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Discontinuous Functional ϕ-Laplacian Boundary Value Problems on Time Scales

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Abstract

The goal of this paper consists in proving the existence of extremal solutions of dynamic functional and discontinuous equations involving the ϕ-Laplacian operator and coupled with nonlinear boundary functional conditions. Such boundary conditions cover, as particular cases, the Dirichlet and multipoint conditions.

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1. Introduction

The existence of solutions of nonlinear boundary value problems is a very well-known field that has been exhaustively studied in the literature for differential and difference equations. The introduction in 1990 by S. Hilger in his classical work [15] of the calculus in arbitrary closed sets gives to the mathematical community a fundamental tool to explore the continuous and the discrete models under the same formulation. This theory has been developed by different authors, we give here special mention to the book of M. Bohner and A. Peterson [6] in 2001, where basic operators for the calculus on time scales are introduced. It is for this that we can study boundary value problems in the framework of the time scales calculus. The obtained results will be valid for some discrete and continuous models but, depending on the graininess function μ, explain the

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different properties between these two kinds of problems. It is important to note that this calculus is applicable for a wider set of situations than the differential and the difference equations, see for instance [2, 4] where equations in the ternary Cantor set are studied.

In this paper we are interested in studying boundary value problems in time scales with functional dependence of the solutions. A particular case of the study given here has been done in [9]. There, the functional dependence was allowed only for the boundary conditions and, such functional dependence is continuous. Here, we use the techniques developed by S. Heikkilä and V. Lakshmikantham [14] in 1994, where are proven sufficient conditions that ensure the existence of extremal fixed points of discontinuous operators defined in abstract spaces.

The problem that we will consider involves the ϕ-Laplacian problem, which arises in the theory of radial solutions for the p-Laplacian equation ($\phi(x) = \mid x \mid^{p-2} x$) on an annular domain (see [13], and references therein) and has been studied recently for differential equations (see, for instance, [10, 11]) and also for difference and dynamic equations [8, 9].

The main tool that we use here is the so-called method of lower and upper solutions. This method is very well-known in the theory of differential and difference equations. For time scales equations one can see the papers [1, 3, 5, 7, 12, 16, 17] and references therein, in which this method is applied to different types of problems.

To be concise, we study the existence of extremal solutions for the following functional boundary value problem

$$\begin{aligned} -[\phi(u^{\Delta}(t))]^{\Delta} &= f(t, u^{\sigma}(t), u), \quad t \in I \equiv \mathbb{T}^{\kappa^2} = [a, b], && (1.1)\\ B_1(u(a), u) &= 0, && (1.2)\\ B_2(u(\sigma^2(b)), u) &= 0. && (1.3) \end{aligned}$$

We assume that the functions that define the equation satisfy the following regularity assumptions:

(H_1) $f : I \times \mathbb{R} \times \mathbb{C}(\mathbb{T}) \to \mathbb{R}$ satisfies the following conditions:

- (a) $f(\cdot, \cdot, v)$ is continuous in $I \times \mathbb{R}$ for all $v \in C(\mathbb{T})$.
- (b) $f(t, x, \cdot)$ is nondecreasing in $C(\mathbb{T})$ for all $(t, x) \in I \times \mathbb{R}$.

(H_2) $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous, strictly increasing, $\phi(0) = 0$ and $\phi(\mathbb{R}) = \mathbb{R}$.

(H_3) $B_1 : \mathbb{R} \times C(\mathbb{T}) \longrightarrow \mathbb{R}$ is such that

- (a) $B_1(\cdot, v)$ is continuous on $\mathbb{R}$ for all $v \in C(\mathbb{T})$.
- (b) $B_1(x, \cdot)$ is nondecreasing on $C(\mathbb{T})$ for all $x \in \mathbb{R}$.

$B_2 : \mathbb{R} \times C(\mathbb{T}) \longrightarrow \mathbb{R}$ satisfies

- (a) $B_2(\cdot, v)$ is continuous on $\mathbb{R}$ for all $v \in C(\mathbb{T})$.
- (b) $B_2(x, \cdot)$ is nonincreasing on $C(\mathbb{T})$ for all $x \in \mathbb{R}$.

Remark 1.1. Note that the assumption $\phi(0) = 0$ is not a restriction. By redefining $\bar{\phi}(x) = \phi(x) - \phi(0)$, the same problem is considered.

