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Stability of a Class of Singular Difference Equations

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Abstract

The aim of this paper is to apply Lyapunov functions to obtain some necessary and sufficient conditions for the stability of singular nonautonomous difference equations.

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1. Introduction

In practice, many problems are modeled by *singular difference equations* (SDEs). Recently, a class of singular nonautonomous difference equations, called index-1 SDEs, has been introduced and the solvability of *initial-value problems* (IVPs) as well as *boundary-value problems* (BVPs) has been studied (cf., [1–3, 5, 6]). Moreover, the Floquet theory has been developed for linear index-1 SDEs [4].

In this paper we apply Lyapunov functions to study stability properties of singular quasilinear difference equations. The paper is organized as follows. In Section 2 we provide basic concepts of index-1 SDEs. The unique solvability of the IVP for a class of SDEs is also established. Section 3 deals with various stability conditions for SDEs. Finally, in Section 4 some illustrative examples are considered.

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2. Basic Concepts

Consider a system

$$A_n x_{n+1} + B_n x_n = f_n(x_n) \quad (n \geqslant 0), \tag{2.1}$$

where $A_n, B_n \in \mathbb{R}^{m\times m}$ and $f_n : \mathbb{R}^m \to \mathbb{R}^m$ are given. Throughout this paper, we assume that the matrices A_n are singular, so Equation (2.1) is called an SDE. In what follows we suppose that the corresponding linear homogeneous equation

$$A_n x_{n+1} + B_n x_n = 0 \quad (n \geqslant 0), \tag{2.2}$$

is of index-1 [1–6], i.e., the following hypotheses are assumed to be fulfilled:

(H1) $\text{rank} A_n = r \quad (n \geqslant 0)$,

(H2) $S_n \cap \ker A_{n-1} = \{0\} \quad (n \geqslant 1)$,

where, $S_n = \{\xi \in \mathbb{R}^m : B_n\xi \in \text{im} A_n\}, (n \geqslant 0)$.

In what follows we always assume that $\dim S_0 = r$ and let $A_{-1} \in \mathbb{R}^{m\times m}$ be a fixed chosen matrix, such that $\mathbb{R}^m = S_0 \oplus \ker A_{-1}$, so the hypothesis H2 is satisfied for all $n \geqslant 0$.

Let $Q_n \in \mathbb{R}^{m\times m}$ be an arbitrary projection onto $\ker A_n$, ($n \geqslant -1$), i.e., $Q_n^2 = Q_n$ and $\text{im} Q_n = \ker A_n$. Then there exists a nonsingular matrix $V_n \in \mathbb{R}^{m\times m}$ such that $Q_n = V_n \tilde{Q} V_n^{-1}$, where $\tilde{Q} := \text{diag}(O_r, I_{m-r})$ and O_r, I_{m-r} stand for $r \times r$ zero and $(m-r)\times(m-r)$ identity matrices, respectively. Further we define the matrix $P_n := I - Q_n$ and the so-called connecting operators $Q_{n-1,n} := V_{n-1}\tilde{Q}V_n^{-1}$ and $Q_{n,n-1} := V_n\tilde{Q}V_{n-1}^{-1}$. Obviously, the connecting operators associated with projections Q_n, Q_{n-1} satisfy the identities $Q_{n-1,n} = Q_{n-1}Q_{n-1,n} = Q_{n-1,n}Q_n,\ Q_{n-1,n}Q_{n,n-1} = Q_{n-1}$ and $Q_{n,n-1}Q_{n-1,n} = Q_n$.

For the next discussion, the following lemma from [4] is needed.

Lemma 2.1. Suppose the hypothesis H1 is fulfilled. Then the hypothesis H2 is equivalent to one of the following statements:

i) the matrix $G_n := A_n + B_n Q_{n-1,n}$ is nonsingular.

ii) $\mathbb{R}^m = S_n \oplus \ker A_{n-1}$.

It is proved [4], that every index-1 SDE (2.2) can be reduced to the Kronecker normal form

$$\text{diag}(I_r, O_{m-r})y_{n+1} + \text{diag}(W_n, I_{m-r})y_n = 0.$$

Let us associate the SDE (2.1) with the initial condition

$$P_{n_0-1}x_{n_0} = P_{n_0-1}\gamma, \quad n_0 \geqslant 0, \tag{2.3}$$

where γ is an arbitrary vector in $\mathbb{R}^m$ and n_0 is a fixed nonnegative integer.

Theorem 2.2. Let $f_n(x)$ be a Lipschitz continuous function with a sufficient small Lipschitz coefficient, i.e.,

$$\|f_n(x) - f_n(\tilde{x})\| \leqslant L_n \|x - \tilde{x}\|, \quad \forall x, \tilde{x} \in \mathbb{R}^m, \tag{2.4}$$

where

$$\omega_n := L_n \|Q_{n-1,n} G_n^{-1}\| < 1, \quad \forall n \geqslant 0. \tag{2.5}$$

Then the IVP (2.1), (2.3) has a unique solution.

