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Differentiation and Differences for Solutions of Nonlocal Boundary Value Problems for Second Order Difference Equations

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Abstract

Solutions of the difference equation $w(m+2) = f(m, w(m), w(m+1))$ satisfying $w(m_1) - w(m_2) = w_1$, $w(m_3) - w(m_4) = w_2$, $m_i \in \mathbb{Z}$, $i = 1, 2, 3, 4$, and $w_1, w_2 \in \mathbb{R}$, are differentiated with respect to w_1 and w_2. In addition, differences of solutions with respect to m_1, m_2, m_3 and m_4 are considered.

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1. Introduction

Given $a < b$ in $\mathbb{Z}$, we use interval notation to denote sets of integers such as $[a, b] = \{a, a+1, \ldots, b\}$, $[a, b) = \{a, \ldots, b-1\}$, $(a, \infty) = \{a+1, a+2, \ldots\}$, etc. This paper is devoted to continuous dependence and differentiation with respect to boundary values,

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and to differences with respect to boundary points, of solutions of the iterative second order forward difference equation,

$$w(m+2) = f(m, w(m), w(m+1)), \quad m \in \mathbb{Z}, \tag{1.1}$$

satisfying

$$w(m_1) - w(m_2) = w_1, \quad w(m_3) - w(m_4) = w_2, \tag{1.2}$$

where $m_1, m_2, m_3, m_4 \in \mathbb{Z}$, $m_i + 1 < m_{i+1}$, $i = 1, 2, 3$, and $w_1, w_2 \in \mathbb{R}$.

We assume throughout this paper:

(A) $f(m, d_1, d_2) : \mathbb{Z} \times \mathbb{R}^2 \to \mathbb{R}$ is continuous.

(B) $\dfrac{\partial f}{\partial d_i}(m, d_1, d_2) : \mathbb{Z} \times \mathbb{R}^2 \to \mathbb{R}$ is continuous, for $i = 1, 2$.

(C) The equation $d_3 = f(m, d_1, d_2)$ can be solved for d_1 as a continuous function of d_2 and d_3, for each $m \in \mathbb{Z}$.

Remark 1.1. We observe that, given a solution $w(m)$ of (1.1), condition (C) yields that $w(m)$ exists on all of $\mathbb{Z}$.

For many of our results, given a solution $w(m)$ of (1.1), we will have need of the *variational equation along* $w(m)$ which is given by

$$z(m+2) = \frac{\partial f}{\partial d_1}(m, w(m), w(m+1))z(m) + \frac{\partial f}{\partial d_2}(m, w(m), w(m+1))z(m+1). \tag{1.3}$$

Interest in multipoint and nonlocal problems for ordinary differential equations has seen a surge of recent activity. Some of this interest has carried over to boundary value problems for finite difference equations; see, for example [1–5, 11, 17, 18, 20, 21]. In addition, many papers have been devoted to smoothness of solutions of boundary value problems, with respect to boundary data, for differential equations, even for the cases of nonlocal conditions; for a few of these, we cite [8, 9, 13, 14] and the references therein. Evolved from some of these latter papers have been results for smoothness and differences, with respect to boundary data, for solutions of difference equations, such as in [6, 7, 15, 19]. This paper is somewhat motivated by the two recent papers on smoothness of solutions, with respect to boundary data, for nonlocal boundary value problems [10, 16].

In Section 2, we state, without proofs, theorems concerning solutions of initial value problems for (1.1), which depend continuously on initial values and which can be differentiated with respect to initial values. We also state a theorem concerning differences of solutions, with respect to initial points, for solutions of initial value problems for (1.1).

In Section 3, we establish analogues of results from Section 2, in terms of smoothness with respect to boundary values, for solutions of the nonlocal problem (1.1), (1.2).

In Section 4, we obtain an analogue of the last result from Section 2, in terms of differences with respect to boundary points, for solutions of the nonlocal problem (1.1), (1.2).

