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Singular Discrete Higher Order Boundary Value Problems

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Abstract

We study singular discrete nth order boundary value problems with mixed boundary conditions. We prove the existence of a positive solution by means of the lower and upper solutions method and the Brouwer fixed point theorem in conjunction with perturbation methods to approximate regular problems.

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1. Preliminaries

This paper is somewhat of an extension of the recent work done by Rachůnková and Rachůnek [19], and the work done by Kunkel [16]. Rachůnková and Rachůnek studied a second order singular boundary value problem for the discrete p-Laplacian, $\phi_p(x) = |x|^{p-2}x, p > 1$. In particular, Rachůnková and Rachůnek dealt with the discrete boundary value problem,

$$\Delta\left(\phi_p(\Delta u(t-1))\right) + f(t, u(t), \Delta u(t-1)) = 0, \quad t \in [1, T+1],$$
$$\Delta u(0) = u(T+2) = 0,$$

in which $f(t, x_1, x_2)$ was singular in x_1. Kunkel's results extend theirs to the third order case, but only for $p = 2$, i.e., $\phi_2(x) = x$. That is, Kunkel's extension focussed on the boundary value problem,

$$-\Delta^3 u(t-2) + f(t, u(t), \Delta u(t-1), \Delta^2 u(t-2)) = 0, \quad t \in [2, T+1],$$
$$\Delta^2 u(0) = \Delta u(T+2) = u(T+3) = 0.$$

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The results of this paper entail an extension of [16] to an nth order singular discrete boundary value problem. The methods of the paper rely heavily on upper and lower solutions methods in conjunction with an application of the Brouwer fixed point theorem [20]. We will provide definitions of appropriate upper and lower solutions. The upper and lower solutions will be applied to nonsingular perturbations of our nonlinear problem, ultimately giving rise to our boundary value problem by passing to the limit. Upper and lower solutions have been used extensively in establishing solutions of boundary value problems for finite difference equations. In addition to [16] and [19], we mention especially the paper by Jiang, *et al.* [12]. in which they dealt with singular discrete boundary value problems using upper and lower solutions methods. For other outstanding results in which upper and lower solutions methods were employed to obtain solutions of boundary value problems for finite difference equations, we refer to [1–3, 5, 6, 8–11, 15, 18, 21].

Singular discrete boundary value problems also have received a good deal of attention. For a list of a few representative works, we suggest the references [3–5, 7, 13, 14, 17]. In this section we will state the definitions that are used in the remainder of the paper.

Definition 1.1. Let $a < b$ be integers. Define the discrete interval

$$[a, b] = \{a, a + 1, \dots, b - 1, b\}.$$

Consider the nth order nonlinear difference equation,

$$(-1)^n \Delta^n u(t-(n-1)) + f(t, u(t), \dots, \Delta^{n-1} u(t-(n-1))) = 0, t \in [n-1, T+1], \quad (1.1)$$

with mixed boundary conditions,

$$\Delta^{n-1} u(0) = \Delta^{n-2} u(T + 2) = \Delta^{n-3} u(T + 3) = \cdots = u(T + n) = 0. \quad (1.2)$$

Here Δ denotes the forward difference operator with step size 1, i.e., $\Delta u(t) = u(t + 1) - u(t)$ and for $n > 1, \Delta^n u(t) = \Delta(\Delta^{n-1} u(t))$. Our goal is to prove the existence of a positive solution of problem (1.1), (1.2).

Definition 1.2. By a solution u of problem (1.1), (1.2) we mean $u : [0, T + n] \to \mathbb{R}$ such that u satisfies the difference equation (1.1) on $[n - 1, T + 1]$ and the boundary conditions (1.2). If $u(t) > 0$ for $t \in [n - 1, T + 1]$, we say u is a positive solution of the problem (1.1), (1.2).

Definition 1.3. Let $\mathcal{D} \subset \mathbb{R}^n$. We say that f is continuous on $[n - 1, T + 1] \times \mathcal{D}$, if $f(\cdot, x_1, \dots, x_n)$ is defined on $[n-1, T+1]$ for each $(x_1, \dots, x_n) \in \mathcal{D}$ and if $f(t, \cdot, \dots, \cdot)$ is continuous on $\mathcal{D}$ for each $t \in [n - 1, T + 1]$.

Definition 1.4. If $\mathcal{D} = \mathbb{R}^n$, problem (1.1), (1.2) is called regular. If $\mathcal{D} \neq \mathbb{R}^n$ and f has singularities on $\partial\mathcal{D}$, then problem (1.1), (1.2) is singular.

We will assume throughout this paper that the following hold:

(A): $\mathcal{D} = (0, \infty) \times \mathbb{R}^{n-1}$.

(B): f is continuous on $[n-1, T+1] \times \mathcal{D}$.

(C): $f(t, x_1, \dots, x_n)$ has a singularity at $x_1 = 0$, i.e., $\limsup_{x_1 \to 0+} |f(t, x_1, \dots, x_n)| = \infty$ for each $t \in [n-1, T+1]$ and for some $(x_2, \dots, x_n) \in \mathbb{R}^{n-1}$.