Proof. The conclusion of this theorem follows from [5, Theorem 1]. However, to make our presentation self-contained we shall provide a straight-forward proof of this fact. Indeed, multiplying on both sides of Equation (2.1) from the left by $P_n G_n^{-1}$ and $Q_n G_n^{-1}$, respectively and observing that $G_n^{-1} A_n = P_n, P_n Q_n = Q_n P_n = O$ we get

$$P_n x_{n+1} + P_n G_n^{-1} B_n x_n = P_n G_n^{-1} f_n(x_n), \tag{2.6}$$

$$Q_n G_n^{-1} B_n x_n = Q_n G_n^{-1} f_n(x_n). \tag{2.7}$$

Putting $u_n := P_{n-1} x_n, v_n := Q_{n-1} x_n, (n \geqslant 0)$ and noting that

$$P_n G_n^{-1} B_n Q_{n-1} x_n = P_n G_n^{-1} B_n Q_{n-1,n} Q_{n,n-1} x_n = P_n Q_{n,n-1} x_n = 0,$$

from (2.6) we find

$$u_{n+1} = -P_n G_n^{-1} B_n u_n + P_n G_n^{-1} f_n(u_n + v_n). \tag{2.8}$$

Now using the fact that $Q_n = G_n^{-1} B_n Q_{n-1,n}$ we can express the left side of (2.7) as

$$\begin{aligned} Q_n G_n^{-1} B_n x_n &= Q_n G_n^{-1} B_n u_n + Q_n G_n^{-1} B_n Q_{n-1,n} Q_{n,n-1} x_n \\ &= Q_n G_n^{-1} B_n u_n + Q_{n,n-1} x_n. \end{aligned}$$

Thus the relation (2.7) becomes

$$Q_{n,n-1} x_n = -Q_n G_n^{-1} B_n u_n + Q_n G_n^{-1} f_n(x_n).$$

Now acting $Q_{n-1,n}$ on both sides of the last relation we get

$$v_n = Q_{n-1} x_n = Q_{n-1,n} G_n^{-1} \left\{ f_n(u_n + v_n) - B_n u_n \right\}. \tag{2.9}$$

Supposing $u := u_n \ (n \geqslant n_0)$ is known, where $u_{n_0} = P_{n_0-1} x_{n_0} = P_{n_0-1} \gamma$ is given, we consider an operator $T_n : \text{im}Q_{n-1} \to \text{im}Q_{n-1}$ defined by

$$T_n(v) := Q_{n-1,n} G_n^{-1} \left\{ f_n(u + v) - B_n u \right\}.$$

Since

$$\|T_n(v) - T_n(\tilde{v})\| \leqslant L_n \|Q_{n-1,n} G_n^{-1}\| \|v - \tilde{v}\| = \omega_n \|v - \tilde{v}\|,$$

the operator T_n is a contraction. Hence there exists an operator $g_n : \text{im}P_{n-1} \to \text{im}Q_{n-1}$ giving the unique solution of (2.9) whenever u_n is known. Moreover, g_n is Lipschitz continuous with the Lipschitz constant $\beta_n := \omega_n(L_n + \|B_n\|)L_n^{-1}(1-\omega_n)^{-1}$. Obviously, the unique solution of the IVP (2.1), (2.3) is given by

$$x_n = u_n + g_n(u_n), \tag{2.10}$$

where $g_n(u_n)$ is a unique solution of (2.9) with $u_{n_0} = P_{n_0-1}\gamma$. ■

In what follows without loss of generality we will assume that $f_n(0) = 0$ for all $n \geqslant 0$. Then $g_n(0) = 0$ and Equation (2.1) always possesses a trivial solution $x_n \equiv 0$ $(n \geqslant 0)$. From (2.10) it implies that each solution x_n of the IVP (2.1), (2.3) satisfies the relation $x_n = P_{n-1}x_n + g_n(P_{n-1}x_n)$ or equivalently,

$$Q_{n-1}x_n = Q_{n-1,n}G_n^{-1}\{f_n(x_n) - B_nP_{n-1}x_n\}.$$

Set $\Delta_n := \left\{x \in \mathbb{R}^m : Q_{n-1}x = Q_{n-1,n}G_n^{-1}[f_n(x) - B_nP_{n-1}x]\right\}$. If $x = \{x_n\}$ is any solution of the IVP (2.1), (2.3), then obviously, $x_n \in \Delta_n$ $(n \geqslant n_0)$. Conversely, for each $\alpha \in \Delta_n$, there exists a solution of (2.1) passing α. Indeed, let $x_k(n;\alpha)$ $(k \geqslant n)$ be a solution of (2.1) satisfying the initial condition $P_{n-1}x_n = P_{n-1}\alpha$. Clearly,

$$x_n(n;\alpha) = P_{n-1}x_n + g_n(P_{n-1}x_n) = P_{n-1}\alpha + g_n(P_{n-1}\alpha) = \alpha.$$

The following lemma shows that the set Δ_n does not depend on the choice of projections.

Lemma 2.3. The following hold:

i) $\Delta_n = \Omega_n := \{x \in \mathbb{R}^m : f_n(x) - B_nx \in \text{im}A_n\}$ $(n \geqslant 0)$.

ii) $\Omega_n \cap \ker A_{n-1} = \{0\}$.

