t)\, d_q(t). \tag{3.32}$$

■

The following table lists some of the smallest Jackson q-Bernoulli numbers:

$n=0$	$n=1$	$n=2$	$n=3$
1	$-q(1+q)^{-1}$	$q^2(\{3\}_q!)^{-1}$	$(q^4-q^3)(\{2\}_q)^{-1}(\{4\}_q)^{-1}$

$n=4$
$q^4(1-q^2-2q^3-q^4+q^6)(\{2\}_q^2\{3\}_q\{5\}_q)^{-1}$

A special case of the Φ_q polynomials are the η_q polynomials of order n, which are obtained by putting $f_n(t)=\dfrac{g(t)2^n}{(\mathrm{E}_q(t)+1)^n}$ in (3.1).

Definition 3.34. A q-analogue of [60, p. 142, (1)]:

$$\frac{2^n}{(\mathrm{E}_q(t)+1)^n}g(t)\mathrm{E}_q(xt)=\sum_{\nu=0}^{\infty}\frac{t^\nu\eta_{\nu,q}^{(n)}(x)}{\{\nu\}_q!}. \tag{3.33}$$

By (3.8) we get a q-analogue of [60], [59, p. 519]:

$$\nabla_{\mathrm{NWA},q}\eta_{\nu,q}^{(n)}(x)=\eta_{\nu,q}^{(n-1)}(x). \tag{3.34}$$

We will now define first q-Euler polynomials, a special case of the η_q polynomials. There are many similar definitions of these, but we will follow [60, p. 143–147], [15, p. 51], because it is equivalent to the q-Appell polynomials from [3].

Definition 3.35. The generating function for the first q-Euler polynomials of degree ν and order n $E_{\mathrm{NWA},\nu,q}^{(n)}(x)$ is the following q-analogue of [74, p. 102], [60, p. 309], [89, p. 345]:

$$\frac{2^n\mathrm{E}_q(xt)}{(\mathrm{E}_q(t)+1)^n}=\sum_{\nu=0}^{\infty}\frac{t^\nu}{\{\nu\}_q!}E_{\mathrm{NWA},\nu,q}^{(n)}(x),\ |t|<\pi. \tag{3.35}$$

This can be generalized to

Definition 3.36. The generating function for the first q-Euler polynomials of degree ν and order n $E_{\mathrm{NWA},\nu,q}^{(n)}(x|\omega_1,\ldots,\omega_n)$ is the following q-analogue of [66, p. 143 (78)]:

$$\begin{gathered}\frac{2^n\mathrm{E}_q(xt)}{\prod_{k=1}^n(\mathrm{E}_q(\omega_k t)+1)}=\sum_{\nu=0}^{\infty}\frac{t^\nu}{\{\nu\}_q!}E_{\mathrm{NWA},\nu,q}^{(n)}(x|\omega_1,\ldots,\omega_n),\\ |t|<\min\left(\left|\frac{\pi}{\omega_1}\right|,\ldots,\left|\frac{\pi}{\omega_n}\right|\right).\end{gathered} \tag{3.36}$$

The following polynomials are influenced by Nörlund, who would maybe have denoted them by C instead of G.

Definition 3.37. The generating function for the q-G polynomials of degree ν and order n $G^{(n)}_{\text{NWA},\nu,q}(x|\omega_1,\ldots,\omega_n)$ is the following formula reminding of [66, p. 143 (78)]:

$$\begin{aligned}&\frac{2^n\mathrm{E}_q(xt)}{\prod_{k=1}^n(\mathrm{E}_q(\omega_k\overline{2}_q)+1)}=\sum_{\nu=0}^{\infty}\frac{t^\nu}{\{\nu\}_q!}G^{(n)}_{\text{NWA},\nu,q}(x|\omega_1,\ldots,\omega_n),\\&|t|<\min\left(\left|\frac{\pi}{2\omega_1}\right|,\ldots,\left|\frac{\pi}{2\omega_n}\right|\right).\end{aligned}\tag{3.37}$$

Obviously, $E^{(n)}_{\text{NWA},\nu,q}(x|\omega_1,\ldots,\omega_n)$ is symmetric in $\omega_1,\ldots,\omega_n$, and the same for $G^{(n)}_{\text{NWA},\nu,q}(x|\omega_1,\ldots,\omega_n)$. In particular

$$E^{(1)}_{\text{NWA},\nu,q}(x|\omega)=\omega^\nu E_{\text{NWA},\nu,q}\left(\frac{x}{\omega}\right).\tag{3.38}$$

From

$$\underset{\omega_1,\ldots,\omega_n}{\nabla^n_{\text{NWA},q}}\;E^{(n)}_{\text{NWA},\nu,q}(x|\omega_1,\ldots,\omega_n)=x^\nu,$$

we obtain

$$\underset{\omega_1,\ldots,\omega_p}{\nabla^p_{\text{NWA},q}}\;E^{(n)}_{\text{NWA},\nu,q}(x|\omega_1,\ldots,\omega_n)=E^{(n-p)}_{\text{NWA},\nu,q}(x|\omega_{p+1},\ldots,\omega_n).$$

From

$$\underset{\omega_1,\ldots,\omega_n}{\nabla^n_{\text{NWA},2,q}}\;G^{(n)}_{\text{NWA},\nu,q}(x|\omega_1,\ldots,\omega_n)=x^\nu,$$

we obtain

$$\underset{\omega_1,\ldots,\omega_p}{\nabla^p_{\text{NWA},2,q}}\;G^{(n)}_{\text{NWA},\nu,q}(x|\omega_1,\ldots,\omega_n)=G^{(n-p)}_{\text{NWA},\nu,q}(x|\omega_{p+1},\ldots,\omega_n).$$

Theorem 3.38. A q-analogue of [60, p. 144, (7)], [66, (7), p. 121], [85, p. 378, (28)]. The corresponding formula for η polynomials is [60, p. 143, (3)]:

$$\sum_{k=0}^{\nu}\binom{\nu}{k}_q E^{(n)}_{\text{NWA},\nu-k,q}(x)+E^{(n)}_{\text{NWA},\nu,q}(x)=2E^{(n-1)}_{\text{NWA},\nu,q}(x).\tag{3.39}$$

With this formula we can compute all first q-Euler polynomials of order n, given knowledge of the polynomials of order $n-1$.

Definition 3.39. A q-analogue of [65, p. 139], [56, p. 252]. The first generalized q-Euler numbers are given by

$$F^{(n)}_{\text{NWA},\nu,q}\equiv E^{(n)}_{\text{NWA},\nu,q}(0).$$

Furthermore we put

$$F_{\text{NWA},k,q}\equiv F^{(1)}_{\text{NWA},k,q};\;E_{\text{NWA},\nu,q}(x)\equiv E^{(1)}_{\text{NWA},\nu,q}(x).$$

Remark 3.40. The numbers $F_{\text{NWA},k,q}$ are q-analogues of the numbers in [59, p. 520], [15, p. 51], which are multiples of the tangent numbers [77, p. 296]. Lucas [56, p. 250] called them G after Genocchi, but the author disagrees with this.

The q-analogues of the original integral Euler numbers (secant numbers), see Salié [77], appear in [4].

Remark 3.41. When $q = 1$, our $E_\nu(x) = \nu! E_{J,\nu}(x)$, where $E_{J,\nu}(x)$ is the Euler polynomial used in [54].

Theorem 3.42. The operator expression is a q-analogue of [15, 3.15 p. 51]:

$$E_{\text{NWA},\nu,q}(x) \equiv \frac{2}{\mathrm{E}_q(D_q)+I}x^\nu = \frac{2}{E(\oplus_q)+I}x^\nu \ddot{=} (x \oplus_q F_{\text{NWA},q})^\nu.$$

The following two recursion formulas are quite useful for the computations of the first q-Euler polynomial.