2. Lower and Upper Solutions Method for Regular Problems

Let us first consider the regular difference equation,

$$(-1)^n \Delta^n u(t-(n-1)) + h(t, u(t), \dots, \Delta^{n-1} u(t-(n-1))) = 0, t \in [n-1, T+1], \quad (2.1)$$

where h is continuous on $[n-1, T+1] \times \mathbb{R}^n$. We establish a lower and upper solutions method for the regular problem (2.1), (1.2).

Definition 2.1. $\alpha : [0, T+n] \to \mathbb{R}$ is called a lower solution of (2.1), (1.2) if,

$$(-1)^n \Delta^n \alpha(t-(n-1)) + h(t, \alpha(t), \dots, \Delta^{n-1} \alpha(t-(n-1))) \geq 0, \quad (2.2)$$

$t \in [n-1, T+1]$, satisfying boundary conditions

$$\begin{array}{rcl} (-1)^{n-1} \Delta^{n-1} \alpha(0) & \leq & 0, \\ (-1)^{n-1} \Delta^{n-2} \alpha(T+2)) & \geq & 0, \\ & \vdots & \\ \alpha(T+n) & \leq & 0. \end{array} \quad (2.3)$$

Definition 2.2. $\beta : [0, T+n] \to \mathbb{R}$ is called an upper solution of (2.1), (1.2) if,

$$(-1)^n \Delta^n \beta(t-(n-1)) + h(t, \beta(t), \dots, \Delta^{n-1} \beta(t-(n-1))) \leq 0, \quad (2.4)$$

$t \in [n-1, T+1]$, satisfying boundary conditions

$$\begin{array}{rcl} (-1)^{n-1} \Delta^{n-1} \beta(0) & \geq & 0, \\ (-1)^{n-1} \Delta^{n-2} \beta(T+2)) & \leq & 0, \\ & \vdots & \\ \beta(T+n) & \geq & 0. \end{array} \quad (2.5)$$

Theorem 2.3. (Lower and Upper Solutions Method) Let α and β be lower and upper solutions of (2.1), (1.2), respectively, and $\alpha \leq \beta$ on $[n-1, T+1]$. Let $h(t, x_0, \dots, x_{n-1})$ be continuous on $[n-1, T+1] \times \mathbb{R}^n$ and nonincreasing in its x_{n-1} variable. Then (2.1), (1.2) has a solution u satisfying,

$$\alpha(t) \leq u(t) \leq \beta(t), \quad t \in [0, T+n].$$

Proof. We proceed through a sequence of steps involving modifications of the function h.

Step 1. For $t \in [n-1, T+1], (x_0, \ldots, x_{n-1}) \in \mathbb{R}^n$, define

$$\widetilde{h}(t, x_0, \ldots, x_{n-2}, x_{n-2} - x_{n-1}) = \begin{cases} h(t, \alpha(t), \ldots, \Delta^{n-2}\alpha(t-(n-2)), \\ \quad \Delta^{n-2}\alpha(t-(n-2)) - \sigma(t-(n-2), x_{n-1})) \\ \quad +(-1)^{n-1}\dfrac{(-1)^{n-1}(x_{n-2} - \Delta^{n-2}\alpha(t-(n-2)))}{(-1)^{n-1}(x_{n-2} - \Delta^{n-2}\alpha(t-(n-2))) + 1}, \\ \qquad (-1)^{n-1}x_{n-2} > (-1)^{n-1}\Delta^{n-2}\alpha(t-(n-2)), \\ \\ h(t, x_0, \ldots, x_{n-2} - \sigma(t-(n-2), x_{n-1})), \\ \qquad (-1)^{n-1}\Delta^{n-2}\beta(t-(n-2)) \le (-1)^{n-1}x_{n-2} \\ \qquad \text{and} \\ \qquad (-1)^{n-1}x_{n-2} \le (-1)^{n-1}\Delta^{n-2}\alpha(t-(n-2)), \\ \\ h(t, \beta(t), \ldots, \Delta^{n-2}\beta(t-(n-2)), \\ \quad \Delta^{n-2}\beta(t-(n-2)) - \sigma(t-(n-2), x_{n-1})) \\ \quad -(-1)^{n-1}\dfrac{(-1)^{n-1}(\Delta^{n-2}\beta(t-(n-2)) - x_{n-2})}{(-1)^{n-1}(\Delta_{n-2}\beta(t-(n-2)) - x_{n-2}) + 1}, \\ \qquad (-1)^{n-1}x_{n-2} < (-1)^{n-1}\Delta^{n-2}\beta(t-(n-2)), \end{cases}$$

where

$$\sigma(t-(n-2), z) = \begin{cases} \Delta^{n-2}\alpha(t-(n-1)), & (-1)^{n-1}z > (-1)^{n-1}\Delta^{n-2}\alpha(t-(n-1)), \\ \\ z, & (-1)^{n-1}\Delta^{n-2}\beta(t-(n-1)) \le (-1)^{n-1}z, \\ & \text{and } (-1)^{n-1}z \le (-1)^{n-1}\Delta^{n-2}\alpha(t-(n-1)), \\ \\ \Delta^{n-2}\beta(t-(n-1)), & (-1)^{n-1}z < (-1)^{n-1}\Delta^{n-2}\beta(t-(n-1)). \end{cases}$$

Thus, $\widetilde{h}$ is continuous on $[n-1, T+1] \times \mathbb{R}^n$ and there exists $M > 0$ so that,

$$|\widetilde{h}(t, x_0, \ldots, x_{n-1})| \le M, \quad t \in [n-1, T+1], (x_0, \ldots, x_{n-1}) \in \mathbb{R}^n.$$

We now study the auxiliary equation,

$$(-1)^{n-1}\Delta^n u(t-(n-1)) + \widetilde{h}(t, u(t), \ldots, \Delta^{n-1}u(t-(n-1))) = 0, \quad t \in [n-1, T+1], \tag{2.6}$$

satisfying boundary conditions (1.2). Our goal now is to prove the existence of a solution of (2.6), (1.2).