2. Differentiation and Differences with Respect to Initial Conditions

The second order difference equation (1.1) along with the conditions,

$$w(m_0) = c_1, \quad w(m_0 + 1) = c_2, \tag{2.1}$$

where $m_0 \in \mathbb{Z}$ and $c_1, c_2 \in \mathbb{R}$, is called an *initial value problem*. In this section, we state two theorems regarding continuous dependence and differentiability of solutions of (1.1), (2.1) with respect to initial values. The proofs of these results follow much along the lines of standard ones for initial value problems for differential equations. For that reason, we will omit their proofs. In addition, we present a theorem involving differences of solutions of (1.1), (2.1) with respect to initial points. The proof of this last result is much like those given in [6] and [7], and so this proof is also omitted. These three theorems are fundamental for the results of Sections 3 and 4.

We remark that, in the presence of condition (C), solutions of (1.1), (2.1) are unique on all of $\mathbb{Z}$. For notation hereafter, given $m_0 \in \mathbb{Z}$ and $c_1, c_2 \in \mathbb{R}$, we will *denote* the *unique solution* of the *initial value problem* (1.1), (2.1) by

$$u(m, m_0, c_1, c_2). \tag{2.2}$$

Theorem 2.1. [Continuous dependence with respect to initial values] Suppose conditions (A) and (C) are satisfied. Let $m_0 \in \mathbb{Z}$ and $c_1, c_2 \in \mathbb{R}$ be given. Then, for each $\epsilon > 0$ and $k \in \mathbb{N}$, there exists a $\delta(\epsilon, k, m_0, c_1, c_2) > 0$ such that, $|c_1 - e_1| < \delta$ and $|c_2 - e_2| < \delta$ imply $|u(m, m_0, c_1, c_2) - u(m, m_0, e_1, e_2)| < \epsilon$, for every $m \in [m_0 - k, m_0 + k]$.

Theorem 2.2. [Differentiation with respect to initial values] Suppose conditions (A), (B) and (C) are satisfied. Let $m_0 \in \mathbb{Z}$ and $c_1, c_2 \in \mathbb{R}$ be given. Then, for $j = 1, 2$, $\beta_j(m) := \dfrac{\partial u}{\partial c_j}(m, m_0, c_1, c_2)$ exists and is the solution of the variational equation (1.3) along $u(m, m_0, c_1, c_2)$, that is,

$$\beta_j(m+2) = \sum_{i=1}^{2} \frac{\partial f}{\partial d_i}(m, u(m, m_0, c_1, c_2), u(m+1, m_0, c_1, c_2))\beta_j(m+i-1), \tag{2.3}$$

and satisfies the initial conditions,

$$\beta_j(m_0 + i - 1) = \delta_{ij}, \quad i = 1, 2. \tag{2.4}$$

Theorem 2.3. [Differences with respect to initial points] Suppose conditions (A), (B) and (C) are satisfied. Let $m_0 \in \mathbb{Z}$ and $c_1, c_2 \in \mathbb{R}$ be given. Then

$$\gamma(m) := \Delta_{m_0} u(m, m_0, c_1, c_2) = u(m, m_0 + 1, c_1, c_2) - u(m, m_0, c_1, c_2)$$

is the solution of the second order linear difference equation,

$$\gamma(m+2) = A_1(m)\gamma(m) + A_2(m)\gamma(m+1),$$

satisfying the initial conditions,

$$\gamma(m_0) = -\Delta_m u(m, m_0+1, c_1, c_2)|_{m=m_0},\ \gamma(m_0+1) = -\Delta_m u(m, m_0, c_1, c_2)|_{m=m_0},$$

where

$$\begin{aligned} A_1(m) = \int_0^1 \frac{\partial f}{\partial d_1}&(m, su(m, m_0+1, c_1, c_2) \\ &+ (1-s)u(m, m_0, c_1, c_2), u(m+1, m_0, c_1, c_2))ds, \end{aligned}$$

and

$$\begin{aligned} A_2(m) = \int_0^1 \frac{\partial f}{\partial d_2}&(m, u(m, m_0+1, c_1, c_2), su(m+1, m_0+1, c_1, c_2) \\ &+ (1-s)u(m+1, m_0, c_1, c_2))ds. \end{aligned}$$