It is clear that, by defining $B_1(x, v) = x - c_0$ and $B_2(x, v) = x - c_1$, these functional conditions include as a particular case, the Dirichlet conditions

$$u(a) = c_0, \quad u(\sigma^2(b)) = c_1.$$

The multipoint boundary value conditions are given by

$$B_1(x, v) = -x + \sum_{i=1}^{n} a_i\, v(t_i), \quad B_2(x, v) = x - \sum_{j=1}^{m} b_j\, v(s_j),$$

with $n,\ m \in \mathbb{N}$, $a_i,\ b_j \geq 0$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$, $a < t_1 < \cdots < t_n \leq \sigma^2(b)$ and $a \leq s_1 < \cdots < s_m < \sigma^2(b)$.

Now, choosing $J_0,\ J_1 \subset \mathbb{T}$ and $l, r \in \mathbb{N}$ odd, it is possible to consider, among others, nonlinear boundary conditions such as

$$u(a) = \int_{J_0} u^l(t)\, \Delta t, \qquad u(\sigma^2(b)) = \int_{J_1} u^r(t)\, \Delta t,$$

or

$$u(a) = \max_{t \in J_0} u(t), \quad u(\sigma^2(b)) = \min_{t \in J_1} u(t).$$

In [9] it is considered the following particular case of problem (1.1)–(1.3):

$$-[\phi(u^\Delta(t))]^\Delta \;=\; f(t, u^\sigma(t)), \quad t \in I, \tag{1.4}$$

coupled with boundary conditions (1.2) and (1.3). For such a problem, it is obtained in [9] an existence result. Before enunciating this result, we define the concept of lower and upper solutions given in [9] as follows.

Definition 1.2. Let $n \geq 0$ be given and $a = t_0 < t_1 < t_2 < \ldots < t_n < t_{n+1} = \sigma(b)$ be fixed. We say that $\alpha \in C(\mathbb{T})$ is a lower solution of problem (1.4), (1.2), (1.3) if the following properties hold:

1. α^Δ is bounded on $\mathbb{T}^\kappa \backslash \{t_1, \ldots, t_n\}$.
2. For all $i \in \{1, \ldots, n\}$ there are $\alpha^\Delta(t_i^-),\ \alpha^\Delta(t_i^+) \in \mathbb{R}$ satisfying the following inequality:
$$\alpha^\Delta(t_i^-) < \alpha^\Delta(t_i^+).$$
3. For all $i = 0, 1, \ldots, n$, $\phi(\alpha^\Delta) \in C^1(t_i, t_{i+1})$ and satisfies
$$\begin{aligned} -[\phi(\alpha^\Delta(t))]^\Delta &\leq f(t, \alpha^\sigma(t)),\ t \in (t_i, t_{i+1}), \\ B_1(\alpha(a), \alpha) \geq\ 0 &\geq B_2(\alpha(\sigma^2(b)), \alpha). \end{aligned}$$

$\beta \in C(\mathbb{T})$ is an upper solution of problem (1.4), (1.2), (1.3) if the reversed inequalities hold for suitable points $a = s_0 < s_1 < s_2 < \ldots < s_n < s_{n+1} = \sigma(b)$.

In the previous definitions, the following notations are used:

$$u(t^+) = \begin{cases} \lim\limits_{s \to t^+} u(s) & \text{if} \quad t \ \text{ is right-dense,} \\ u(t) & \text{if} \quad t \ \text{ is right-scattered,} \end{cases}$$

and

$$u(t^-) = \begin{cases} \lim\limits_{s \to t^-} u(s) & \text{if} \quad t \ \text{ is left-dense,} \\ u(\rho(t)) & \text{if} \quad t \ \text{ is left-scattered.} \end{cases}$$

A solution of problem (1.4), (1.2), (1.3) will be a function that belongs to the set

$$S = \{u \in C(\mathbb{T}) : \quad u \in C^1(\mathbb{T}^\kappa) : \quad \phi(u^\Delta) \in C^1(I)\}, \tag{1.5}$$

and fulfills equations (1.4), (1.2), (1.3).

The existence result given in [9] is the following.

Theorem 1.3. Let α and β be a lower and an upper solution, respectively, for problem (1.4), (1.2), (1.3) such that $\alpha \le \beta$ in $\mathbb{T}$. Assume that hypotheses (H_1)–(H_3) are satisfied for f, B_1 and B_2 continuous. Then problem (1.4), (1.2), (1.3) has at least one solution

$$u \in [\alpha, \beta] = \{v \in C(\mathbb{T}) \ : \ \alpha(t) \le v(t) \le \beta(t) \text{ for all } t \in \mathbb{T}\}.$$

A solution $v \in V$ of problem (1.4), (1.2), (1.3) will be denoted as the maximal solution of this problem in V, if satisfies that any other solution $w \in V$ is such that $w \le v$ on $\mathbb{T}$. The concept of minimal solution in V is defined by reversing the inequalities. We refer to both functions as extremal solutions of problem (1.4), (1.2), (1.3) in V.