Theorem 3.43. A q-analogue of [66, (27), p. 24], [85, p. 378, (29)]:

$$E_{\text{NWA},\nu,q}(x) + \sum_{k=0}^{\nu} \binom{\nu}{k}_q E_{\text{NWA},k,q}(x) = 2x^\nu. \tag{3.40}$$

Theorem 3.44. A q-analogue of [15, 3.16 p. 51], [56, p. 252]:

$$(1 \oplus_q F_{\text{NWA},q})^n + (F_{\text{NWA},q})^n \ddot{=} 2\delta_{0,n}.$$

Theorem 3.45. A q-analogue of [12, p. 6 (4.3)], [65, (19), p. 136], a corrected version of [56, p. 261]:

$$f(x \oplus_q F_{\text{NWA},q} \oplus_q 1) + f(x \oplus_q F_{\text{NWA},q}) \ddot{=} 2f(x). \tag{3.41}$$

We will also state the corresponding equation for $E^{(n)}_{\text{NWA},\nu,q}$ written in two different forms.

Theorem 3.46. A q-analogue of [65, (19), p. 150, p. 155], [66, (29) p. 126]:

$$\nabla_{\text{NWA},q} f(x \oplus_q F^{(n)}_{\text{NWA},q}) \ddot{=} f(x \oplus_q F^{(n-1)}_{\text{NWA},q}) \ddot{=} \nabla_{\text{NWA},q} f(E^{(n)}_{\text{NWA},q}(x)) \ddot{=} f(E^{(n-1)}_{\text{NWA},q}(x)). \tag{3.42}$$

The following table lists some of the first q-Euler numbers $F_{\text{NWA},n,q}$:

$n=0$	$n=1$	$n=2$	$n=3$
1	-2^{-1}	$2^{-2}(-1+q)$	$2^{-3}(-1+2q+2q^2-q^3)$

$n=4$
$2^{-4}(q-1)\{3\}_q!(q^2-4q+1)$

The following table lists some of the first $F_{\text{JHC},n,q}$:

$n=0$	$n=1$	$n=2$	$n=3$
1	-2^{-1}	$2^{-2}(1-q)$	$2^{-3}(-1+2q+2q^2-q^3)$

$n=4$
$2^{-4}(1-3q-3q^2+3q^4+3q^5-q^6)$

Theorem 3.47. We have the following operational representations, a q-analogue of [87]:

$$E^{(n)}_{\text{NWA},\nu,q}(\omega_1,\ldots,\omega_n)\doteq(\oplus_{q,l=1}^{n}\omega_l E_{\text{NWA},l,q})^{\nu},$$

$$E^{(n)}_{\text{JHC},\nu,q}(\omega_1,\ldots,\omega_n)\doteq(\oplus_{q,l=1}^{n}\omega_l E_{\text{JHC},l,q})^{\nu}.$$

Corollary 3.48.

$$\mathrm{E}_q(tE_{\text{NWA},q})\doteq\frac{2}{\mathrm{E}_q(t)+1}, \tag{3.43}$$

$$\mathrm{E}_q(tE_{\text{JHC},q})\doteq\frac{2}{\mathrm{E}_{\frac{1}{q}}(t)+1},$$

$$\mathrm{E}_q(tG_{\text{NWA},q})\doteq\frac{2}{\mathrm{E}_q(t\overline{2}_q)+1}.$$

Theorem 3.49. The first q-Euler polynomial can be expressed as a finite sum of differentiable operators on x^n. Almost a q-analogue of [54, p. 289]:

$$E_{\text{NWA},n,q}(x)=\sum_{m=0}^{n}\frac{(-1)^m}{2^m}\triangle^m_{\text{NWA},q}x^n. \tag{3.44}$$

Theorem 3.50. A generalization of (3.40):

$$2^{-n}\sum_{k=0}^{n}\binom{n}{k}E^{(n)}_{\text{NWA},\nu,q}(x\oplus_q\overline{k}_q)=x^{\nu}. \tag{3.45}$$

Proof. Develop $\nabla^n_{\text{NWA},q}E^{(n)}_{\text{NWA},\nu,q}(x)$. ∎

Definition 3.51. A q-analogue of [46, p. 88]. The notation from N. Nielsen (1865–1925) [63, p. 401] is a slightly modified variant of the original paper by Lucas [56]:

$$\sigma_{\text{NWA},m,q}(n)\equiv\sum_{k=0}^{n-1}(-1)^k(\overline{k}_q)^m.$$

Theorem 3.52. A q-analogue of [15, p. 53], [60, p. 307], [37, p. 136]:

$$\sigma_{\text{NWA},m,q}(n)=\frac{(-1)^{n-1}E_{\text{NWA},m,q}(\overline{n}_q)+E_{\text{NWA},m,q}(\overline{0}_q)}{2}. \tag{3.46}$$

Proof.

$$\begin{aligned}LHS =&\sum_{k=0}^{n-1}(-1)^k\nabla_{\text{NWA},q}E_{\text{NWA},m,q}(\overline{k}_q)\\ =&\frac{1}{2}\sum_{k=0}^{n-1}(-1)^k(E_{\text{NWA},m,q}(\overline{k}_q\oplus_q 1)+E_{\text{NWA},m,q}(\overline{k}_q)) = RHS.\end{aligned}$$

■

So far we considered only q-Bernoulli polynomials and q-Euler polynomials of positive order n. As the sequel shows, it will be useful to allow n also to be a negative integer. The following calculations are q-analogues of Nörlund [66, p. 133 ff].

Definition 3.53. As a q-analogue of [66, (50) p. 133], [68, p. 1226, xvi] and [85, p. 378, (19)], we define first q-Bernoulli polynomials of two variables as

$$\begin{aligned}&B_{\text{NWA},\nu,q}^{(n+p)}(x\oplus_q y|\omega_1,\ldots,\omega_{n+p})\\ &= (B_{\text{NWA},q}^{(n)}(x|\omega_1,\ldots,\omega_n)\oplus_q B_{\text{NWA},q}^{(p)}(y|\omega_{n+1},\ldots,\omega_{n+p}))^{\nu},\end{aligned} \tag{3.47}$$

where we assume that n and p operate on x and y respectively, and the same for any q-polynomial.

The relation (3.17) together with (3.47) show, that $B_{\text{NWA},\nu,q}^{(n)}(x|\omega_1,\ldots,\omega_n)$ is a homogeneous function of $x,\omega_1,\ldots,\omega_n$ of degree ν, a q-analogue of [66, p. 134 (55)], i.e.,

$$B_{\text{NWA},\nu,q}^{(n)}(\lambda x|\lambda\omega_1,\ldots,\lambda\omega_n) = \lambda^{\nu}B_{\text{NWA},\nu,q}^{(n)}(x|\omega_1,\ldots,\omega_n),\ \lambda\in\mathbb{C}. \tag{3.48}$$

And the same for q-Euler, Lucas and G polynomials. This can be generalized in at least two ways.