3. Differentiation with Respect to Nonlocal Boundary Values

In this section, we establish analogues of Theorems 2.1 and 2.2. For the remainder of this paper, to distinguish from the initial value notation in (2.2), we denote, for $m_1, m_2, m_3, m_4 \in \mathbb{Z}$ and $w_1, w_2 \in \mathbb{R}$, solutions of (1.1), (1.2) by

$$w(m, m_1, m_2, m_3, m_4, w_1, w_2). \tag{3.1}$$

Remark 3.1. Sometimes, when a particular variable, such as some m_i, is to be emphasized, we may abbreviate the notation in (3.1) by

$$w(m, \cdots, m_i, \cdots) \text{ or } w(m, \cdot, m_i, \cdot). \tag{3.2}$$

Our results hereafter rely heavily on uniqueness of solutions of nonlocal boundary value problems. For this, we make use of Hartman's definition of generalized zero [12].

Definition 3.2. Let $v : \mathbb{Z} \to \mathbb{R}$. We say that v *has a generalized zero (gz)* at $n_0 \in \mathbb{Z}$ provided, either $v(n_0) = 0$, or there exists $k \in \mathbb{Z}$ such that $(-1)^k u(n_0 - k)u(n_0) > 0$, and if $k > 1$, $u(n_0 - k + 1) = \cdots = u(n_0 - 1) = 0$.

Definition 3.3. The nonlinear difference equation (1.1) is said to *satisfy property (U) on* $\mathbb{Z}$, whenever $w_1(m)$ and $w_2(m)$ are solutions of (1.1) such that $[w_1(m_1) - w_2(m_1)] - [w_1(m) - w_2(m)]$ has a gz at m_2, and $[w_1(m_3) - w_2(m_3)] - [w_1(m) - w_2(m)]$ has a gz at m_4, where $m_1 < m_1 + 1 < m_2 < m_2 + 1 < m_3 < m_3 + 1 < m_4$ in $\mathbb{Z}$, it follows that $w_1(m) - w_2(m) \equiv 0$ on $\mathbb{Z}$.

Remark 3.4. If property (U) holds for (1.1), then solutions of the boundary value problem (1.1), (1.2) are unique.

Since our results of this section also involve property (U) with respect to the variational equation (1.3), we include for completeness a definition of the property relative to linear difference equations.

Definition 3.5. The linear difference equation,

$$r(m+2) = \alpha_1(m)r(m) + \alpha_2(m)r(m+1), \tag{3.3}$$

where $\alpha_i : \mathbb{Z} \to \mathbb{R}$, $i = 1, 2$, is said to *satisfy property (U) on* $\mathbb{Z}$, provided there is no nontrivial solution, $r(m)$, of (3.3) such that $r(m_1) - r(m)$ has a gz at m_2, and $r(m_3) - r(m)$ has a gz at m_4, where $m_1 < m_1 + 1 < m_2 < m_2 + 1 < m_3 < m_3 + 1 < m_4$ in $\mathbb{Z}$.

Our first result of the section deals with continuous dependence of solutions of (1.1), (1.2) on boundary values. Its proof entails a rather straightforward application of the Brouwer theorem on invariance of domain. We omit the details of the proof, but for a typical argument, we suggest [7, 15].

Theorem 3.6. [Continuous dependence with respect to boundary values] Suppose conditions (A) and (C) are satisfied and that (1.1) satisfies property (U) on $\mathbb{Z}$. Let $y(m)$ be a solution of (1.1) on $\mathbb{Z}$, and let $m_1 < m_1 + 1 < m_2 < m_2 + 1 < m_3 < m_3 + 1 < m_4$ in $\mathbb{Z}$ be given. Then, there exists an $\epsilon > 0$ such that, if $\delta_1, \delta_2 \in \mathbb{R}$ with $|\delta_i| < \epsilon$, $i = 1, 2$, the boundary value problem for (1.1) satisfying