Step 2. We lay the foundation to use the Brouwer fixed point theorem. To this end, define

$$E = \{u : [0, T+3] \to \mathbb{R} : \Delta^{n-1}u(0) = \Delta^{n-2}u(T+2) = \cdots = u(T+n) = 0\}$$

and also define

$$\|u\| = \max\{|u(t)| : t \in [n-1, T+1]\}.$$

E is a Banach space. Further, we define an operator $\mathcal{T} : E \to E$ by,

$$(\mathcal{T}u)(t) = -\sum_{j_{n-1}=t+n-2}^{T+n-1} \cdots \sum_{j_1=j_2}^{T+n-1} \sum_{i=n-1}^{j_1} \widetilde{h}(i, u(i), \ldots, \Delta^{n-1}u(i-(n-1))). \quad (2.7)$$

$\mathcal{T}$ is a continuous operator. Moreover, from the bounds placed on $\widetilde{h}$ in Step 1 and from (2.7), if

$$r > \sum_{j_{n-1}=1}^{T+n-1} \cdots \sum_{s=j_2}^{T+n-1} (s-(n-2))M,$$

then $\mathcal{T}(\overline{B(r)}) \subset \overline{B(r)}$, where $B(r) = \{u \in E : \|u\| < r\}$. Therefore, by the Brouwer fixed point theorem [20], there exists $u \in \overline{B(r)}$ such that $u = \mathcal{T}u$.

Step 3. We now show that u is a fixed point of $\mathcal{T}$ iff u is a solution of (2.6), (1.2).First assume $u = \mathcal{T}u$. Then $u \in E$ and thus, satisfies (1.2). Further,

$$\begin{aligned}
\Delta u(t) &= u(t+1) - u(t) \\
&= -\sum_{j_{n-1}=t+n-1}^{T+n-1} \cdots \sum_{j_1=j_2}^{T+n-1} \sum_{i=n-1}^{j_1} \widetilde{h}(i, u(i), \ldots, \Delta^{n-1}u(i-(n-1))) \\
&\quad - \left(-\sum_{j_{n-1}=t+n-2}^{T+n-1} \cdots \sum_{j_1=j_2}^{T+n-1} \sum_{i=n-1}^{j_1} \widetilde{h}(i, u(i), \ldots, \Delta^{n-1}u(i-(n-1))) \right) \\
&= \sum_{j_{n-2}=t+n-2}^{T+n-1} \cdots \sum_{j_1=j_2}^{T+n-1} \sum_{i=n-1}^{j_1} \widetilde{h}(i, u(i), \ldots, \Delta^{n-1}u(i-(n-1))).
\end{aligned}$$

And,

$$\begin{aligned}
\Delta^2 u(t-1) &= \Delta u(t) - \Delta u(t-1) \\
&= \sum_{j_{n-2}=t+n-2}^{T+n-1} \cdots \sum_{j_1=j_2}^{T+n-1} \sum_{i=n-1}^{j_1} \widetilde{h}(i, u(i), \ldots, \Delta^{n-1}u(i-(n-1))) \\
&\quad - \sum_{j_{n-2}=t+n-3}^{T+n-1} \cdots \sum_{j_1=j_2}^{T+n-1} \sum_{i=n-1}^{j_1} \widetilde{h}(i, u(i), \ldots, \Delta^{n-1}u(i-(n-1))) \\
&= -\sum_{j_{n-3}=t+n-3}^{T+n-1} \cdots \sum_{j_1=j_2}^{T+n-1} \sum_{i=n-1}^{j_1} \widetilde{h}(i, u(i), \ldots, \Delta^{n-1}u(i-(n-1))).
\end{aligned}$$

Continuing on in this manner, we see that,

$$\begin{aligned}\Delta^{n-1}u(t-n) &= \Delta^{n-2}u(t-(n-1)) - \Delta^{n-2}u(t-n)\\ &= (-1)^{n-2}\sum_{j_1=t+1}^{T+n-1}\sum_{i=n-1}^{j_1}\widetilde{h}(i,u(i),\ldots,\Delta^{n-1}u(i-(n-1)))\\ &\quad -(-1)^{n-2}\sum_{j_1=t}^{T+n-1}\sum_{i=n-1}^{j_1}\widetilde{h}(i,u(i),\ldots,\Delta^{n-1}u(i-(n-1)))\\ &= (-1)^{n-1}\sum_{i=n-1}^{t}\widetilde{h}(i,u(i),\ldots,\Delta^{n-1}u(i-(n-1))).\end{aligned}$$