Theorem 3.54. A q-analogue of [76, p. 704], [66, p. 133]. If $\sum_{l=1}^{s} n_l = n$,

$$B_{\text{NWA},k,q}^{(n)}(x_1\oplus_q\ldots\oplus_q x_s) = \sum_{m_1+\ldots+m_s=k}\binom{k}{m_1,\ldots,m_s}_q\prod_{j=1}^{s}B_{\text{NWA},m_j,q}^{(n_j)}(x_j), \tag{3.49}$$

where we assume that n_j operates on x_j. And the same for any q-polynomial.
Proof. In umbral notation we have, as in the classical case

$$(x_1\oplus_q\ldots\oplus_q x_s\oplus_q\tilde{n}_q\gamma)^k\sim((x_1\oplus_q\tilde{n_1}_q\gamma')\oplus_q\ldots\oplus_q(x_s\oplus_q\tilde{n_s}_q\gamma''))^k,$$

where $\gamma',\ldots,\gamma''$ are distinct umbrae, each equivalent to γ. ■

Theorem 3.55. A q-analogue of [76, p. 704], [66, p. 133]. If $\sum_{l=1}^{s} n_l = n$,

$$B_{\text{NWA},k,q}^{(n)}(x_1 \boxplus_q \ldots \boxplus_q x_s) = \sum_{m_1+\ldots+m_s=k} \binom{k}{m_1,\ldots,m_s}_q \prod_{j=1}^{s} B_{\text{NWA},m_j,q}^{(n_j)}(x_j) q^{\binom{\vec{m}}{2}}, \tag{3.50}$$

$\vec{m} = (m_2, \ldots, m_n)$. We assume that n_j operates on x_j. And the same for any q-polynomial.

By (3.10) and (3.34), we get

$$\triangle_{\text{NWA},q}^{n} B_{\text{NWA},\nu,q}^{(n)}(x) = \frac{\{\nu\}_q!}{\{\nu-n\}_q!} x^{\nu-n},$$

$$\nabla_{\text{NWA},q}^{n} E_{\text{NWA},\nu,q}^{(n)}(x) = x^{\nu},$$

and we have

Definition 3.56. A q-analogue of [65, p. 177], [66, (66), p. 138]. The first q-Bernoulli polynomials of negative order $-n$ are given by

$$B_{\text{NWA},\nu,q}^{(-n)}(x|\omega_1,\ldots,\omega_n) \equiv \frac{\{\nu\}_q!}{\{\nu+n\}_q!} \underset{\omega_1,\ldots,\omega_n}{\triangle_{\text{NWA},q}^{n}} x^{\nu+n}, \tag{3.51}$$

and the first q-Euler polynomial of negative order $-n$ by the following q-analogue of [66, (67) p. 138]

$$E_{\text{NWA},\nu,q}^{(-n)}(x|\omega_1,\ldots,\omega_n) \equiv \underset{\omega_1,\ldots,\omega_n}{\nabla_{\text{NWA},q}^{n}} x^{\nu}, \tag{3.52}$$

where $\nu, n \in \mathbb{N}$. This defines q-Bernoulli and q-Euler polynomials of negative order as iterated $\triangle_{\text{NWA},q}$ and $\nabla_{\text{NWA},q}$ operating on positive integer powers of x.

To save space in the following, we are only going to write down equations for q-Bernoulli polynomials and numbers and not for the corresponding q-Euler polynomials $E_{\text{NWA},\nu,q}$ and numbers $F_{\text{NWA},\nu,q}$. Any $\triangle_{\text{NWA}}$ will have to be changed to ∇_{NWA} in the dual. Also q-Lucas polynomials and numbers will have to be changed to q-G polynomials and numbers in the dual. This applies to equations (3.53) to (3.59) and (3.100) to (3.105). Furthermore,

$$B_{\text{NWA},\nu,q}^{(-n)} \equiv B_{\text{NWA},\nu,q}^{(-n)}(0). \tag{3.53}$$

A calculation shows that formulas (3.10) and (3.34) hold for negative orders too, and we get [2, p. 255 10.9]:

$$B_{\text{NWA},\nu,q}^{(-n-p)}(x \oplus_q y) \doteq (B_{\text{NWA},q}^{(-n)}(x) \oplus_q B_{\text{NWA},q}^{(-p)}(y))^{\nu}. \tag{3.54}$$

A special case is the following q-analogue of [66, p. 139, (71)]:

$$B_{\text{NWA},\nu,q}^{(-n)}(x \oplus_q y) \doteq (B_{\text{NWA},q}^{(-n)}(x) \oplus_q y)^{\nu}.$$

Theorem 3.57. A q-analogue of [66, p. 140 (72), (73)], [68, p. 1226, xvii]. This equation first occurred in [2, p. 255, 10.10]:

$$(x \oplus_q y)^\nu \doteqdot (B^{(-n)}_{\mathrm{NWA},q}(x) \oplus_q B^{(n)}_{\mathrm{NWA},q}(y))^\nu. \tag{3.55}$$

In particular for $y = 0$, we get a q-analogue of [68, p. 1226, xviii]:

$$x^\nu \doteqdot (B^{(-n)}_{\mathrm{NWA},q} \oplus_q B^{(n)}_{\mathrm{NWA},q}(x))^\nu. \tag{3.56}$$

These recurrence formulas express first q-Bernoulli and q-Euler polynomials of order n without mentioning polynomials of negative order. These equations can also be expressed in the form

$$x^\nu = \sum_{s=0}^{\nu} \frac{B^{(-n)}_{\mathrm{NWA},s,q}}{\{s\}_q!} D_q^s B^{(n)}_{\mathrm{NWA},\nu,q}(x). \tag{3.57}$$

Hence the first q-Bernoulli and q-Euler polynomials satisfy linear q-difference equations with constant coefficients.

The following theorem is useful for the computation of first q-Bernoulli and q-Euler polynomials of positive order. This is because the polynomials of negative order are of simpler nature and can easily be computed. When the $B^{(-n)}_{\mathrm{NWA},s,q}$ etc. are known, (3.58) can be used to compute the $B^{(n)}_{\mathrm{NWA},s,q}$.

Theorem 3.58.

$$\sum_{s=0}^{\nu} \binom{\nu}{s}_q B^{(n)}_{\mathrm{NWA},s,q} B^{(-n)}_{\mathrm{NWA},\nu-s,q} = \delta_{\nu,0}. \tag{3.58}$$

Proof. Put $x = y = 0$ in (3.55). ∎

Theorem 3.59. A q-analogue of [66, p. 142]. Assume that $f(x)$ is analytic with q-Taylor expansion

$$f(x) = \sum_{\nu=0}^{\infty} D_q^\nu f(0) \frac{x^\nu}{\{\nu\}_q!}.$$

Then we can express powers of $\triangle_{\mathrm{NWA},q}$ and $\nabla_{\mathrm{NWA},q}$ operating on $f(x)$ as powers of D_q as follows. These series converge when the absolute value of x is small enough:

$$\underset{\omega_1,\ldots,\omega_n}{\triangle^n_{\mathrm{NWA},q}} f(x) = \sum_{\nu=0}^{\infty} D_q^{\nu+n} f(0) \frac{B^{(-n)}_{\mathrm{NWA},\nu,q}(x|\omega_1,\ldots,\omega_n)}{\{\nu\}_q!}. \tag{3.59}$$

Proof. Use (3.10) and (3.2). ∎

Now put $f(x) = \mathrm{E}_q(xt)$ to obtain the generating function of the q-Bernoulli and q-Euler polynomials of negative order:

$$\frac{\prod_{k=1}^n (\mathrm{E}_q(\omega_k t) - 1)\mathrm{E}_q(xt)}{t^n \prod_{k=1}^n \omega_k} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} B^{(-n)}_{\mathrm{NWA},\nu,q}(x|\omega_1,\ldots,\omega_n),$$

$$\frac{\prod_{k=1}^{n}(\mathrm{E}_q(\omega_k t)+1)\mathrm{E}_q(xt)}{2^n}=\sum_{\nu=0}^{\infty}\frac{t^{\nu}}{\{\nu\}_q!}E^{(-n)}_{\mathrm{NWA},\nu,q}(x|\omega_1,\ldots,\omega_n).$$

The reason for the difference in appearance compared to the original for the following equation is that one of the function arguments is a Ward number.