$$\begin{aligned} w(m_1) - w(m_2) &= y(m_1) - y(m_2) + \delta_1, \\ w(m_3) - w(m_4) &= y(m_3) - y(m_4) + \delta_2, \end{aligned}$$

has a unique solution $w(m, m_1, m_2, m_3, m_4, y(m_1) - y(m_2) + \delta_1, y(m_3) - y(m_4) + \delta_2)$. Furthermore, as $\epsilon \to 0$, the solutions $w(m, m_1, m_2, m_3, m_4, y(m_1) - y(m_2) + \delta_1, y(m_3) - y(m_4) + \delta_2)$ converge to $y(m)$ on $\mathbb{Z}$.

We now prove an analogue of Theorem 2.2 for nonlocal boundary value problems.

Theorem 3.7. [Differentiation with respect to boundary values] Assume that conditions (A), (B) and (C) are satisfied, that (1.1) satisfies property (U) on $\mathbb{Z}$, and that the variational equation (1.3) satisfies property (U) along all solutions of (1.1). Assume $w(m) = w(m, m_1, m_2, m_3, m_4, w_1, w_2)$ is a solution of (1.1), (1.2) on $\mathbb{Z}$. Then, for $j = 1, 2$, $\dfrac{\partial w}{\partial w_j}$ exists on $\mathbb{Z}$, and $z_j(m) := \dfrac{\partial w}{\partial w_j}(m)$ is the solution of the variational equation (1.3) along $w(m)$, and satisfies, respectively,

$$\begin{aligned} &z_1(m_1) - z_1(m_2) = 1, \quad z_1(m_3) - z_1(m_4) = 0, \\ &z_2(m_1) - z_2(m_2) = 0, \quad z_2(m_3) - z_2(m_4) = 1. \end{aligned}$$

Proof. We make the argument for only $\dfrac{\partial w}{\partial w_1}$, since the argument for $\dfrac{\partial w}{\partial w_2}$ is almost identical. Let $\epsilon > 0$ be as in Theorem 3.6. Let $0 < |h| < \epsilon$ be given and consider the quotient

$$z_{1h}(m) := \frac{1}{h}\left[w(m, m_1, m_2, m_3, m_4, w_1 + h, w_2) - w(m, m_1, m_2, m_3, m_4, w_1, w_2)\right].$$

We note first that, for every $h \neq 0$,

$$z_{1h}(m_1) - z_{1h}(m_2) = \frac{1}{h}[w_1 + h - w_1] = 1, \tag{3.4}$$

$$z_{1h}(m_3) - z_{1h}(m_4) = \frac{1}{h}[w_2 - w_2] = 0. \tag{3.5}$$

It now suffices to show that $\lim_{h\to 0} z_{1h}(m)$ exists on $\mathbb{Z}$. We will express $z_{1h}(m)$ in terms of solutions of initial value problems for (1.1). We let

$$\begin{aligned}
\sigma_1 &= w(m_1, m_1, m_2, m_3, m_4, w_1, w_2),\\
\sigma_2 &= w(m_1+1, m_1, m_2, m_3, m_4, w_1, w_2),\\
\epsilon_1 = \epsilon_1(h) &= w(m_1, m_1, m_2, m_3, m_4, w_1 + h, w_2) - \sigma_1,\\
\epsilon_2 = \epsilon_2(h) &= w(m_1+1, m_1, m_2, m_3, m_4, w_1 + h, w_2) - \sigma_2.
\end{aligned}$$

By Theorem 3.6, $\epsilon_i(h) \to 0$, as $h \to 0$, $i = 1, 2$. Now invoking the notation (2.2) for solutions of initial value problems for (1.1), we have

$$z_{1h}(m) = \frac{1}{h}\left[u(m, m_1, \sigma_1 + \epsilon_1, \sigma_2 + \epsilon_2) - u(m, m_1, \sigma_1, \sigma_2)\right].$$