Proof. i) Letting $x \in \Delta_n$ we have $Q_{n-1}x = Q_{n-1,n}G_n^{-1}\{f_n(x) - B_nP_{n-1}x\}$, hence

$$x = P_{n-1}x + Q_{n-1}x = Q_{n-1,n}G_n^{-1}f_n(x) + (I - Q_{n-1,n}G_n^{-1}B_n)P_{n-1}x.$$

From the last relation we get

$$f_n(x) - B_nx = (I - B_nQ_{n-1,n}G_n^{-1})f_n(x) - B_n(I - Q_{n-1,n}G_n^{-1}B_n)P_{n-1}x.$$

Observing that

$$B_n(I - Q_{n-1,n}G_n^{-1}B_n)P_{n-1}x = (I - B_nQ_{n-1,n}G_n^{-1})B_nP_{n-1}x,$$

we find

$$f_n(x) - B_nx = (I - B_nQ_{n-1,n}G_n^{-1})\{f_n(x) - B_nP_{n-1}x\}.$$

Since $B_nQ_{n-1,n}G_n^{-1} = (G_n - A_n)G_n^{-1} = I - A_nG_n^{-1}$, it implies that

$$f_n(x) - B_nx = A_nG_n^{-1}\{f_n(x) - B_nP_{n-1}x\} \in \mathrm{im}A_n,$$

hence $x \in \Omega_n$. Conversely, let $x \in \Omega_n$, i.e., $f_n(x) - B_nx = A_n\xi$, for some $\xi \in \mathbb{R}^m$. We have to prove that $Q_{n-1}x = Q_{n-1,n}G_n^{-1}(f_n(x) - B_nP_{n-1}x)$, or equivalently,

$$x = Q_{n-1,n}G_n^{-1}[f_n(x) - B_nx] + Q_{n-1,n}G_n^{-1}B_nQ_{n-1}x + P_{n-1}x.$$

Denoting the right-hand side of the last relation by w_n and observing that

$$Q_{n-1,n}G_n^{-1}\{f_n(x) - B_nx\} = Q_{n-1,n}G_n^{-1}A_n\xi = Q_{n-1,n}P_n\xi = 0$$

we find

$$\begin{aligned} w_n &= Q_{n-1,n}G_n^{-1}B_nQ_{n-1}x + P_{n-1}x \\ &= Q_{n-1,n}(G_n^{-1}B_nQ_{n-1,n})Q_{n,n-1}x + P_{n-1}x = Q_{n-1}x + P_{n-1}x = x. \end{aligned}$$

Thus, $x \in \Delta_n$ and the first part of Lemma 2.3 is proved.
ii) Let $x \in \Omega_n \cap \ker A_{n-1}$. Then $P_{n-1}x = 0$ and $x \in \Delta_n$, hence $x = P_{n-1}x + g_n(P_{n-1}x) = 0$. The proof of Lemma 2.3 is complete. ■

We end this section by observing that the initial condition (2.3) is equivalent to the condition

$$A_{n_0-1}x_{n_0} = A_{n_0-1}\gamma \quad (n_0 \geqslant 1), \tag{2.11}$$

which is independent of the choice of projections. Indeed, acting on both sides of (2.11) by $G_{n_0-1}^{-1}$ and using the equality $G_{n_0-1}^{-1}A_{n_0-1} = P_{n_0-1}$ we get (2.3). Conversely, multiplying on both sides of (2.3) by A_{n_0-1} and noting that $A_{n_0-1}P_{n_0-1} = A_{n_0-1}$ we obtain (2.11).

For the sake of convenience, we choose a matrix $B_{-1} \in \mathbb{R}^{m\times m}$ such that the matrix pencil $\{A_{-1}, B_{-1}\}$ is of index-1. Then the matrix $G_{-1} = A_{-1} + B_{-1}Q_{-1}$ is nonsingular. Moreover, $A_{-1} = A_{-1}P_{-1}$ and $G_{-1}^{-1}A_{-1} = P_{-1}$. Thus both initial conditions (2.3) and (2.11) are equivalent for all $n_0 \geqslant 0$. The unique solution of the IVP (2.1), (2.3) or (2.1), (2.11) will be denoted by $x_n(n_0;\gamma)$.

3. Stability of Singular Difference Equations

In this section the notions of stability of the trivial solution are introduced and some necessary and sufficient conditions for the stability are established.

We shall restrict ourselves to the canonical projection onto kerA_{n-1}, i.e., the projection from $\mathbb{R}^{m\times m}$ into kerA_{n-1} along S_n and will denote it again by Q_{n-1}. Then $P_{n-1} := I - Q_{n-1}$ is the canonical projection from $\mathbb{R}^{m\times m}$ into S_n along kerA_{n-1}. Thanks to the decomposition $\mathbb{R}^m = S_n \oplus \ker A_{n-1}$ the canonical projections are determined uniquely from the data A_n, B_n and A_{n-1}. Note that if $\tilde{Q}_{n-1}$ is an any projection onto ker$A_{n-1}(n \geqslant 1)$ and $\tilde{Q}_{n-1,n}$ is the associate connecting operator, then the canonical projection Q_n can be computed as $Q_n = \tilde{Q}_{n-1,n}\tilde{G}_n^{-1}B_n$, where $\tilde{G}_n := A_n + B_n\tilde{Q}_{n-1,n}$. We should refer to the work [4] for details.

Let $\mathbb{R}_+$ and $\mathbb{Z}_+$ be the set of nonnegative real numbers and integers, respectively.