Finally,

$$\begin{aligned}\Delta^{n}u(t-(n-1)) &= \Delta^{n-1}u(t-(n-2)) - \Delta^{n-1}u(t-(n-1))\\ &= (-1)^{n-1}\sum_{i=n-1}^{t}\widetilde{h}(i,u(i),\ldots,\Delta^{n-1}u(i-(n-1)))\\ &\quad -(-1)^{n-1}\sum_{i=n-1}^{t+1}\widetilde{h}(i,u(i),\ldots,\Delta^{n-1}u(i-(n-1)))\\ &= (-1)^{n}\widetilde{h}(t+1,u(t+1),\ldots,\Delta^{n-1}u(t-(n-2))).\end{aligned}$$

In particular, $(-1)^{n-1}\Delta^n u(t-(n-1)) + \widetilde{h}(t,u(t),\ldots,\Delta^{n-1}u(t-(n-1))) = 0$ and (2.6) is satisfied. Now assume $u(t)$ solves (2.6), (1.2). Then $u \in E$ and from (1.2) we get

$$\begin{aligned}(-1)^n\Delta^n u(0) &= (-1)^n\Delta^{n-1}u(1) - (-1)^n\Delta^{n-1}u(0)\\ &= (-1)^n\Delta^{n-1}u(1)\\ &= \widetilde{h}(n-1,u(n-1),\ldots,\Delta^{n-1}u(0)).\end{aligned}$$

Thus, $(-1)^n\Delta^{n-1}u(1) = \widetilde{h}(n-1,u(n-1),\ldots,\Delta^{n-1}u(0))$. Also,

$$\begin{aligned}(-1)^n\Delta^n u(1) &= (-1)^n\Delta^{n-1}u(2) - (-1)^n\Delta^{n-1}u(1)\\ &= (-1)^n\Delta^{n-1}u(2) - \widetilde{h}(n-1,u(n-1),\ldots,\Delta^{n-1}u(0))\\ &= \widetilde{h}(n,u(n),\ldots,\Delta^{n-1}u(1)).\end{aligned}$$

Thus, $(-1)^n\Delta^{n-1}u(2) = \sum_{i=n-1}^{n}\widetilde{h}(i,u(i),\ldots,\Delta^{n-1}u(i-(n-1)))$. Continuing inductively, we conclude

$$(-1)^n\Delta^{n-1}u(t-(n-2)) = \sum_{i=n-1}^{t}\widetilde{h}(i,u(i),\ldots,\Delta^{n-1}u(i-(n-1))). \tag{2.8}$$

Note here that if $a > b$, we will use the convention that $\sum_{a}^{b} = 0$. From (2.8) and (1.2), we get

$$\begin{aligned}(-1)^n\Delta^{n-1}u(T+1) &= (-1)^n\Delta^{n-2}u(T+2) - (-1)^n\Delta^{n-2}u(T+1)\\ &= -(-1)^n\Delta^{n-2}u(T+1)\\ &= \sum_{i=n-1}^{T+n-1}\widetilde{h}(i,u(i),\ldots,\Delta^{n-1}u(i-(n-1))).\end{aligned}$$

Thus, $(-1)^{n-1}\Delta^{n-2}u(T+1) = \sum_{i=n-1}^{T+n-1}\widetilde{h}(i,u(i),\ldots,\Delta^{n-1}u(i-(n-1)))$. Also,

$$\begin{aligned}(-1)^n\Delta^{n-1}u(T) &= (-1)^n\Delta^{n-2}u(T+1) - (-1)^n\Delta^{n-2}u(T)\\ &= -\sum_{i=n-1}^{T+n-1}\widetilde{h}(i,u(i),\ldots,\Delta^{n-1}u(i-(n-1)))\\ &\quad -(-1)^n\Delta^{n-2}u(T)\\ &= \sum_{i=n-1}^{T+n-2}\widetilde{h}(i,u(i),\ldots,\Delta^{n-1}u(i-(n-1))).\end{aligned}$$

Thus, $(-1)^{n-1}\Delta^{n-2}u(T) = \sum_{j_1=T+n-2}^{T+n-1}\sum_{i=n-1}^{j_1}\widetilde{h}(i,u(i),\ldots,\Delta^{n-1}u(i-(n-1)))$. Continuing inductively, we conclude

$$(-1)^{n-1}\Delta^{n-2}u(t-(n-1)) = \sum_{j_1=t-1}^{T+n-1}\sum_{i=n-1}^{j_1}\widetilde{h}(i,u(i),\ldots,\Delta^{n-1}u(i-(n-1))). \quad (2.9)$$

From (2.9) and (1.2), we get

$$\begin{aligned}(-1)^{n-1}\Delta^{n-2}u(T+2) &= (-1)^{n-1}\Delta^{n-3}u(T+3) - (-1)^{n-1}\Delta^{n-3}u(T+2)\\ &= -(-1)^{n-1}\Delta^{n-3}u(T+2)\\ &= 0.\end{aligned}$$