Next, by employing a telescoping sum and applying Theorem 2.2,

$$\begin{aligned}
z_{1h}(m) &= \frac{1}{h}\{[u(m, m_1, \sigma_1 + \epsilon_1, \sigma_2 + \epsilon_2) - u(m, m_1, \sigma_1, \sigma_2 + \epsilon_2)]\\
&\quad +[u(m, m_1, \sigma_1, \sigma_2 + \epsilon_2) - u(m, m_1, \sigma_1, \sigma_2)]\}\\
&= \frac{1}{h}[\beta_1(m, u(m, m_1, \sigma_1 + \overline{\epsilon}_1, \sigma_2 + \epsilon_2))\epsilon_1\\
&\quad + \beta_2(m, u(m, m_1, \sigma_1, \sigma_2 + \overline{\epsilon}_2))\epsilon_2],
\end{aligned}$$

where $\beta_i(m, u(\cdot))$, $i = 1, 2$, denotes the solution of the variational equation (1.3) along $u(\cdot)$ and satisfies the respective initial conditions,

$$\begin{aligned}
&\beta_1(m_1, u(\cdot)) = 1, \quad \beta_1(m_1 + 1, u(\cdot)) = 0,\\
&\beta_2(m_1, u(\cdot)) = 0, \quad \beta_2(m_1 + 1, u(\cdot)) = 1,
\end{aligned}$$

and where $\overline{\epsilon}_i$ is between 0 and ϵ_i, $i = 1, 2$. To show $\lim_{h\to 0} z_{1h}(m)$ exists, it now suffices to show that $\lim_{h\to 0} \frac{\epsilon_i}{h}$ exists, for $i = 1, 2$. From (3.4) and (3.5), we have that

$$\begin{aligned}
&\frac{\epsilon_1}{h}[\beta_1(m_1, u(m, \cdot, \sigma_1 + \overline{\epsilon}_1, \cdot)) - \beta_1(m_2, u(m, \cdot, \sigma_1 + \overline{\epsilon}_1, \cdot))]\\
&\quad + \frac{\epsilon_2}{h}[\beta_2(m_1, u(m, \cdot, \sigma_2 + \overline{\epsilon}_2)) - \beta_2(m_2, u(m, \cdot, \sigma_2 + \overline{\epsilon}_2))] = 1,
\end{aligned} \tag{3.6}$$

and

$$\frac{\epsilon_1}{h}[\beta_1(m_3, u(m, \cdot, \sigma_1 + \overline{\epsilon}_1, \cdot)) - \beta_1(m_4, u(m, \cdot, \sigma_1 + \overline{\epsilon}_1, \cdot))]$$
$$+\frac{\epsilon_2}{h}[\beta_2(m_3, u(m, \cdot, \sigma_2 + \overline{\epsilon}_2)) - \beta_2(m_4, u(m, \cdot, \sigma_2 + \overline{\epsilon}_2))] = 0. \quad (3.7)$$

Now, since $\beta_j(m, u(m, m_1, \sigma_1, \sigma_2))$, $j = 1, 2$, are nontrivial solutions of (1.3), and since (1.3) satisfies property (U) along all solutions of (1.1), it follows from the initial conditions at m_1 satisfied by β_1 and β_2 along $u(m, m_1, \sigma_1, \sigma_2)$ that $\det D \neq 0$, where D is the 2×2 matrix with first column,

$$[1-\beta_1(m_2, u(m, m_1, \sigma_1, \sigma_2)), \beta_1(m_3, u(m, m_1, \sigma_1, \sigma_2))-\beta_1(m_4, u(m, m_1, \sigma_1, \sigma_2))]^T,$$

and second column,

$$[-\beta_2(m_2, u(m, m_1, \sigma_1, \sigma_2)), \beta_2(m_3, u(m, m_1, \sigma_1, \sigma_2)) - \beta_2(m_4, u(m, m_1, \sigma_1, \sigma_2))]^T,$$