Definition 3.1. The trivial solution of (2.1) is said to be

i) A-stable (P-stable) if for each $\epsilon > 0$ and any $n_0 \geqslant 0$ there exists a $\delta = \delta(\epsilon, n_0) \in (0, \epsilon]$ such that $\|A_{n_0-1}\gamma\| < \delta$ ($\|P_{n_0-1}\gamma\| < \delta$) implies $\|x_n(n_0;\gamma)\| < \epsilon$ for all $n \geqslant n_0$.

ii) A-uniformly (P-uniformly) stable if it is A-stable (P-stable) and the number δ mentioned in part i) of this definition does not depend on n_0.

iii) A-asymptotically (P-asymptotically) stable if for any $n_0 \geqslant 0$ there exists a $\delta_0 = \delta_0(n_0) > 0$ such that the inequality $\|A_{n_0-1}\gamma\| < \delta_0$ ($\|P_{n_0-1}\gamma\| < \delta_0$) implies $\|x_n(n_0;\gamma)\| \longrightarrow 0$ ($n \longrightarrow 0$).

Remark 3.2. From the relation $G_n^{-1}A_n = P_n$ and $A_nP_n = A_n$, it is easy to show that the notions A-stability and P-stability are equivalent. The same conclusion is true for the A-asymptotical stability and P-asymptotical stability. That is why in what follows we will drop the prefixes A and P when talking about the stability or asymptotical stability. Further, if the matrices A_n are uniformly bounded, then A-uniform stability implies P-uniform stability. Conversely, if G_n^{-1} have uniformly bounded inverses, then A-uniform stability follows from P-uniform stability.

Denote by $\mathcal{K}$ the class of all continuous and strictly increasing functions ψ from $[0,\infty)$ into itself, such that $\psi(0) = 0$.

Lemma 3.3. The trivial solution of (2.1) is A-uniformly (P-uniformly) stable if and only if there exists a function $\psi \in \mathcal{K}$, such that for any solution x_n of (2.1) and for any nonnegative integer n_0, there holds the inequality

$$\|x_n\| \leqslant \psi(\|A_{n_0-1}x_{n_0}\|) \quad \forall n \geqslant n_0 \tag{3.1}$$

$$(\|x_n\| \leqslant \psi(\|P_{n_0-1}x_{n_0}\|) \quad \forall n \geqslant n_0).$$

Proof. We provide a proof of the lemma for the A-uniform stability case. The remaining P-uniform stability case can be considered similarly.

Suppose first that there exists a function $\psi \in \mathcal{K}$ satisfying the condition (3.1). For each positive ϵ we choose $\delta = \delta(\epsilon) \in (0, \epsilon]$ such that $\psi(\delta) < \epsilon$. If x_n is an arbitrary solution of (2.1) and $\|A_{n_0-1}x_{n_0}\| < \delta$, then

$$\|x_n\| \leqslant \psi(\|A_{n_0-1}x_{n_0}\|) < \psi(\delta) < \epsilon, \quad \forall n \geqslant n_0.$$

Conversely, suppose that the trivial solution of (2.1) is A-uniformly stable, i.e., for each positive ϵ there exists a $\delta = \delta(\epsilon) \in (0, \epsilon]$, such that if x_n is any solution of (2.1) satisfying the inequality $\|A_{n_0-1}x_{n_0}\| < \delta$, where n_0 is a fixed nonnegative integer, then $\|x_n\| < \epsilon$ for all $n \geqslant n_0$. Denote by $\alpha(\epsilon)$ the supremum of such $\delta(\epsilon)$. Clearly, if $\|A_{n_0-1}x_{n_0}\| < \alpha(\epsilon)$ for some n_0, then $\|x_n\| < \epsilon$ for all $n \geqslant n_0$. Further, the function $\alpha(\epsilon)$ is positive and increasing and moreover, $\alpha(\epsilon) \leqslant \epsilon$. Consider a function $\beta(\epsilon)$ defined by $\beta(\epsilon) := \frac{1}{\epsilon}\int_0^{\epsilon} \alpha(t)dt$ for positive ϵ and $\beta(0) := 0$. It is easy to prove that $\beta \in \mathcal{K}$ and $0 < \beta(\epsilon) < \alpha(\epsilon) \leqslant \epsilon$. Then the inverse of β, denoted by ψ will belong to $\mathcal{K}$. Let x_n be a solution of (2.1) and n_0 be a fixed nonnegative integer. Set $\epsilon_n := \|x_n\|$ and consider two possibilities. If $\|x_n\| = 0$, then $\|x_n\| = 0 \leqslant \psi(\|A_{n_0-1}x_{n_0}\|)$, since ψ is nonnegative. Now suppose $\epsilon_n := \|x_n\| > 0$. If $\|A_{n_0-1}x_{n_0}\| < \beta(\epsilon_n)$, then $\|x_n\| < \epsilon_n = \|x_n\|,\ \forall n \geqslant n_0$, which is impossible, hence $\|A_{n_0-1}x_{n_0}\| \geqslant \beta(\epsilon_n)$, therefore

$$\|x_n\| = \epsilon_n \leqslant \beta^{-1}(\|A_{n_0-1}x_{n_0}\|) = \psi(\|A_{n_0-1}x_{n_0}\|),$$

which was to be proved. The proof of Lemma 3.3 is complete. ■

By arguing as in the proof of Lemma 3.3 we come to the following result.