In [9] it is ensured the existence of extremal solutions of problem (1.4), (1.2), (1.3) in the particular case of the boundary condition (1.3) being replaced by

$$B_2(u(a), u(\sigma^2(b))) = 0. \tag{1.6}$$

The obtained result is the following.

Theorem 1.4. Let α and β be a lower and an upper solution, respectively, for problem (1.4), (1.2), (1.6) (with obvious notation) such that $\alpha \le \beta$ in $\mathbb{T}$. Assume that the hypotheses of Theorem 1.3 hold. Then problem (1.4), (1.2), (1.6) has extremal solutions in $[\alpha, \beta]$.

In Section 2, we prove the existence of extremal solutions of problem (1.1)–(1.3). In this case we do not assume the existence of lower and upper solutions. The conditions that we will impose to the functions that define the problem allow us to ensure the existence of extremal solutions in bounded sets of continuous functions. In Section 3 we present an example to illustrate the given results.

We note that problem (1.1)–(1.3) has functional dependence in all of the three functions that define the problem and, moreover, such dependence is discontinuous.

2. Main Results

In this section we prove the existence of extremal solutions in bounded sets of problem (1.1)–(1.3). In this case we assume the following conditions:

(F_1) $\liminf_{x\to-\infty} f(t,x,x) > 0 > \limsup_{x\to\infty} f(t,x,x)$;

(B_1) $\liminf_{x\to-\infty} B_1(x,x) > 0 > \limsup_{x\to\infty} B_1(x,x)$;

(B_2) $\liminf_{x\to-\infty} B_2(x,x) < 0 < \limsup_{x\to\infty} B_2(x,x)$.

To deduce the existence of solutions, we use the following result, which is a consequence of [14, Theorem 1.2.2].

Lemma 2.1. Let Y be a subset of an ordered metric space X, $[a,b]$ a nonempty order interval in Y and $G : [a,b] \to [a,b]$ a nondecreasing mapping. If $\{Gx_n\}$ converges in Y whenever $\{x_n\}$ is a monotone sequence in $[a,b]$, then function G has the minimal fixed point $x_* \in [a,b]$ and the maximal fixed point $x^* \in [a,b]$. Moreover, they satisfy the expressions

$$x_* = \min\{y \,:\, Gy \le y\}, \quad x^* = \max\{y \,:\, Gy \ge y\}.$$

Now, we are in a position to prove the main result of this paper.

Theorem 2.2. Assume that conditions (H_1), (H_2), (H_3), (F_1), (B_1) and (B_2) hold. Then for any $N > 0$ large enough, problem (1.1)–(1.3) has extremal solutions in the set $S \cap [-N, N]$, with S defined in (1.5).

Proof. From hypotheses (F_1), (B_1) and (B_2), we know that there are two constants $\alpha_0 << 0 << \beta_0$ such that for all $\alpha < \alpha_0$ and $\beta > \beta_0$, it is verified that

$$f(t,\alpha,\alpha) > 0, \quad t \in I; \qquad B_1(\alpha,\alpha) > 0 > B_2(\alpha,\alpha)$$

and

$$f(t,\beta,\beta) < 0, \quad t \in I; \qquad B_1(\beta,\beta) < 0 < B_2(\beta,\beta).$$

Let $v \in [\alpha_0, \beta_0]$ be an arbitrarily fixed function in $C(\mathbb{T})$. Consider the problem

$$(P_v)\begin{cases} -[\phi(u^\Delta(t))]^\Delta &= f(t,u^\sigma(t),v), \quad t \in I,\\ B_1(u(a),v) &= 0,\\ B_2(u(\sigma^2(b)),v) &= 0.\end{cases}$$

From conditions (H_1) and (H_3) we deduce that α_0 and β_0 are a pair of lower and upper solutions of problem (P_v). So, we are in the conditions of Theorem 1.4 and we deduce that problem (P_v) has extremal solutions in $[\alpha_0, \beta_0]$.

As a consequence, we can define the mapping $G : [\alpha_0,\beta_0] \to [\alpha_0,\beta_0]$ as

$$Gv := \text{ maximal solution in } [\alpha_0,\beta_0] \text{ of problem } (P_v). \tag{2.1}$$

It is clear that the solutions of problem (1.1)–(1.3) are the fixed points of function G. So, we must ensure that such fixed points exist. To this end, we verify that function G fulfills the conditions of Lemma 2.1.

To prove that G is nondecreasing, let $v_1,\ v_2 \in [\alpha_0, \beta_0]$ with $v_1 \le v_2$ on $\mathbb{T}$, and put $u_1 := Gv_1$ and $u_2 := Gv_2$. Let us see that $u_1 \le u_2$ on $\mathbb{T}$.