Thus, $(-1)^{n-2}\Delta^{n-3}u(T+2) = 0$. Also,

$$\begin{aligned}\Delta(-1)^{n-1}\Delta^{n-2}u(T+1) &= (-1)^{n-1}\Delta^{n-3}u(T+2) - (-1)^{n-1}\Delta^{n-3}u(T+1)\\ &= -(-1)^{n-1}\Delta^{n-3}u(T+1)\\ &= \sum_{j_1=T+n-1}^{T+n-1}\sum_{i=n-1}^{j_1}\widetilde{h}(i,u(i),\ldots,\Delta^{n-1}u(i-(n-1))).\end{aligned}$$

Thus, $(-1)^{n-2}\Delta^{n-3}u(T+1) = \sum_{j_1=T+n-1}^{T+n-1}\sum_{i=n-1}^{j_1}\widetilde{h}(i,u(i),\ldots,\Delta^{n-1}u(i-(n-1)))$. Also,

$$\begin{aligned}(-1)^{n-1}\Delta^{n-2}u(T) &= (-1)^{n-1}\Delta^{n-3}u(T+1)-(-1)^{n-1}\Delta^{n-3}u(T)\\ &= -\sum_{j_1=T+n-1}^{T+n-1}\sum_{i=n-1}^{j_1}\widetilde{h}(i,u(i),\ldots,\Delta^{n-1}u(i-(n-1)))\\ &\quad -(-1)^{n-1}\Delta^{n-3}u(T)\\ &= \sum_{j_1=T+n-2}^{T+n-1}\sum_{i=n-1}^{j_1}\widetilde{h}(i,u(i),\ldots,\Delta^{n-1}u(i-(n-1))).\end{aligned}$$

Hence, $(-1)^{n-2}\Delta^{n-3}u(T) = \sum_{j_2=T+n-2}^{T+n-1}\sum_{j_1=j_2}^{T+n-1}\sum_{i=n-1}^{j_1}\widetilde{h}(i,u(i),\ldots,\Delta^{n-1}u(i-(n-1)))$. Continuing inductively, we conclude

$$(-1)^{n-2}\Delta^{n-3}u(t-(n-1)) = \sum_{j_2=t-1}^{T+n-1}\sum_{j_1=j_2}^{T+n-1}\sum_{i=n-1}^{j_1}\widetilde{h}(i,u(i),\ldots,\Delta^{n-1}u(i-(n-1))).$$

Continuing in this manner, we notice that

$$\Delta u(t-(n-1)) = \sum_{j_{n-2}=t-1}^{T+n-1}\sum_{j_{n-3}=j_{n-2}}^{T+n-1}\cdots\sum_{j_1=j_2}^{T+n-1}\sum_{i=n-1}^{j_1}\widetilde{h}(i,u(i),\ldots,\Delta^{n-1}u(i-(n-1))).$$

Thus, we see that

$$\begin{aligned}\Delta u(T+n-1) &= u(T+n)-u(T+n-1)\\ &= -u(T+n-1)\\ &= 0.\end{aligned}$$

In a similar manner,

$$u(T+n-2) = \ldots = u(T+3) = 0.$$

Thus,

$$\begin{aligned}\Delta u(T+2) &= u(T+3)-u(T+2)\\ &= -u(T+2)\\ &= 0,\end{aligned}$$

which implies that $u(T+2)=0$. And

$$\begin{aligned}\Delta u(T+1) &= u(T+2)-u(T+1)\\ &= -u(T+1)\\ &= \sum_{j_{n-2}=T+n-1}^{T+n-1}\sum_{j_{n-3}=j_{n-2}}^{T+n-1}\cdots\sum_{j_1=j_2}^{T+n-1}\sum_{i=n-1}^{j_1}\widetilde{h}(i,u(i),\ldots,\Delta^{n-1}u(i-(n-1))).\end{aligned}$$

Thus,

$$-u(T+1) = \sum_{j_{n-2}=T+n-1}^{T+n-1} \sum_{j_{n-3}=j_{n-2}}^{T+n-1} \cdots \sum_{j_1=j_2}^{T+n-1} \sum_{i=n-1}^{j_1} \widetilde{h}(i, u(i), \ldots, \Delta^{n-1}u(i-(n-1))).$$

Also,

$$\begin{aligned} \Delta u(T) &= u(T+1) - u(T) \\ &= -\sum_{j_{n-2}=T+n-1}^{T+n-1} \sum_{j_{n-3}=j_{n-2}}^{T+n-1} \cdots \\ &\quad \times \sum_{j_1=j_2}^{T+n-1} \sum_{i=n-1}^{j_1} \widetilde{h}(i, u(i), \ldots, \Delta^{n-1}u(i-(n-1))) - u(T) \\ &= \sum_{j_{n-2}=T+n-2}^{T+n-1} \sum_{j_{n-3}=j_{n-2}}^{T+n-1} \cdots \sum_{j_1=j_2}^{T+n-1} \sum_{i=n-1}^{j_1} \widetilde{h}(i, u(i), \ldots, \Delta^{n-1}u(i-(n-1))). \end{aligned}$$

Thus,

$$-u(T) = \sum_{j_{n-1}=T+n-2}^{T+n-1} \sum_{j_{n-2}=j_{n-1}}^{T+n-1} \cdots \sum_{j_1=j_2}^{T+n-1} \sum_{i=n-1}^{j_1} \widetilde{h}(i, u(i), \ldots, \Delta^{n-1}u(i-(n-1))).$$

Continuing the pattern, we notice that

$$-u(t-(n-1)) = \sum_{j_{n-1}=t-1}^{T+n-1} \sum_{j_{n-2}=j_{n-1}}^{T+n-1} \cdots \sum_{j_1=j_2}^{T+n-1} \sum_{i=n-1}^{j_1} \widetilde{h}(i, u(i), \ldots, \Delta^{n-1}u(i-(n-1))).$$

Therefore, $u = Tu$ and this step is completed.