Lemma 3.4. The trivial solution of (2.1) is stable if and only if there exist functions $\psi_n \in \mathcal{K}$, ($n \geqslant 0$), such that for any solution x_n of (2.1) and for each nonnegative integer n_0, there holds the inequality

$$\|x_n\| \leqslant \psi_{n_0}(\|A_{n_0-1}x_{n_0}\|), \quad \forall n \geqslant n_0.$$

Theorem 3.5. The existence of the Lyapunov function $V : \mathbb{Z}_+ \times \mathbb{R}^m \longrightarrow \mathbb{R}_+$ being continuous in the second variable at $\gamma = 0$ and functions $\psi_n \in \mathcal{K}$, such that

i) $V(n, 0) = 0, \quad n \geqslant 0.$

ii) $\|y\| \leqslant V(n, P_{n-1}y) \leqslant \psi_n(\|P_{n-1}y\|), \quad \forall y \in \Delta_n, \quad n \geqslant 0.$

iii) $\Delta V(n, P_{n-1}y_n) := V(n+1, P_n y_{n+1}) - V(n, P_{n-1}y_n) \leqslant 0$

for any solution y_n of (2.1), is a necessary and sufficient condition for the stability of the trivial solution of the SDE (2.1).

Proof. *Necessity*. Suppose that the trivial solution of (2.1) is stable. Then according to Lemma 3.4, there exist functions $\psi_n \in \mathcal{K}$ ($n \geqslant 0$), such that for any solution x_n of (2.1) $\|x_n\| \leqslant \psi_{n_0}(\|A_{n_0-1}x_{n_0}\|),\ \forall n \geqslant n_0$. Define the functions $\varphi_n(t) = \psi_n(\|A_{n-1}\|t),\ t \in [0, \infty)$. Clearly, $\varphi_n \in \mathcal{K}$ and

$$\begin{aligned}\|x_n\| &\leqslant \psi_{n_0}(\|A_{n_0-1}x_{n_0}\|) = \psi_{n_0}(\|A_{n_0-1}P_{n_0-1}x_{n_0}\|)\\ &\leqslant \psi_{n_0}(\|A_{n_0-1}\|\|P_{n_0-1}x_{n_0}\|).\end{aligned}$$

Thus,

$$\|x_n\| \leqslant \varphi_{n_0}(\|P_{n_0-1}x_{n_0}\|) \quad \forall n \geqslant n_0. \tag{3.2}$$

Further, we define the Lyapunov function

$$V(n,\gamma) := \sup_{k\in\mathbb{Z}_+} \|x_{n+k}(n;\gamma)\|; \quad \gamma\in\mathbb{R}^m, \quad n\in\mathbb{Z}_+, \tag{3.3}$$

where $x_{n+k} := x_{n+k}(n;\gamma)$ is the unique solution of (2.1) satisfying the initial condition $P_{n-1}x_n = P_{n-1}\gamma$. The inequality (3.2) ensures the correctness of the definition (3.3). Moreover, $V(n,\gamma) \leqslant \varphi_n(\|P_{n-1}\gamma\|)$, which implies that $V(n,0)=0$ and the continuity of the function V w.r.t. the second variable at $\gamma = 0$. For each $y\in\Delta_n$ we have

$$V(n,P_{n-1}y) := \sup_{l\in\mathbb{Z}_+} \|x_{n+l}(n;P_{n-1}y)\| \geqslant \|x_n(n;P_{n-1}y)\|,$$

where $x_k(n;P_{n-1}y)$ denotes the solution of (2.1) satisfying the initial condition $P_{n-1}x_n(n;P_{n-1}y) = P_{n-1}(P_{n-1}y) = P_{n-1}y$. Since $x_n, y \in \Delta_n$, it follows $x_n(n;P_{n-1}y) = y$, hence

$$V_n(n,P_{n-1}y) \geqslant \|x_n(n;P_{n-1}y)\| = \|y\|.$$

Further, the inequality (3.2) gives

$$V(n,P_{n-1}y) = \sup_{l\in\mathbb{Z}_+} \|x_{n+l}(n;P_{n-1}y)\| \leqslant \varphi_n(\|P_{n-1}y\|).$$

On the other hand, for an arbitrary solution y_n of (2.1), due to the unique solvability of the IVP (2.1), (2.3) we have

$$V(n,P_{n-1}y_n) = \sup_{l\in\mathbb{Z}_+} \|x_{n+l}(n;P_{n-1}y_n)\| = \sup_{l\geqslant 0}\|y_{n+l}\|.$$

Thus

$$\begin{aligned} V(n+1,P_{n+1}y_n) &= \sup_{l\geqslant 0}\|y_{n+l+1}\| = \sup_{l\geqslant 1}\|y_{n+l}\| \\ &\leqslant \sup_{l\geqslant 0}\|y_{n+l}\| = V(n,P_{n-1}y_n), \end{aligned}$$

hence $\Delta V(n,P_{n-1}y_n) \leqslant 0$. The necessity part is proved.
Sufficiency. We argue by contradiction by assuming that the trivial solution of (2.1) is not stable, i.e., there exist a positive ϵ_0 and a nonnegative integer n_0, such that for all $\delta\in(0,\epsilon_0]$, there exists a solution x_n of (2.1) satisfying the inequalities $\|P_{n_0-1}x_{n_0}\| < \delta$ and $\|x_{n_1}\| \geqslant \epsilon_0$ for some $n_1 \geqslant n_0$.