Theorem 3.60. A q-analogue of [65, p. 191, (10)], [57, p. 21 (18)]:

$$B^{(m)}_{\mathrm{NWA},\nu,q}(x\oplus_q \overline{n}_q)=\sum_{k=0}^{\min(\nu,n)}\binom{n}{k}\frac{\{\nu\}_q!}{\{\nu-k\}_q!}B^{(m-k)}_{\mathrm{NWA},\nu-k,q}(x). \tag{3.60}$$

Proof. Use (2.29) and (3.10). ■

We can put $m=x=0$ in (3.60) to obtain a q-analogue of [60, p. 133, (3)].

Theorem 3.61.

$$\frac{\{\nu\}_q!}{\{\nu-n\}_q!}x^{\nu-n}=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}B^{(n)}_{\mathrm{NWA},\nu,q}(x\oplus_q \overline{k}_q). \tag{3.61}$$

Proof. Use equation (2.32). ■

Theorem 3.62. A q-analogue of [65, (21) p. 163], [68, p. 1220]. The corresponding formula for $n=1$ occurred in [2, p. 254]:

$$\sum_{k=0}^{\infty}\frac{B^{(n)}_{\mathrm{NWA},k,q}(x)}{\{k\}_q!}\triangle^{n}_{\mathrm{NWA},q}D^k_q f(y)=D^n_{q,x}f(x\oplus_q y). \tag{3.62}$$

Proof. As in [65, p. 163] replace $f(x)$ by $f(x\oplus_q y)$ in (3.24):

$$f(B^{(n)}_{\mathrm{NWA},q}(x)\oplus_q y\oplus_q 1)-f(B^{(n)}_{\mathrm{NWA},q}(x)\oplus_q y)=D_q f(B^{(n-1)}_{\mathrm{NWA},q}(x)\oplus_q y). \tag{3.63}$$

Use the umbral formula (2.1) to get

$$\sum_{k=0}^{\infty}\frac{B^{(n)}_{\mathrm{NWA},k,q}(x)}{\{k\}_q!}\triangle_{\mathrm{NWA},q}D^k_q f(y)=\sum_{k=0}^{\infty}\frac{B^{(n-1)}_{\mathrm{NWA},k,q}(x)}{\{k\}_q!}D^{k+1}_q f(y).$$

Apply the operator $\triangle^{n-1}_{\mathrm{NWA},q}$ with respect to y to both sides and use (3.59):

$$\sum_{k=0}^{\infty}\frac{B^{(n)}_{\mathrm{NWA},k,q}(x)}{\{k\}_q!}\triangle^{n}_{\mathrm{NWA},q}D^k_q f(y)=\sum_{k=0}^{\infty}\frac{B^{(n-1)}_{\mathrm{NWA},k,q}(x)}{\{k\}_q!}\sum_{l=0}^{\infty}D^{k+l+n}_q f(0)\frac{B^{(-n+1)}_{\mathrm{NWA},l,q}(y)}{\{l\}_q!}.$$

Finally use (2.1), (2.8), (3.55) to rewrite the right-hand side. ■

Remark 3.63. The RHS of (3.62) can also be written $D^n_{q,y}f(x\oplus_q y)$ or $D^n_q f(x\oplus_q y)$.

If we put $n = q = 1$ in (3.62), we get an Euler–Maclaurin expansion known from [60, p. 140]. If we also put $y = 0$, we get an expansion of a polynomial in terms of Bernoulli polynomials known from [54, p. 248].

Corollary 3.64. A q-analogue of Szegö [87]. Let $\varphi(x)$ be a polynomial of degree ν. A solution $f(x)$ of the q-difference equation

$$\underset{\omega_1,\ldots,\omega_n}{\triangle^n_{\text{NWA},q}} f(x) = D^n_q \varphi(x)$$

is given by

$$f(x \oplus_q y) = \sum_{k=0}^{\nu} \frac{B^{(n)}_{\text{NWA},k,q}(x|\omega_1,\ldots,\omega_n)}{\{k\}_q!} D^k_q \varphi(y). \tag{3.64}$$

Proof. The LHS of (3.64) can be written as $\varphi(B^{(n)}_{\text{NWA},q}(x|\omega_1,\ldots,\omega_n) \oplus_q y)$, because if we apply $\triangle^n_{\text{NWA},q,x}$ to both sides we get

$$\underset{\omega_1,\ldots,\omega_n}{\triangle^n_{\text{NWA},q}} f(x \oplus_q y) = D^n_{q,x}\varphi(x \oplus_q y) = \underset{\omega_1,\ldots,\omega_n}{\triangle^n_{\text{NWA},q}} \varphi(B^{(n)}_{\text{NWA},q}(x|\omega_1,\ldots,\omega_n) \oplus_q y).$$

∎

Theorem 3.65. A q-analogue of Nörlund [65, p. 156], [66, p. 127 (31)]. Compare [65, p. 147]. A special case is found in [54, p. 307]:

$$\sum_{k=0}^{\infty} \frac{E^{(n)}_{\text{NWA},k,q}(x)}{\{k\}_q!} \nabla^n_{\text{NWA},q} D^k_q f(y) = f(x \oplus_q y). \tag{3.65}$$

Proof. As in [65, p. 155] replace $f(x)$ by $f(x \oplus_q y)$ in (3.42):

$$\frac{1}{2}\Big(f(E^{(n)}_{\text{NWA},q}(x) \oplus_q y \oplus_q 1) + f(E^{(n)}_{\text{NWA},q}(x) \oplus_q y)\Big) = f(E^{(n-1)}_{\text{NWA},q}(x) \oplus_q y). \tag{3.66}$$

Use the umbral formula (2.1) to get

$$\sum_{k=0}^{\infty} \frac{E^{(n)}_{\text{NWA},k,q}(x)}{\{k\}_q!} \nabla_{\text{NWA},q} D^k_q f(y) = \sum_{k=0}^{\infty} \frac{E^{(n-1)}_{\text{NWA},k,q}(x)}{\{k\}_q!} D^k_q f(y).$$

Apply the operator $\nabla^{n-1}_{\text{NWA},q}$ with respect to y to both sides and use (3.59):

$$\sum_{k=0}^{\infty} \frac{E^{(n)}_{\text{NWA},k,q}(x)}{\{k\}_q!} \nabla^n_{\text{NWA},q} D^k_q f(y) = \sum_{k=0}^{\infty} \frac{E^{(n-1)}_{\text{NWA},k,q}(x)}{\{k\}_q!} \sum_{l=0}^{\infty} D^{k+l}_q f(0) \frac{E^{(-n+1)}_{\text{NWA},l,q}(y)}{\{l\}_q!}.$$

Finally use (2.1), (2.8), (3.55) to rewrite the right-hand side. ∎

If we put $n = q = 1$ in (3.65), we get the Euler–Boole theorem known from [37, p. 128], [60, p. 149].

Corollary 3.66. A q-analogue of Szegö [87]. Let $\varphi(x)$ be a polynomial of degree ν. A solution $f(x)$ of the q-difference equation

$$\underset{\omega_1,\ldots,\omega_n}{\nabla^n_{\text{NWA},q}} f(x) = \varphi(x)$$

is given by

$$f(x \oplus_q y) = \sum_{k=0}^{\nu} \frac{E^{(n)}_{\text{NWA},k,q}(x|\omega_1,\ldots,\omega_n)}{\{k\}_q!} D_q^k \varphi(y). \tag{3.67}$$

Proof. The LHS of (3.67) can be written as $\varphi(E^{(n)}_{\text{NWA},q}(x|\omega_1,\ldots,\omega_n) \oplus_q y)$, because if we apply $\nabla^n_{\text{NWA},q,x}$ to both sides we get

$$\underset{\omega_1,\ldots,\omega_n}{\nabla^n_{\text{NWA},q}} f(x \oplus_q y) = \varphi(x \oplus_q y) = \underset{\omega_1,\ldots,\omega_n}{\nabla^n_{\text{NWA},q}} \varphi(E^{(n)}_{\text{NWA},q}(x|\omega_1,\ldots,\omega_n) \oplus_q y). \tag{3.68}$$

■

There are a few formulas similar to the Leibniz theorem. We can express the NWA difference operator in terms of the mean value operator and vice versa.