Step 4. We now show that solutions $u(t)$ of (2.6), (1.2) satisfy,

$$\alpha(t) \le u(t) \le \beta(t), \quad t \in [0, T+2].$$

Consider the case of obtaining $u(t) \le \beta(t)$. Let $v(t) = \beta(t) - u(t)$ and then consider $\Delta^{n-2}v(t) = \Delta^{n-2}\beta(t) - \Delta^{n-2}u(t)$. For the sake of establishing a contradiction, assume that $\max\{\Delta^{n-2}v(t) : t \in [0, T+2]\} = \Delta^{n-2}v(l) > 0$. Conditions (1.2) and (2.5) imply that $l \in [1, T+1]$. Thus, $\Delta^{n-2}v(l+1) \le \Delta^{n-2}v(l)$ and $\Delta^{n-2}v(l-1) \le \Delta^{n-2}v(l)$. Consequently, $\Delta^{n-1}v(l) \le 0$ and $\Delta^{n-1}v(l-1) \ge 0$. This in turn implies that $\Delta^n v(l-1) \le 0$. Therefore,

$$\Delta^n \beta(l-1) \le \Delta^n u(l-1). \tag{2.10}$$

On the other hand, since h is nonincreasing in its x_{n-1} variable, from (2.1) we have

$$\begin{aligned}\Delta^n\beta(l-1)-\Delta^n u(l-1) &= (-1)^{n-1}\widetilde{h}(l+n,u(l+n),\ldots,\Delta^{n-1}u(l-1)) \\ &\quad +\Delta^n\beta(l-1) \\ &\geq (-1)^{n-1}h(l+n,\beta(l+n),\ldots,\Delta^{n-1}\beta(l-1)) \\ &\quad +\frac{\Delta^{n-2}v(l)}{\Delta^{n-2}v(l)+1}+\Delta^n\beta(l-1) \\ &\geq -\Delta^n\beta(l-1)+\frac{\Delta^{n-2}v(l)}{\Delta^{n-2}v(l)+1}+\Delta^n\beta(l-1) \\ &= \frac{\Delta^{n-2}v(l)}{\Delta^{n-2}v(l)+1} \\ &> 0.\end{aligned}$$

Hence, $\Delta^n\beta(l-1) > \Delta^n u(l-1)$, but this contradicts (2.10). Therefore, $\Delta^{n-2}v(l) \leq 0$. This implies that $\Delta^{n-2}\beta(l) \leq \Delta^{n-2}u(l)$, and hence, from repeated applications of boundary conditions (1.2) and (2.5) along with summing both sides of $\Delta^{n-2}\beta(l) \leq \Delta^{n-2}u(l)$, we see that $u(t) \leq \beta(t)$. A similar argument shows that $\alpha(t) \leq u(t)$. Thus, the conclusion of the theorem holds and our proof is complete. ■

3. Main Result

In this section, we make use of Theorem 2.3 to obtain positive solutions of the singular problem (1.1), (1.2). In particular, in applying Theorem 2.3, we deal with a sequence of regular perturbations of (1.1), (1.2). Ultimately, we obtain a desired solution of (1.1), (1.2) by passing to the limit on a sequence of solutions for the perturbations.

Theorem 3.1. Assume conditions (A), (B), and (C) hold, along with the following:

(D): There exists $c \in (0,\infty)$ so that $(-1)^n f(t,c,0,\ldots,0) \leq 0$ for all $t \in [n-1,T+1]$.

(E): $f(t,x_0,\ldots,x_{n-1})$ is nonincreasing in its x_{n-1} variable for $t \in [n-1,T+1]$ and $x_0 \in (0,c]$.

(F): $\lim_{x_0\to 0^+} f(t,x_0,\ldots,x_{n-1}) = \infty$ for $t \in [n-1,T+1], x_1 \in (-c,c)$.

Then (1.1), (1.2) has a solution u satisfying,

$$0 < u(t) \leq c, \quad t \in [0,T+1].$$

Proof. Again for the proof, we proceed through a sequence of steps.