Since $V(n_0,0) = 0$ and $V(n_0,\gamma)$ is continuous at $\gamma = 0$, there exists a $\delta_0' = \delta_0'(\epsilon,n_0) > 0$, such that for all $\xi\in\mathbb{R}^m, \quad \|\xi\| < \delta_0'$ we have $V(n_0,\xi) < \epsilon_0$. Choosing $\delta_0 \leqslant \{\delta_0',\epsilon_0\}$ we can find a solution x_n of (2.1) satisfying $\|P_{n_0-1}x_{n_0}\| < \delta_0$, however $\|x_{n_1}\| \geqslant \epsilon_0$ for some $n_1 \geqslant n_0$. Since $\|P_{n_0-1}x_{n_0}\| < \delta_0 \leqslant \delta_0'$, one gets $V(n_0,P_{n_0-1}x_{n_0}) < \epsilon_0$. On the other hand, using the properties i) and iii) of the function V, we find

$$V(n_0,P_{n_0-1}x_{n_0}) \geqslant V(n_1,P_{n_1-1}x_{n_1}) \geqslant \|x_{n_1}\| \geqslant \epsilon_0,$$

which leads to a contradiction. The proof of Theorem 3.5 is complete. ∎

A similar argument as in the proof of the sufficiency part of Theorem 3.5 leads to the next result.

Theorem 3.6. Assume that there exists a function $\psi \in \mathcal{K}$ and a Lyapunov function $V : \mathbb{Z}_+ \times \mathbb{R}^m \longrightarrow \mathbb{R}_+$, which is continuous w.r.t. the second variable at $\gamma = 0$, such that

i) $V(n, 0) = 0, \quad \forall n \geqslant 0.$

ii) $\psi(\|x\|) \leqslant V(n, A_{n-1}x), \quad \forall x \in \Omega_n, \quad n \geqslant 0.$

iii) $\Delta V(n, A_{n-1}x_n) := V(n+1, A_n x_{n+1}) - V(n, A_{n-1}x_n) \leqslant 0$

for any solution x_n of (2.1). Then the trivial solution of (2.1) is stable.

Theorem 3.7. The trivial solution of (2.1) is P-uniformly stable if and only if there exist two functions $a, b \in \mathcal{K}$ and a Lyapunov function $V : \mathbb{Z}_+ \times \mathbb{R}^m \longrightarrow \mathbb{R}_+$, such that

i) $a(\|x\|) \leqslant V(n, P_{n-1}x) \leqslant b(\|P_{n-1}x\|), \quad \forall x \in \Delta_n, \quad n \geqslant 0.$

ii) $\Delta V(n, P_{n-1}x_n) := V(n+1, P_n x_{n+1}) - V(n, P_{n-1}x_n) \leqslant 0$

for any solution x_n of (2.1).

Proof. The proof of the necessity part is based on Lemma 3.4 and is similar to the corresponding part of Theorem 3.5.

Now suppose that the trivial solution of (2.1) is not P-uniformly stable, hence there exists ϵ_0, such that for all $\delta \in (0, \epsilon_0]$, there exist a solution x_n of (2.1) and two nonnegative integers $n_1 \leqslant n_2$, such that $\|P_{n_1-1}x_{n_1}\| < \delta$ however $\|x_{n_2}\| \geqslant \epsilon_0$. Choose $\delta_0 > 0$, such that $\delta_0 \leqslant \epsilon_0$ and $b(\delta_0) < a(\epsilon_0)$. According to our assumption there exist a solution x_n of (2.1) and two nonnegative integers $n_1 \leqslant n_2$, such that $\|P_{n_1-1}x_{n_1}\| < \delta_0$ and $\|x_{n_2}\| \geqslant \epsilon_0$. Using the first property of the Lyapunov function we have

$$V(n_2, P_{n_2-1}x_{n_2}) \geqslant a(\|x_{n_2}\|) \geqslant a(\epsilon_0)$$

and

$$V(n_1, P_{n_1-1}x_{n_1}) \leqslant b(\|P_{n_1-1}x_{n_1}\|) < b(\delta_0) < a(\epsilon_0).$$

On the other hand, taking into account the second property of V, we get

$$a(\epsilon_0) > V(n_1, P_{n_1-1}x_{n_1}) \geqslant V(n_2, P_{n_2-1}x_{n_2}) \geqslant a(\|x_{n_2}\|) \geqslant a(\epsilon_0),$$

which was the desired contradiction. Thus the trivial solution of (2.1) is P-uniformly stable. Theorem 3.7 is proved. ∎

We end this section by stating a theorem on the A-uniform stability and asymptotical stability of the trivial solution of (2.1). Its proof is similar to those of Theorems 3.5, 3.7, and therefore will be omitted.

Theorem 3.8. Suppose that there exist two functions $a, b \in \mathcal{K}$ and a Lyapunov function $V : \mathbb{Z}_+ \times \mathbb{R}^m \longrightarrow \mathbb{R}_+$, such that

i) $a(\|x\|) \leqslant V(n, A_{n-1}x) \leqslant b(\|A_{n-1}x\|), \quad \forall x \in \Omega_n, \quad n \geqslant 0.$

ii) $\Delta V(n, A_{n-1}x_n) := V(n+1, A_n x_{n+1}) - V(n, A_{n-1}x_n) \leqslant 0$

for any solution x_n of (2.1). Then the trivial solution of (2.1) is A-uniformly stable. If Condition ii) is replaced by

iii) $\Delta V(n, A_{n-1}x_n) := V(n+1, A_n x_{n+1}) - V(n, A_{n-1}x_n) \leqslant -c(\|A_{n-1}x_n\|)$

for any solution x_n of (2.1), where c is a certain function of the class $\mathcal{K}$, then the trivial solution of (2.1) is asymptotically stable.

4. Examples

In this section we will use the Euclidean norms of vectors and matrices.