Theorem 3.67.

$$\triangle^n_{\text{NWA},q}(fg) = 2^n \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} \nabla^i_{\text{NWA},q} f(\nabla^{n-i}_{\text{NWA},q} E(\oplus_q)^i) g. \tag{3.69}$$

Proof. Same as Jordan [54, p. 98, (13)]. ■

Theorem 3.68.

$$\nabla^n_{\text{NWA},q}(fg) = \left(-\frac{1}{2}\right)^n \sum_{i=0}^{n} (-2)^i \binom{n}{i} \nabla^i_{\text{NWA},q} f(\triangle^{n-i}_{\text{NWA},q} E(\oplus_q)^i) g.$$

Proof. Same as [54, p. 99, (2)]. ■

Theorem 3.69.

$$\nabla^n_{\text{NWA},q}(fg) = \sum_{i=0}^{n} \left(\frac{1}{2}\right)^i \binom{n}{i} \triangle^i_{\text{NWA},q} f(\nabla^{n-i}_{\text{NWA},q} E(\oplus_q)^i) g.$$

Proof. Same as [54, p. 99, (3)]. ■

Theorem 3.70. A q-analogue of Lagrange 1772, [54, p. 101], [15, p. 19], [60, p. 37]. The inverse NWA difference is given by

$$\triangle_{\text{NWA},q}^{-1} f(\overline{x}_q)|_0^n = \sum_{k=0}^{n-1} f(\overline{k}_q) \equiv \sum_0^n f(\overline{x}_q)\delta_q(x). \tag{3.70}$$

Proof. Use the same idea as Euler, reproduced by F. Schweins [82, p. 9]. ■

Theorem 3.71. The inverse ∇ is given by

$$\nabla_{\text{NWA},q}^{-1}\left(\frac{f(\overline{0}_q) + (-1)^{n-1} f(\overline{n}_q)}{2}\right) = \sum_{k=0}^{n-1} (-1)^k f(\overline{k}_q). \tag{3.71}$$

Theorem 3.72. The inverse NWA-difference is given by

$$\triangle_{\text{NWA},2,q}^{-1} f(\overline{2x+1}_q)|_{x=0}^n = \sum_{k=0}^{n-1} f(\overline{2k+1}_q). \tag{3.72}$$

Theorem 3.73. The inverse $\nabla_{\text{NWA},2,q}$ is given by

$$\nabla_{\text{NWA},2,q}^{-1}\left(\frac{f(\overline{1}_q) + (-1)^{n-1} f(\overline{2n+1}_q)}{2}\right) = \sum_{k=0}^{n-1} (-1)^k f(\overline{2k+1}_q). \tag{3.73}$$

Theorem 3.74. The analogue of integration by parts is a q-analogue of [54, p. 105], [15, p. 21], [60, p. 41], [67, p. 19]:

$$\sum_{k=0}^{n-1} f(\overline{k}_q)\triangle_{\text{NWA},q} g(\overline{k}_q) = [f(\overline{x}_q)g(\overline{x}_q)]_0^n - \sum_{k=0}^{n-1} E(\oplus_q) g(\overline{k}_q)\triangle_{\text{NWA},q} f(\overline{k}_q). \tag{3.74}$$

Theorem 3.75. A q-analogue of Euler's symbolic formula from Glaisher [35, p. 303], Lucas [56, p. 242, (1)] is given by

$$\begin{aligned}\sum_{k=0}^{n-1} f(\overline{k}_q) &\doteq \int_{\overline{0}_q}^{\overline{n}_q} f(x \oplus_q B_{\text{NWA},q})\, d_q(x) \equiv \int_{\overline{0}_q}^{\overline{n}_q} f(B_{\text{NWA},q}(x))\, d_q(x) \\ &\equiv \int_{B_{\text{NWA},q}}^{B_{\text{NWA},q}\oplus_q \overline{n}_q} f(x)\, d_q(x).\end{aligned} \tag{3.75}$$

Proof. Apply $\triangle_{\text{NWA},q}$ to both sides to get

$$f(x)|_{\overline{0}_q}^{\overline{n}_q} \doteq \triangle_{\text{NWA},q} \int_{\overline{0}_q}^{\overline{n}_q} f(x \oplus_q B_{\text{NWA},q})\, d_q(x).$$

Then apply (3.12). ■

Corollary 3.76. A q-analogue of Rota & Taylor [76, p. 701]:

$$J_{\text{B,N},q} f(x \oplus_q B_{\text{NWA},q}) \doteq f(x). \tag{3.76}$$

We immediately get a proof of the following formula, which is of considerable use in integration theory.

Theorem 3.77. The q-Euler–Maclaurin summation theorem for formal power series. A q-analogue of [15, p. 54], [35, p. 303], [67, p. 25], [76, p. 706], [54, p. 253]:

$$\sum_{k=0}^{n-1} f(\overline{k}_q) = \int_{\overline{0}_q}^{\overline{n}_q} f(x)\, d_q(x) - \frac{1}{\{2\}_q}(f(\overline{n}_q) - f(0)) + \sum_{k=2}^{\infty} \frac{B_{\text{NWA},k,q}}{\{k\}_q!}(D_q^{k-1} f(\overline{n}_q) - D_q^{k-1} f(0)). \tag{3.77}$$

Example 3.78. Put $f(x) = x^m$ in (3.77) to get formula (3.25).

Corollary 3.79. A dual to the q-Euler–Maclaurin summation theorem (3.77). The q-integral on the RHS shall be interpreted in the following way: First q-integrate f in the form $f(x)$. Then put in the values in the umbral sense according to (2.1):

$$\sum_{k=0}^{n-1} f(\overline{k}_q) = \int_0^{\overline{n}_q} f(B_{\text{NWA},q})\, d_q(x) + \sum_{k=0}^{\infty} \frac{(\overline{x}_q)^{k+1}}{\{k+1\}_q!} D_q^k f(B_{\text{NWA},k,q})|_0^n. \tag{3.78}$$

We will now derive analogous results for q-Euler numbers. We start with

Theorem 3.80. A generalization of (3.46). Compare Goldstine [37, p. 136] ($q = 1$):

$$\sum_{k=0}^{n-1} (-1)^k f(\overline{k}_q) \doteq \frac{(-1)^{x-1}}{2} f(E_{\text{NWA},q}(x))|_{\overline{0}_q}^{\overline{n}_q}. \tag{3.79}$$

Proof. Apply $\nabla_{\text{NWA},q}$ to both sides to get by (3.71)

$$\frac{f(\overline{0}_q) + (-1)^{n-1} f(\overline{n}_q)}{2} \doteq \nabla_{\text{NWA},q} \frac{(-1)^{x-1}}{2} f(E_{\text{NWA},q}(x))|_{\overline{0}_q}^{\overline{n}_q}. \tag{3.80}$$

Finally use (3.42) with $n = 1$. ∎

Theorem 3.81. Almost a q-analogue of Boole's first summation formula for alternate functions [54, p. 316, (2)]. A q-analogue of the Euler formula from Glaisher [35, p. 310], Lucas [56, p. 252]:

$$\begin{aligned}\sum_{k=0}^{n-1} (-1)^k f(\overline{k}_q) &= \sum_{k=0}^{\infty} \frac{F_{\text{NWA},k,q}}{\{k\}_q!} D_q^k \left(\frac{f(\overline{0}_q)}{2} + (-1)^{n-1} \frac{f(\overline{n}_q)}{2} \right) \\ &\doteq \frac{1}{2} \left[f(F_{\text{NWA},q}) + (-1)^{n-1} f(F_{\text{NWA},q} \oplus_q \overline{n}_q) \right].\end{aligned} \tag{3.81}$$

Proof. Use (3.65). ∎

We will continue this part with a theorem involving both q-Bernoulli and q-Euler polynomials.