Step 1. For $k \in \mathbb{N}, t \in [n-1,T+1], (x_0,\ldots,x_{n-1}) \in \mathbb{R}^n$, define

$$f_k(t,x_0,\ldots,x_{n-1}) = \begin{cases} f(t,|x_0|,x_1,\ldots,x_{n-1}), & |x_0| \geq \dfrac{1}{k}, \\ f\left(t,\dfrac{1}{k},x_1,\ldots,x_{n-1}\right), & |x_0| < \dfrac{1}{k}. \end{cases}$$

Then f_k is continuous on $[n-1, T+1]\times\mathbb{R}^n$ and non-increasing for $t \in [n-1, T+1], x_0 \in [-c, c]$. Assumption (F) implies that there exists k_0, such that, for all $k \geq k_0$,

$$f_k(t, 0, \ldots, 0) = f\left(t, \frac{1}{k}, 0, \ldots, 0\right) > 0, \quad t \in [n-1, T+1].$$

Consider,

$$\begin{aligned}(-1)^n \Delta^n u(t-(n-1)) + f_k(t, u(t), \ldots, \Delta^{n-1}u(t-(n-1))) = 0,\\ t \in [n-1, T+1]. \end{aligned} \tag{3.1}$$

Define $\alpha(t) = 0$ and $\beta(t) = c$. Then α and β are lower and upper solutions for (3.1), (1.2) and $\alpha(t) \leq \beta(t)$ on $[0, T+n]$. Thus, by Theorem 2.3, there exists u_k a solution of (3.1), (1.2) satisfying $0 \leq u_k(t) \leq c, t \in [0, T+n], k \geq k_0$. Consequently,

$$|\Delta u_k(t)| \leq c, \quad t \in [0, T+(n-1)]. \tag{3.2}$$

Step 2. Let $k \in \mathbb{N}, k \geq k_0$. Since $u_k(t)$ solves (3.1), we get from our work in Theorem 2.3,

$$\Delta u_k(t) = \sum_{j_{n-2}=t+n-2}^{T+n-1} \sum_{j_{n-3}=j_{n-2}}^{T+n-1} \cdots \sum_{j_1=j_2}^{T+n-1} \sum_{i=n-1}^{j_1} f_k(i, u_k(i), \ldots, \Delta^{n-1}u_k(i-(n-1))). \tag{3.3}$$

By assumption (F), there exists $\varepsilon_1 \in \left(0, \dfrac{1}{k_0}\right)$ such that if $k \geq \dfrac{1}{\varepsilon_1}$,

$$f_k(n-1, x_0, \ldots, x_{n-1}) > c, \quad x_0 \in (0, \varepsilon_1], x_1 \in [-c, c]. \tag{3.4}$$

Assume $k \geq \dfrac{1}{\varepsilon_1}$ and that $u_k(1) < \varepsilon_1$. Then, by (3.3) and (3.4),

$$\begin{aligned}
\Delta u_k(1) &= \sum_{j_{n-2}=n-1}^{T+n-1} \sum_{j_{n-3}=j_{n-2}}^{T+n-1} \cdots \sum_{j_1=j_2}^{T+n-1} \sum_{i=n-1}^{j_1} f_k(i, u_k(i), \ldots, \Delta^{n-1}u_k(i-(n-1))) \\
&= \sum_{j_{n-2}=n}^{T+n-1} \sum_{j_{n-3}=j_{n-2}}^{T+n-1} \cdots \sum_{j_1=j_2}^{T+n-1} \sum_{i=n-1}^{j_1} f_k(i, u_k(i), \ldots, \Delta^{n-1}u_k(i-(n-1))) \\
&\quad + f_k(n-1, u_k(n-1), \ldots, \Delta^{n-1}u_k(0)) \\
&> c + \sum_{j_{n-2}=n}^{T+n-1} \sum_{j_{n-3}=j_{n-2}}^{T+n-1} \cdots \sum_{j_1=j_2}^{T+n-1} \sum_{i=n-1}^{j_1} f_k(i, u_k(i), \ldots, \Delta^{n-1}u_k(i-(n-1))) \\
&> c.
\end{aligned}$$

But this contradicts (3.2). Hence $u_k(1) \geq \varepsilon_1$, for all $k \geq \dfrac{1}{\varepsilon_1}$. Denote

$$m_2 = \max\{|f_k(n-1, x_0, \ldots, x_{n-1})| : x_0 \in [\varepsilon_1, c], x_1 \in [-c, c]\}.$$

By assumption (F), there exists $\varepsilon_2 \in (0, \varepsilon_1]$ such that, if $k \geq \dfrac{1}{\varepsilon_2}$ and $u_k < \varepsilon_2$, then

$$f_k(n, x_0, \ldots, x_{n-1}) > c - T^{n-2} \cdot m_2, \quad x_0 \in (0, \varepsilon_2], x_1 \in [-c, c].$$

Hence,

$$\begin{aligned}
\Delta u_k(2) &= \sum_{j_{n-2}=n}^{T+n-1} \sum_{j_{n-3}=j_{n-2}}^{T+n-1} \cdots \sum_{j_1=j_2}^{T+n-1} \sum_{i=n-1}^{j_1} f_k(i, u_k(i), \ldots, \Delta^{n-1} u_k(i-(n-1))) \\
&= \sum_{j_{n-2}=n}^{T+n-1} \sum_{j_{n-3}=j_{n-2}}^{T+n-1} \cdots \sum_{j_1=j_2}^{T+n-1} \sum_{i=n}^{j_1} f_k(i, u_k(i), \ldots, \Delta^{n-1} u_k(i-(n-1))) \\
&\quad + T^{n-2} \cdot f_k(n-1, u_k(n-1), \ldots, \Delta^{n-1} u_k(0)) \\
&= \sum_{j_{n-2}=n+1}^{T+n-1} \sum_{j_{n-3}=j_{n-2}}^{T+n-1} \cdots \sum_{j_1=j_2}^{T+n-1} \sum_{i=n}^{j_1} f_k(i, u_k(i), \ldots, \Delta^{n-1} u_k(i-(n-1))) \\
&\quad + f_k(n, u_k(n), \ldots, \Delta^{n-1} u_k(1)) \\
&\quad + T^{n-2} \cdot f_k(n-1, u_k(n-1), \ldots, \Delta^{n-1} u_k(0)) \\
&> \sum_{j_{n-2}=n}^{T+n-1} \sum_{j_{n-3}=j_{n-2}}^{T+n-1} \cdots \sum_{j_1=j_2}^{T+n-1} \sum_{i=n}^{j_1} f_k(i, u_k(i), \ldots, \Delta^{n-1} u_k(i-(n-1))) \\
&\quad + f_k(n, u_k(n), \ldots, \Delta^{n-1} u_k(1)) + T^{n-2} m_2 \\
&> f_k(n, u_k(n), \ldots, \Delta^{n-1} u_k(1)) + T^{n-2} m_2 \\
&> c.
\end{aligned}$$