Example 4.1. Consider the SDE (2.1) with the following data:

$$A_n = (n+2)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}; \quad B_n = \begin{pmatrix} 1 & 0 \\ 0 & n+2 \end{pmatrix}, \quad n \geqslant -1,$$

and

$$f_n(x) = \frac{\sin x_1}{n+2}(0,1)^T; \quad x = (x_1, x_2)^T, \quad (n \geqslant 0).$$

As kerA_n = span$\{(0,1)^T\}$, imA_n = span$\{(1,0)^T\}$, $n \geqslant -1$ and S_n = span$\{(1,0)^T\}$, $n \geqslant 0$, the hypotheses H1, H2 are fulfilled, hence the SDE (2.2) is of index-1. Clearly, the canonical projections are $P_n = P := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}; Q_n = Q := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, therefore $Q_{n-1,n} = Q$. A simple calculation shows that $G_n = A_n + B_n Q_{n-1,n} = (n+2)I$, hence $G_n^{-1} = (n+2)^{-1}I$. Further, the function $f_n(x)$ is Lipschitz continuous with the Lipschitz coefficient $L_n = (n+2)^{-1}$. Moreover, $f_n(0) = 0$ and $\omega_n := L_n\|Q_{n-1,n}G_n^{-1}\| = (n+2)^{-2} < 1$ for all nonnegative n. According to Theorem 2.2, the IVP (2.1), (2.3) has a unique solution. From the definition of Δ_n, we have $x \in \Delta_n$ if and only if

$$Q_{n-1}x = Q_{n-1,n}G_n^{-1}\{f_n(x) - B_n P_{n-1}x\},$$

or $Qx = QG_n^{-1}\{f_n(x) - B_n Px\}$. The last relation leads to $x_2 = (n+2)^{-2}\sin x_1$. Thus,

$$\Delta_n = \Omega_n = \{x = (x_1, x_2)^T : x_2 = (n+2)^{-2}\sin x_1\}, \quad n \geqslant 0.$$

Introducing a function $V(n, \gamma) := 2\|\gamma\|$, we get for each $x \in \Delta_n$,

$$\|x\| = (x_1 + x_2)^{1/2} = (x_1^2 + (n+2)^{-4}\sin^2 x_1)^{1/2} \leqslant 2|x_1| = 2\|P_{n-1}x\|.$$

Further, $V(n, P_{n-1}x) = 2\|P_{n-1}x\| = 2|x_1|$. Thus, $\|x\| \leqslant V(n, P_{n-1}x) = 2\|P_{n-1}x\|$ for all $x \in \Delta_n$. Supposing that x_n is a solution of (2.1) and putting $u_n = P_{n-1}x_n = Px_n;\ v_n = Q_{n-1}x_n = Qx_n$ we have

$$\begin{aligned}\Delta V(n, P_{n-1}x_n) &= V(n+1, P_n x_{n+1}) - V(n, P_{n-1}x_n)\\ &= 2(\|Px_{n+1}\| - \|Px_n\|) = 2(\|u_{n+1}\| - \|u_n\|).\end{aligned}$$

Using Equation (2.8) we find

$$u_{n+1} = -P_n G_n^{-1} B_n u_n + P_n G_n^{-1} f_n(x_n) = -(n+2)^{-1} u_n,$$

hence $\|u_{n+1}\| - \|u_n\| = -(n+1)(n+2)^{-1}\|u_n\| \leqslant -2^{-1}\|u_n\|$, consequently $\Delta V(n, P_{n-1}x_n) \leqslant -\|P_{n-1}x_n\|$. According to Theorem 3.7, the trivial solution of (2.1) is P-uniformly stable. Moreover, it is also asymptotically stable.

Example 4.2. Let the data in (2.1) be as follows:

$$A_n = \begin{pmatrix} n+3 & 1 \\ n+3 & 1 \end{pmatrix}; \quad B_n = \begin{pmatrix} n+2 & 1 \\ n+1 & n+1 \end{pmatrix}, \quad n \geqslant -1,$$

and

$$f_n(x) = \frac{\sin((n+2)x_1 + x_2)}{2(n+1)(n+2)}(1,0)^T, \quad x = (x_1, x_2)^T, \quad (n \geqslant 0).$$

In this case, kerA_n = span$\{(1, -n-3)^T\}$, imA_n = span$\{(1,1)^T\},\ n \geqslant -1$ and S_n = span$\{(n,1)^T\},\ n \geqslant 0$. Clearly, $S_n \cap \ker A_{n-1} = \{0\},\ n \geqslant 0$, hence Equation (2.2) is of index-1. Consider the projections

$$Q_n = \begin{pmatrix} 1 & 0 \\ -n-3 & 0 \end{pmatrix} \text{ and } P_n = I - Q_n = \begin{pmatrix} 0 & 0 \\ n+3 & 1 \end{pmatrix}.$$

A simple calculation shows that

$$V_n = \begin{pmatrix} 0 & -1 \\ 1 & -n-3 \end{pmatrix}; \quad V_n^{-1} = \begin{pmatrix} n+3 & 1 \\ 1 & 0 \end{pmatrix},$$

hence

$$Q_{n-1,n} = V_{n-1}\tilde{Q}V_n^{-1} = \begin{pmatrix} 1 & 0 \\ -n-2 & 0 \end{pmatrix}.$$