(where T denotes matrix transpose). By continuous dependence provided by Theorems 2.1 and 3.6, for $h \neq 0$ sufficiently small, $\det D(h) \neq 0$, where $D(h)$ is the 2×2 matrix whose columns are the coefficients of $\frac{\epsilon_1}{h}$ and $\frac{\epsilon_2}{h}$, respectively, in the system (3.6) and (3.7), that is, whose first column is

$$[1-\beta_1(m_2, u(m, \cdot, \sigma_1+\overline{\epsilon}_1, \cdot), \beta_1(m_3, u(m, \cdot, \sigma_1+\overline{\epsilon}_1, \cdot))-\beta_1(m_4, u(m, m_1, \sigma_1+\overline{\epsilon}_1, \cdot))]^T,$$

and whose second column is

$$[-\beta_2(m_2, u(m, \cdot, \sigma_2 + \overline{\epsilon}_2)), \beta_2(m_3, u(m, \cdot, \sigma_2 + \overline{\epsilon}_2)) - \beta_2(m_4, u(m, \cdot, \sigma_2 + \overline{\epsilon}_2))]^T.$$

As a consequence, the system (3.6) and (3.7) can be solved uniquely for $\frac{\epsilon_1}{h}$ and $\frac{\epsilon_2}{h}$, and by Theorems 2.1 and 3.6,

$$\lim_{h\to 0} \frac{\epsilon_i}{h} := L_i, \quad i = 1, 2.$$

So, $z_1(m) := \lim_{h\to 0} z_{1h}(m)$ exists. By construction, $z_1(m) = \frac{\partial w}{\partial w_1}(m)$. Moreover,

$$\begin{aligned} z_1(m) &= L_1\beta_1(m, u(m, m_1, \sigma_1, \sigma_2)) + L_2\beta_2(m, u(m, m_1, \sigma_1, \sigma_2)) \\ &= L_1\beta_1(m, w(m)) + L_2\beta_2(m, w(m)), \end{aligned}$$

and as such, $z_1(m)$ is a solution of the variational equation (1.3) along $w(m)$. In addition, from (3.4) and (3.5),

$$z_1(m_1) - z_1(m_2) = 1, \quad z_1(m_3) - z_1(m_4) = 0.$$

The proof is complete. ■

4. Differences with Respect to Boundary Points

In this section, we establish an analogue of Theorem 2.3 for solutions of (1.1), (1.2) with respect to the boundary points.

Theorem 4.1. [Differences with respect to boundary points] Assume that conditions (A), (B) and (C) are satisfied, and that (1.1) satisfies property (U) on $\mathbb{Z}$. Assume $w(m, m_1, m_2, m_3, m_4, w_1, w_2)$ is a solution of (1.1), (1.2) on $\mathbb{Z}$. Then, for $i = 1, 2, 3, 4$,

$$v_i(m) := \Delta_{m_i} w(m, m_1, m_2, m_3, m_4, w_1, w_2) = w(m, \cdot, m_i + 1, \cdot) - w(m, \cdot, m_i, \cdot)$$

is a solution of

$$v_i(m + 2) = A_{1i}(m)v_i(m) + A_{2i}(m)v_i(m + 1),$$

where

$$\begin{aligned} A_{1i}(m) = \int_0^1 \frac{\partial f}{\partial d_1}(m, sw(m, \cdot, m_i + 1, \cdot) \\ + (1 - s)w(m, \cdot, m_i, \cdot), w(m + 1, \cdot, m_i + 1, \cdot))ds, \end{aligned}$$

and

$$\begin{aligned} A_{2i}(m) = \int_0^1 \frac{\partial f}{\partial d_2}(m, w(m, \cdot, m_i, \cdot), sw(m + 1, \cdot, m_i + 1, \cdot) \\ + (1 - s)w(m + 1, \cdot, m_i, \cdot))ds. \end{aligned}$$