But this contradicts (3.2). Hence $u_k(2) \geq \varepsilon_2$, for all $k \geq \dfrac{1}{\varepsilon_2}$. Continuing similarly for $t = 3, 4, \ldots, T$, we get $0 < \varepsilon_T < \cdots < \varepsilon_2 < \varepsilon_1$ such that $u_k(t) \geq \varepsilon_T$, for $t \in [1, T]$. For $2 \leq i \leq T$, denote $m_i = \max\{|f_k(n+i-3, x_0, \ldots, x_{n-1})| : x_0 \in [\varepsilon_i, c], x_1 \in [-c, c]\}$. By assumption (F), there exists $\varepsilon_{T+1} \in (0, \varepsilon_T]$ such that, if $k \geq \dfrac{1}{\varepsilon_{T+1}}$ and $u_k(T+1) < \varepsilon_{T+1}$, then

$$f_k(T+n-2, x_0, \ldots, x_{n-1}) > c - \sum_{i=2}^{T} m_i, \quad x_0 \in (0, \varepsilon_T], x_1 \in [-c, c].$$

Hence,

$$\Delta u_k(T+1) = \sum_{j_{n-2}=T+n-1}^{T+n-1} \sum_{j_{n-3}=j_{n-2}}^{T+n-1} \cdots \sum_{j_1=j_2}^{T+n-1} \sum_{i=n-1}^{j_1} f_k(i, u_k(i), \ldots,$$

$$
\begin{aligned}
&\Delta^{n-1}u_k(i-(n-1))) \\
= & \sum_{i=n-1}^{T+n-1} f_k(i, u_k(i), \ldots, \Delta^{n-1}u_k(i-(n-1))) \\
= & f_k(T+n-1, u(T+n-1), \ldots, \Delta^{n-1}u(T)) \\
& + f_k(T+n-2, u(T+n-2), \ldots, \Delta^{n-1}u(T-1)) \\
& + \sum_{i=2}^{T} f_k(n+i-3, u_k(n+i-3), \ldots, \Delta^{n-1}u_k(i-2)) \\
> & f_k(T+n-2, u(T+n-2), \ldots, \Delta^{n-1}u(T-1)) \\
& + \sum_{i=2}^{T} m_i \\
> & c.
\end{aligned}
$$

But this contradicts (3.2). Hence $u_k(T+1) \geq \varepsilon_{T+1}$, for all $k \geq \dfrac{1}{\varepsilon_{T+1}}$. Therefore, by letting $\varepsilon = \varepsilon_{T+1}$, we get

$$
0 < \varepsilon \leq u_k(t) \leq c, \quad t \in [0, T+2], k \geq \frac{1}{\varepsilon}. \tag{3.5}
$$

Since $u_k(t)$ satisfies (3.5) and (1.2), we can choose a subsequence $\{u_{k_n}(t)\} \subset \{u_k(t)\}$ such that $\lim_{n\to\infty} u_{k_n}(t) = u(t), \quad t \in [0, T+n], u(t) \in E$, where E is as defined in Step 2 of Theorem 2.3. Now,

$$
\Delta u_{k_n}(t) = \sum_{j_{n-2}=t+n-2}^{T+n-1} \sum_{j_{n-3}=j_{n-2}}^{T+n-1} \cdots \sum_{j_1=j_2}^{T+n-1} \sum_{i=n-1}^{j_1} f(i, u_{k_n}(i), \ldots, \Delta^{n-1}u_{k_n}(i-(n-1))),
$$

and so letting $n \to \infty$ and from the continuity of f, we get that

$$
\Delta u(t) = \sum_{j_{n-2}=t+n-2}^{T+n-1} \sum_{j_{n-3}=j_{n-2}}^{T+n-1} \cdots \sum_{j_1=j_2}^{T+n-1} \sum_{i=n-1}^{j_1} f(i, u(i), \ldots, \Delta^{n-1}u(i-(n-1))).
$$

Consequently, via similar methods used in Step 3 of Theorem 2.3,

$$
(-1)^{n-1}\Delta^n u(t-(n-1)) = f(t, u(t), \ldots, \Delta^{n-1}u(t-(n-1))).
$$

Therefore, u solves (1.1), and by (3.5), our theorem holds. ■

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