Further,

$$G_n = A_n + B_n Q_{n-1,n} = \begin{pmatrix} n+3 & 1 \\ -n^2-n+2 & 1 \end{pmatrix}$$

and

$$G_n^{-1} = (n+1)^{-2}\begin{pmatrix} 1 & -1 \\ n^2+n-2 & n+3 \end{pmatrix}.$$

Observe that the function $f_n(x)$ is Lipschitz continuous with a Lipschitz coefficient $L_n = (2(n+1)(n+2))^{-1}$. Since $\omega_n = L_n\|Q_{n-1,n}G_n^{-1}\| < 1$, the IVP (2.1), (2.3) has a unique solution. Denoting $\xi_n := (n+2)x_1 + x_2$ for any $x = (x_1, x_2)^T$, we can rewrite $P_{n-1}x = (0, \xi_n)^T; A_{n-1}x = (\xi_n, \xi_n)^T$, which leads to the relations $\|P_{n-1}x\| = |\xi_n|$; $\|A_{n-1}x\| = \sqrt{2}|\xi_n|$. Now note that $x = (x_1, x_2)^T$ belongs to Δ_n if and only if $Q_{n-1}x = Q_{n-1,n}G_n^{-1}\{f_n(x) - B_nP_{n-1}x\}$. A direct computation shows that the last relation is equivalent to the equality

$$x_1 = \varphi_n(x_1, x_2) := \frac{1}{(n+1)^2}\left\{\frac{\sin\xi_n}{2(n+1)(n+2)} + n\xi_n\right\}. \tag{4.1}$$

Thus, $\Delta_n = \{(x_1, x_2)^T : x_1 = \varphi_n(x_1, x_2)\}$. Let us define a Lyapunov function

$$V(n, \gamma) := \left(1 + \frac{1}{n+1}\right)\|\gamma\|$$

for all nonnegative integers n and all $\gamma \in \mathbb{R}^2$. For each $x = (x_1, x_2)^T \in \Delta_n$ we put $u = P_{n-1}x = (0, \xi_n)^T$ and $v = Q_{n-1}x = (x_1, -(n+2)x_1)^T$. From (4.1) we have

$$\begin{aligned}\|v\| &= (x_1^2 + (n+2)^2x_1^2)^{1/2} = [1 + (n+2)^2]^{1/2}|x_1| \\ &\leqslant (1 + (n+2)^2)^{1/2}(n+1)^{-2}\left(\frac{1}{2(n+1)(n+2)} + n\right)|\xi_n| \leqslant 2|\xi_n|.\end{aligned}$$

Since $\|u\| = |\xi_n|$ and taking into account the last inequality we find $\|v\| \leqslant 2\|u\|$, which leads to the relation $\|x\| = \|u + v\| \leqslant 3\|u\|$. Thus

$$\frac{1}{3}\|x\| \leqslant \|u\| = \|P_{n-1}x\| \quad \forall x \in \Delta_n. \tag{4.2}$$

Observing that $\|A_{n-1}x\| = \sqrt{2}|\xi_n|$ for any $x \in \Delta_n$ we get

$$\begin{aligned}V(n, A_{n-1}x) &= \left(1 + \frac{1}{n+1}\right)|A_{n-1}x| \\ &\geq |A_{n-1}x| = \sqrt{2}|\xi_n| = \sqrt{2}|u| \geq \frac{\sqrt{2}}{3}|x|\end{aligned}$$

for all $x \in \Delta_n$ and $n \geqslant 0$. Now suppose $x_n = (x_{n,1}, x_{n,2})^T$ is any solution of Equation (2.1). Let $u_n := P_{n-1}x_n,\ v_n := Q_{n-1}x_n$ and $\xi_n := (n+2)x_{n,1} + x_{n,2}$. Then u_n is a solution of (2.8). Noting that $P_nG_n^{-1}B_nu_n = (0, \xi_n)^T$ and $P_nG_n^{-1}f_n(x_n) = \left(0, \frac{\sin\xi_n}{2(n+1)(n+2)}\right)^T$, we can rewrite Equation (2.8) as

$$u_{n+1} = \left(0, -\xi_n + \frac{\sin\xi_n}{2(n+1)(n+2)}\right)^T,$$

which gives $\|u_{n+1}\| \leqslant |\xi_n| + \dfrac{|\sin\xi_n|}{2(n+1)(n+2)}$. Further computation gives

$$\begin{aligned}
&\Delta V(n, A_{n-1}x_n) = V(n+1, A_n x_{n+1}) - V(n, A_{n-1}x_n)\\
&= \left(1+\frac{1}{n+2}\right)\|A_n x_{n+1}\| - \left(1+\frac{1}{n+1}\right)\|A_{n-1}x_n\|\\
&= \sqrt{2}\left\{\left(1+\frac{1}{n+2}\right)\|u_{n+1}\| - \left(1+\frac{1}{n+1}\right)\|u_n\|\right\}\\
&\leqslant \sqrt{2}\left\{\left(1+\frac{1}{n+2}\right)\left(|\xi_n| + \frac{|\sin\xi_n|}{2(n+1)(n+2)}\right) - \left(1+\frac{1}{n+1}\right)|\xi_n|\right\}\\
&= \sqrt{2}\left\{\frac{n+3}{2(n+1)(n+2)^2}|\sin\xi_n| - \frac{|\xi_n|}{(n+1)(n+2)}\right\}\\
&\leqslant \frac{\sqrt{2}}{(n+1)(n+2)}(|\sin\xi_n| - |\xi_n|) \leqslant 0.
\end{aligned}$$

Theorem 3.5 ensures the stability of the trivial solution of Equation (2.1).

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