Theorem 3.82. A q-analogue of Srivastava & Pintér [85, p. 379]:

$$B_{\text{NWA},\nu,q}^{(n)}(x \oplus_q y) = \sum_{k=0}^{\nu} \binom{\nu}{k}_q \left(B_{\text{NWA},k,q}^{(n)}(y) + \frac{\{k\}_q}{2} B_{\text{NWA},k-1,q}^{(n-1)}(y) \right) E_{\text{NWA},\nu-k,q}(x). \tag{3.82}$$

The proofs of the following formulas are made through the generating function. Observe that we have to change to JHC on the LHS.

Theorem 3.83. A q-analogue of the Raabe–Bernoulli complementary argument theorem [70, p. 354], [60, p. 128, (1)]:

$$B_{\text{JHC},\nu,q}(x) = (-1)^{\nu} B_{\text{NWA},\nu,q}(1 \ominus_q x).$$

Theorem 3.84. A q-analogue of the Euler complementary argument theorem from Milne-Thomson [60, p. 145, (1)]:

$$E_{\text{JHC},\nu,q}(x) = (-1)^{\nu} E_{\text{NWA},\nu,q}(1 \ominus_q x).$$

We now continue the study of q-Lucas and q-G polynomials in much the same way as for q-Bernoulli and q-Euler polynomials. As there are no complementary argument theorems for these, we do not need the JHC-versions.

Theorem 3.85. The successive differences of q-Lucas polynomials can be expressed as q-Lucas polynomials:

$$\underset{\omega_1,\ldots,\omega_p}{\triangle^{p}_{\text{NWA},2,q}} L_{\text{NWA},\nu,q}^{(n)}(x|\omega_1,\ldots,\omega_n) = \frac{\{\nu\}_q!}{\{\nu-p\}_q!} L_{\text{NWA},\nu-p,q}^{(n-p)}(x|\omega_{p+1},\ldots,\omega_n). \tag{3.83}$$

The following invertible operator will be useful in this connection.

Definition 3.86. Compare Cigler [15, p. 32] $(n=1)$. The operator $S_{\text{L,N},q}^{n} \in \mathbb{C}(D_q)$ is

$$S_{\text{L,N},q}^{n} \equiv \frac{(\text{E}_q(\overline{2}_q D_q) - I)^n}{(2D_q)^n}. \tag{3.84}$$

Theorem 3.87. Compare [15, p. 43, 3.3]:

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} L_{\text{NWA},\nu,q}^{(n)}(x \oplus_q \overline{2k}_q) = \{\nu-n+1\}_{n,q} x^{\nu-n} 2^n. \tag{3.85}$$

By the generating function

$$\frac{1}{2} \triangle_{\text{NWA},2,q} L_{\text{NWA},\nu,q}^{(n)}(x) = \{\nu\}_q L_{\text{NWA},\nu-1,q}^{(n-1)}(x) = D_q L_{\text{NWA},\nu,q}^{(n-1)}(x). \tag{3.86}$$

The following symbolic relations are obtained.

Theorem 3.88.

$$\frac{1}{2}\left[(L^{(n)}_{\text{NWA},q}\oplus_q x\oplus_q\overline{2}_q)^{\nu}-(L^{(n)}_{\text{NWA},q}\oplus_q x)^{\nu}\right]\doteq\{\nu\}_q(L^{(n-1)}_{\text{NWA},q}\oplus_q x)^{\nu-1}. \tag{3.87}$$

Theorem 3.89.

$$\begin{aligned}\triangle_{\text{NWA},q}\nabla_{\text{NWA},q}f(L^{(n)}_{\text{NWA},\nu,q}(x))\equiv&\frac{1}{2}\left[f(L^{(n)}_{\text{NWA},\nu,q}(x)\oplus_q\overline{2}_q)-f(L^{(n)}_{\text{NWA},\nu,q}(x))\right]\\ \doteq&D_q f(L^{(n-1)}_{\text{NWA},\nu,q}(x)).\end{aligned} \tag{3.88}$$

Theorem 3.90. The q-Lucas polynomials of degree ν and order n can be expressed as

$$L^{(n)}_{\text{NWA},\nu,q}(t)=S^{-n}_{\text{L,N},q}t^{\nu}. \tag{3.89}$$

The first q-Lucas numbers have the following values:

$$L_{\text{NWA},0,q}=1;\,L_{\text{NWA},1,q}=(-3-q)(2+2q)^{-1},$$

$$\begin{gathered}L_{\text{NWA},2,q}=(1+3q+8q^2+3q^3+q^4)(4+8q+8q^2+4q^3)^{-1},\\ L_{\text{NWA},3,q}=(-1-2q-2q^2-9q^3+9q^4+2q^5+2q^6+q^7)[8(q+1)^2(q^2+1)]^{-1},\\ L_{\text{NWA},4,q}=f(q)(16(1+q)^2\{3\}_q\{5\}_q)^{-1},\end{gathered}$$

where

$$\begin{aligned}f(q)=&1+3q+q^2-11q^3-18q^4-63q^5-104q^6-130q^7\\ &-104q^8-63q^9-18q^{10}-11q^{11}+q^{12}+3q^{13}+q^{14}.\end{aligned}$$

We will now give a few other equations for q-Lucas polynomials. We start with

Definition 3.91. A q-analogue of Vandiver [90, p. 575], Nielsen [63, p. 401 (7)], Lucas [56, p. 253]:

$$t_{\text{NWA},m,q}(n)\equiv\sum_{k=0}^{n-1}(\overline{2k+1}_q)^m. \tag{3.90}$$

Theorem 3.92. A q-analogue of Lucas [56, p. 261]:

$$t_{\text{NWA},m,q}(n)=\frac{L_{\text{NWA},m+1,q}(\overline{(2n+1)}_q)-L_{\text{NWA},m+1,q}(\overline{(1)}_q)}{\{m+1\}_q}. \tag{3.91}$$

Theorem 3.93.

$$2x^n=\int_x^{x\oplus_q\overline{2}_q}L_{\text{NWA},n,q}(t)\,d_q(t)=\frac{L_{\text{NWA},n+1,q}(x\oplus_q\overline{2}_q)-L_{\text{NWA},n+1,q}(x)}{\{n+1\}_q}. \tag{3.92}$$

This can be rewritten as

$$2x^n = \frac{1}{\{n+1\}_q} \sum_{k=0}^{n} \binom{n+1}{k}_q L_{\text{NWA},k,q}(x)(\overline{2}_q)^{n+1-k}. \tag{3.93}$$

The following symbolic relations for q-G polynomials are obtained.