Moreover, $v_i(m)$, $i = 1, 2, 3, 4,$ satisfies the respective boundary conditions,

$$\begin{aligned} v_1(m_1) - v_1(m_2) &= -\Delta_m w(m, m_1 + 1, m_2, m_3, m_4, w_1, w_2)|_{m=m_1}, \\ v_1(m_3) - v_1(m_4) &= 0, \\ v_2(m_1) - v_2(m_2) &= \Delta_m w(m, m_1, m_2 + 1, m_3, m_4, w_1, w_2)|_{m=m_2}, \\ v_2(m_3) - v_2(m_4) &= 0, \\ v_3(m_1) - v_3(m_2) &= 0, \\ v_3(m_3) - v_3(m_4) &= -\Delta_m w(m, m_1, m_2, m_3 + 1, m_4, w_1, w_2)|_{m=m_3}, \\ v_4(m_1) - v_4(m_2) &= 0, \\ v_4(m_3) - v_4(m_4) &= \Delta_m w(m, m_1, m_2, m_, m_4 + 1, w_1, w_2)|_{m=m_4}. \end{aligned}$$

Proof. The proof involves the mean value theorem in conjunction with difference calculus. Again, we will establish the result for $i = 1$, that is, we will deal with

$$v_1(m) := \Delta_{m_1} w(m, m_1, m_2, m_3, m_4, w_1, w_2) = w(m, m_1 + 1, \cdot) - w(m, m_1, \cdot).$$

First, using a telescoping sum and the mean value theorem,

$$\begin{aligned}
v_1(m+2) &= f(m, w(m, m_1+1, \cdot), w(m+1, m_1+1, \cdot)) \\
&\quad - f(m, w(m, m_1, \cdot), w(m+1, m_1, \cdot)) \\
&= f(m, w(m, m_1+1, \cdot), w(m+1, m_1+1, \cdot)) \\
&\quad - f(m, w(m, m_1, \cdot), w(m+1, m_1+1, \cdot)) \\
&\quad + f(m, w(m, m_1, \cdot), w(m+1, m_1+1, \cdot)) \\
&\quad - f(m, w(m, m_1, \cdot), w(m+1, m_1, \cdot)) \\
&= \int_0^1 \frac{\partial f}{\partial d_1}(m, sw(m, m_1+1, \cdot) + (1-s)w(m, m_1, \cdot), \\
&\quad\ w(m+1, m_1+1, \cdot))ds[w(m, m_1+1, \cdot) - w(m, m_1, \cdot)] \\
&\quad + \int_0^1 \frac{\partial f}{\partial d_2}(m, w(m, m_1, \cdot), sw(m+1, m_1+1, \cdot) \\
&\quad + (1-s)w(m+1, m_1, \cdot))ds[w(m+1, m_1+1, \cdot) - w(m+1, m_1, \cdot)].
\end{aligned}$$

That is,

$$v_1(m+2) = A_{11}(m)v_1(m) + A_{21}(m)v_1(m+1).$$

Next,

$$\begin{aligned}
v_1(m_1) - v_1(m_2) &= [w(m_1, m_1+1, \cdot) - w(m_1, m_1, \cdot)] \\
&\quad - [w(m_2, m_1+1, \cdot) - w(m_2, m_1, \cdot)] \\
&= [w(m_1, m_1+1, \cdot) - w(m_1+1, m_1+1, \cdot)] \\
&\quad + [w(m_1+1, m_1+1, \cdot) - w(m_2, m_1+1, \cdot)] \\
&\quad - [w(m_1, m_1, \cdot) - w(m_2, m_1, \cdot)] \\
&= -[w(m_1+1, m_1+1, \cdot) - w(m_1, m_1+1, \cdot)] + w_1 - w_1 \\
&= -\Delta_m w(m, m_1+1, \cdot)|_{m=m_1},
\end{aligned}$$

and

$$\begin{aligned}
v_1(m_3) - v_1(m_4) &= [w(m_3, m_1+1, \cdot) - w(m_3, m_1, \cdot)] \\
&\quad - [w(m_4, m_1+1, \cdot) - w(m_4, m_1, \cdot)] \\
&= [w(m_3, m_1+1, \cdot) - w(m_4, m_1+1, \cdot)] \\
&\quad - [w(m_3, m_1, \cdot) - w(m_4, m_1, \cdot)] \\
&= w_2 - w_2 \\
&= 0.
\end{aligned}$$

The proof is complete. ■

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