Theorem 3.94. A q-analogue of Nörlund [65, p. 138] ($n = 1$), [66, p. 125]:

$$\frac{1}{2}\left[(G^{(n)}_{\text{NWA},q} \oplus_q x \oplus_q \overline{2}_q)^\nu + (G^{(n)}_{\text{NWA},q} \oplus_q x)^\nu\right] \doteq (G^{(n-1)}_{\text{NWA},q} \oplus_q x)^\nu. \tag{3.94}$$

A q-analogue of [65, p. 137] ($n = 1$), [66, p. 124]:

$$\begin{aligned}\nabla_{\text{NWA},2,q} f(G^{(n)}_{\text{NWA},q}(x)) \equiv& \frac{1}{2}\left[f(G^{(n)}_{\text{NWA},q}(x) \oplus_q \overline{2}_q) + f(G^{(n)}_{\text{NWA},q}(x))\right] \\ \doteq& f(G^{(n-1)}_{\text{NWA},q}(x)).\end{aligned} \tag{3.95}$$

The following table lists some of the first $G_{\text{NWA},n,q}$. A q-analogue of the integers in [66, p. 27]:

$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 4$
1	-1	$2^{-1}(-1+q)$	$q(1+q)$	$-2^{-2}(q^3-1)(1+q)^3$

We will now give a few other equations for q-G polynomials. We start with

Definition 3.95. A q-analogue of J. Herschel [46, p. 91]:

$$\tau_{\text{NWA},m,q}(n) \equiv \sum_{k=0}^{n-1} (-1)^k (\overline{2k+1}_q)^m.$$

Theorem 3.96. A q-analogue of [56, p. 237], [63, p. 401]:

$$\tau_{\text{NWA},m,q}(n) = \frac{(-1)^{n-1} G_{\text{NWA},m+1,q}(\overline{(2n+1)}_q) + G_{\text{NWA},m+1,q}(\overline{(1)}_q)}{2}. \tag{3.96}$$

Proof.

$$\begin{aligned} LHS =& \sum_{k=0}^{n-1} (-1)^k \nabla_{\text{NWA},2,q} G_{\text{NWA},m,q}(\overline{(2k+1)}_q) \\ =& \sum_{k=0}^{n-1} \frac{(-1)^k}{2} (G_{\text{NWA},m,q}(\overline{2k+1}_q) + G_{\text{NWA},m,q}(\overline{2k+3}_q)) = RHS. \end{aligned}$$

∎

Theorem 3.97. Another integration by parts formula:

$$\sum_{k=0}^{n-1} f(\overline{2k+1_q})\triangle_{\text{NWA},q} g(\overline{2k+1_q})$$
$$= [f(\overline{2x+1_q})g(\overline{2x+1_q})]_0^n - \sum_{k=0}^{n-1} E(\oplus_q)^{\overline{2}_q} g(\overline{2k+1_q})\triangle_{\text{NWA},2,q} f(\overline{2k+1_q}).$$

Theorem 3.98. Compare [37, p. 136] ($q=1$):

$$\sum_{k=0}^{n-1}(-1)^k f(\overline{2k+1_q}) = \frac{(-1)^{x-1}}{2} f(G_{\text{NWA},q}(\overline{2x+1_q}))|_0^n. \tag{3.97}$$

Definition 3.99. A q-analogue of the Lucas polynomial of negative order $-n$ is given by

$$L^{(-n)}_{\text{NWA},\nu,q}(x) \equiv \frac{\{\nu\}_q!}{\{\nu+n\}_q!}\triangle^n_{\text{NWA},2,q} x^{\nu+n}, \tag{3.98}$$

and the q-G polynomial of negative order $-n$ is given by

$$G^{(-n)}_{\text{NWA},\nu,q}(x) \equiv \nabla^n_{\text{NWA},2,q} x^{\nu}, \tag{3.99}$$

where $\nu, n \in \mathbb{N}$. This defines q-Lucas- and q-G polynomials of negative order as iterated $\triangle_{\text{NWA},q}$ and $\nabla_{\text{NWA},q}$ operating on positive integer powers of x.

Furthermore,

Theorem 3.100.

$$L^{(-n-p)}_{\text{NWA},\nu,q}(x \oplus_q y) \doteq (L^{(-n)}_{\text{NWA},q}(x) \oplus_q L^{(-p)}_{\text{NWA},q}(y))^{\nu}. \tag{3.100}$$

A special case is the following

Theorem 3.101.

$$(x \oplus_q y)^{\nu} \doteq (L^{(-n)}_{\text{NWA},q}(x) \oplus_q L^{(n)}_{\text{NWA},q}(y))^{\nu}, \tag{3.101}$$

Proof. Put $p=-n$ in (3.100). ■

In particular for $y=0$

$$x^{\nu} \doteq (L^{(-n)}_{\text{NWA},q} \oplus_q L^{(n)}_{\text{NWA},q}(x))^{\nu}, \tag{3.102}$$

These recurrence formulas express q-Lucas- and q-G polynomials of order n without mentioning polynomials of negative order. These can also be expressed in the form

$$x^{\nu} = \sum_{s=0}^{\nu} \frac{L^{(-n)}_{\text{NWA},s,q}}{\{s\}_q!} D^s_q L^{(n)}_{\text{NWA},\nu,q}(x). \tag{3.103}$$

We conclude that the q-Lucas and q-G polynomials satisfy linear q-difference equations with constant coefficients. The next equation is similar to (3.58).

$$\sum_{s=0}^{\nu}\binom{\nu}{s}_q L_{\text{NWA},s,q}^{(n)} L_{\text{NWA},\nu-s,q}^{(-n)} = \delta_{\nu,0}. \tag{3.104}$$

Proof. Put $x = y = 0$ in (3.101). ∎

Theorem 3.102. Assume that $f(x)$ is analytic with q-Taylor expansion

$$f(x) = \sum_{\nu=0}^{\infty} D_q^{\nu} f(0) \frac{x^{\nu}}{\{\nu\}_q!}.$$

Then we can express powers of $\triangle_{\text{NWA},q}$ and $\nabla_{\text{NWA},q}$ operating on $f(x)$ as powers of D_q as follows. These series converge when the absolute value of x is small enough:

$$\triangle_{\text{NWA},2,q}^{n} f(x) = \sum_{\nu=0}^{\infty} D_q^{\nu+n} f(0) \frac{L_{\text{NWA},\nu,q}^{(-n)}(x)}{\{\nu\}_q!}. \tag{3.105}$$

Proof. Use (3.86) and (3.2). ∎

With $f(x) = \mathrm{E}_q(xt)$ we get the generating function for $L_{\text{NWA},\nu,q}^{(-n)}(x)$ and $G_{\text{NWA},\nu,q}^{(-n)}(x)$:

$$(\mathrm{E}_q(\overline{2}_q t) - 1)^n \mathrm{E}_q(xt) = \sum_{\nu=0}^{\infty} \frac{t^{\nu+n}}{\{\nu\}_q!} L_{\text{NWA},\nu,q}^{(-n)}(x), \tag{3.106}$$

$$\frac{(\mathrm{E}_q(\overline{2}_q t) + 1)^n}{2^n} \mathrm{E}_q(xt) = \sum_{\nu=0}^{\infty} \frac{t^{\nu}}{\{\nu\}_q!} G_{\text{NWA},\nu,q}^{(-n)}(x).$$

The reason for the difference in appearance compared to the original for the following equation is that one of the function arguments is a Ward number.

Theorem 3.103.

$$L_{\text{NWA},\nu,q}^{(n)}(x \oplus_q \overline{2n}_q) = \sum_{k=0}^{n} \binom{n}{k} \frac{\{\nu\}_q!}{\{\nu-k\}_q!} L_{\text{NWA},\nu-k,q}^{(n-k)}(x).$$

Proof. Use (2.29) and (3.86). ∎

Theorem 3.104.

$$\frac{\{\nu\}_q!}{\{\nu-n\}_q!} x^{\nu-n} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} L_{\text{NWA},\nu,q}^{(n)}(x \oplus_q \overline{2k}_q).$$

Proof. Use equation (2.32). ∎

Theorem 3.105.

$$\frac{1}{2}\sum_{k=0}^{\infty}\frac{L_{\text{NWA},k,q}^{(n)}(x)}{\{k\}_q!}\triangle_{\text{NWA},2,q}^{n}D_q^k f(y)=D_q^n f(x\oplus_q y). \tag{3.107}$$

Proof. Replace $f(x)$ by $f(x\oplus_q y)$ in (3.88):

$$\frac{1}{2}\Big(f(L_{\text{NWA},q}^{(n)}(x)\oplus_q \overline{2}_q\oplus_q y)-f(L_{\text{NWA},q}^{(n)}(x)\oplus_q y)\Big)\ddot{=}D_q f(L_{\text{NWA},q}^{(n-1)}(x)\oplus_q y). \tag{3.108}$$

Use the umbral formula (2.1) to get

$$\frac{1}{2}\sum_{k=0}^{\infty}\frac{L_{\text{NWA},k,q}^{(n)}(x)}{\{k\}_q!}\triangle_{\text{NWA},2,q}D_q^k f(y)=\sum_{k=0}^{\infty}\frac{L_{\text{NWA},k,q}^{(n-1)}(x)}{\{k\}_q!}D_q^{k+1}f(y).$$

Apply the operator $\triangle_{\text{NWA},2,q}^{n-1}$ with respect to y to both sides and use (3.105):

$$\frac{1}{2}\sum_{k=0}^{\infty}\frac{L_{\text{NWA},k,q}^{(n)}(x)}{\{k\}_q!}\triangle_{\text{NWA},q}^{n}D_q^k f(y)=\sum_{k=0}^{\infty}\frac{L_{\text{NWA},k,q}^{(n-1)}(x)}{\{k\}_q!}\sum_{l=0}^{\infty}D_q^{k+l+n}f(0)\frac{L_{\text{NWA},l,q}^{(-n+1)}(y)}{\{l\}_q!}.$$

Finally use (3.101) to rewrite the right-hand side. ∎

Corollary 3.106. Let $\varphi(x)$ be a polynomial of degree ν. A solution $f(x)$ of the q-difference equation

$$\frac{1}{2}\begin{matrix}\triangle_{\text{NWA},2,q}^{n}\\ \omega_1,\ldots,\omega_n\end{matrix}\;f(x)=D_q^n\varphi(x)$$

is given by

$$f(x\oplus_q y)=\sum_{k=0}^{\nu}\frac{L_{\text{NWA},k,q}^{(n)}(x|\omega_1,\ldots,\omega_n)}{\{k\}_q!}D_q^k\varphi(y). \tag{3.109}$$

Proof. The LHS of (3.109)) can be written as $\varphi(L_{\text{NWA},q}^{(n)}(x|\omega_1,\ldots,\omega_n)\oplus_q y)$, because if we apply $\frac{1}{2}\triangle_{\text{NWA},2,q,x}^{n}$ to both sides, we get

$$\begin{aligned}&\frac{1}{2}\begin{matrix}\triangle_{\text{NWA},2,q}^{n}\\ \omega_1,\ldots,\omega_n\end{matrix}\;f(x\oplus_q y)=D_{q,x}^n\varphi(x\oplus_q y)\\ &=\frac{1}{2}\begin{matrix}\triangle_{\text{NWA},2,q}^{n}\\ \omega_1,\ldots,\omega_n\end{matrix}\;\varphi(L_{\text{NWA},q}^{(n)}(x|\omega_1,\ldots,\omega_n)\oplus_q y).\end{aligned}$$

∎

Theorem 3.107.

$$\sum_{k=0}^{\infty}\frac{G_{\text{NWA},k,q}^{(n)}(x)}{\{k\}_q!}\nabla_{\text{NWA},2,q}^{n}D_q^k f(y)=f(x\oplus_q y). \tag{3.110}$$

Proof. Replace $f(x)$ by $f(x \oplus_q y)$ in (3.95):

$$\frac{1}{2}\Big(f(G^{(n)}_{\text{NWA},q}(x) \oplus_q \overline{2}_q \oplus_q y) + f(G^{(n)}_{\text{NWA},q}(x) \oplus_q y)\Big) \doteq f(G^{(n-1)}_{\text{NWA},q}(x) \oplus_q y). \quad (3.111)$$

Use the umbral formula (2.1) to get

$$\sum_{k=0}^{\infty} \frac{G^{(n)}_{\text{NWA},k,q}(x)}{\{k\}_q!} \nabla_{\text{NWA},2,q} D_q^k f(y) = \sum_{k=0}^{\infty} \frac{G^{(n-1)}_{\text{NWA},k,q}(x)}{\{k\}_q!} D_q^k f(y).$$

Apply the operator $\nabla^{n-1}_{\text{NWA},2,q}$ with respect to y to both sides and use (3.105):

$$\sum_{k=0}^{\infty} \frac{G^{(n)}_{\text{NWA},k,q}(x)}{\{k\}_q!} \nabla^n_{\text{NWA},2,q} D_q^k f(y) = \sum_{k=0}^{\infty} \frac{G^{(n-1)}_{\text{NWA},k,q}(x)}{\{k\}_q!} \sum_{l=0}^{\infty} D_q^{k+l} f(0) \frac{G^{(-n+1)}_{\text{NWA},l,q}(y)}{\{l\}_q!}.$$

Finally use (3.101) to rewrite the right-hand side. ■

Corollary 3.108. Let $\varphi(x)$ be a polynomial of degree ν. A solution $f(x)$ of the q-difference equation

$$\underset{\omega_1,\ldots,\omega_n}{\nabla^n_{\text{NWA},2,q}} f(x) = \varphi(x) \quad (3.112)$$

is given by

$$f(x \oplus_q y) = \sum_{k=0}^{\nu} \frac{G^{(n)}_{\text{NWA},k,q}(x|\omega_1,\ldots,\omega_n)}{\{k\}_q!} D_q^k \varphi(y). \quad (3.113)$$

Proof. The LHS of (3.113)) can be written as $\varphi(G^{(n)}_{\text{NWA},q}(x|\omega_1,\ldots,\omega_n) \oplus_q y)$, because if we apply $\nabla^n_{\text{NWA},2,q,x}$ to both sides we get

$$\underset{\omega_1,\ldots,\omega_n}{\nabla^n_{\text{NWA},2,q}} f(x \oplus_q y) = \varphi(x \oplus_q y) = \underset{\omega_1,\ldots,\omega_n}{\nabla^n_{\text{NWA},2,q}} \varphi(G^{(n)}_{\text{NWA},q}(x|\omega_1,\ldots,\omega_n) \oplus_q y).$$

■

There are a few formulas similar to the Leibniz theorem. We can express the NWA difference operator in terms of the mean value operator and vice versa.

Theorem 3.109.

$$\triangle^n_{\text{NWA},2,q}(fg) = 2^n \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} \nabla^i_{\text{NWA},2,q} f(\nabla^{n-i}_{\text{NWA},2,q} E(\oplus_q)^{\overline{2i}_q}) g. \quad (3.114)$$

Proof. Same as Jordan [54, p. 98, (13)]. ■

Theorem 3.110.

$$\nabla^n_{\mathrm{NWA},2,q}(fg) = \left(-\frac{1}{2}\right)^n \sum_{i=0}^{n} (-2)^i \binom{n}{i} \nabla^i_{\mathrm{NWA},2,q} f(\triangle^{n-i}_{\mathrm{NWA},2,q} E(\oplus_q)^{\overline{2i}_q}) g.$$

Proof. Same as [54, p. 99, (2)]. ∎

Theorem 3.111.

$$\nabla^n_{\mathrm{NWA},2,q}(fg) = \sum_{i=0}^{n} \left(\frac{1}{2}\right)^i \binom{n}{i} \triangle^i_{\mathrm{NWA},2,q} f(\nabla^{n-i}_{\mathrm{NWA},2,q} E(\oplus_q)^{\overline{2i}_q}) g. \tag{3.115}$$

Proof. Same as [54, p. 99, (3)]. ∎

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