From the definition of G and the monotonicity assumptions imposed to functions f, B_1 and B_2, we arrive at the following inequalities:

$$-[\phi(u_1^{\Delta}(t))]^{\Delta} = f(t, u_1^{\sigma}(t), v_1) \le f(t, u_1^{\sigma}(t), v_2), \quad t \in I,$$

$$0 = B_1(u_1(a), v_1) \le B_1(u_1(a), v_2),$$
$$0 = B_2(u_1(\sigma^2(b)), v_1) \ge B_2(u_1(\sigma^2(b)), v_2).$$

This implies that u_1 is a lower solution for problem (P_{v_2}).

As a consequence, we know that u_2, the maximal solution in $[\alpha_0, \beta_0]$ of problem (P_{v_2}), is over u_1, i.e., $u_1 \le u_2$ on $\mathbb{T}$.

Now, let $\{v_n\}$ be a monotone sequence in $[\alpha_0, \beta_0]$ of continuous functions on $\mathbb{T}$. Since function G is monotone nondecreasing, we know that the sequence $\{Gv_n\}$ is monotone and bounded on $\mathbb{T}$. As a consequence it has a pointwise limit w in $\mathbb{T}$.

Now, following the proof of [9, Lemma 2.3], we have that any solution $u_v \in [\alpha_0, \beta_0]$ of problem (P_v) satisfies the following equality:

$$u_v^{\Delta}(t) = \phi^{-1}\left(\tau_u^v - \int_a^r f(s, u_v^{\sigma}(s), v)\,\Delta s\right)\Delta r, \tag{2.2}$$

with τ_v^u the unique solution of the expression

$$\begin{aligned}\int_a^{\sigma(b)} \phi^{-1}\left(\tau_v^u - \int_a^r f(s, u_v^{\sigma}(s), v)\,\Delta s\right)\Delta r \\ = u_v(\sigma^2(b)) - B_2(u_v(\sigma^2(b)), v) - u_v(a) - B_1(u_v(a), v).\end{aligned} \tag{2.3}$$

Now, by using the regularity and monotonicity assumptions on functions f, B_1 and B_2, one can verify that there exists a positive constant C, depending only on α_0 and β_0 such that $|\tau_v^u| \le C$ for all $v \in [\alpha_0, \beta_0]$. Now, equality (2.2) implies that for all $v \in [\alpha_0, \beta_0]$, it is satisfied that $|u_v^{\Delta}| \le Q$, with Q a positive constant depending on α_0 and β_0.

From the definition of function G, we deduce that for all $n \in \mathbb{N}$ the following inequality holds:

$$|G\,v_n(t) - G\,v_n(s)| = \left|\int_s^t (G\,v_n)^{\Delta}(r)\,\Delta r\right| \le Q\,|t-s| \quad \text{for all } s,\ t \in \mathbb{T}.$$

Now, by using the inequality

$$|w(t)-w(s)| \le |w(t)-G\,v_n(t)|+|G\,v_n(t)-G\,v_n(s)|+|G\,v_n(s)-w(s)| \quad \text{for all } s,\ t \in \mathbb{T},$$

we conclude that $w \in C(\mathbb{T})$ and, as a consequence the hypotheses of Lemma 2.1 are fulfilled. So, we have that the function G has a maximal fixed point u^* that is a solution of problem (1.1)–(1.3).

The fact that u^* is the maximal solution in $[\alpha_0, \beta_0]$ of this problem follows from the property

$$u^* = \max \{u \in [\alpha_0, \beta_0] \ : u \le Gu\}.$$

If there is a solution $u \in [u^*, \beta_0]$ of problem (1.1)–(1.3), then it is a solution of problem (P_u) and, from the monotonicity properties imposed to function f, we have that function u^* is a lower solution of such a problem. The definition of G tells us that $u \le Gu$, which contradicts the definition of u^*.

The existence of the minimal solution in $[\alpha_0, \beta_0]$ of problem (1.1)–(1.3) is proved similarly. ∎

3. An Example

In this section we present an example that ensures the existence of extremal solutions on bounded sets of a discontinuous functional problem. Moreover we obtain the exact expression of the maximal solution. To deduce this expression we use the following result, which is a direct consequence of [14, Corollary 1.2.2].

Lemma 3.1. Assume that the hypotheses of Theorem 2.2 hold. Let G be defined as in the proof of Theorem 2.2 and define the sequence $\{\beta_n\}$ as $\beta_{n+1} = G\,\beta_n$.

If functions f, B_1 and B_2 are right continuous, then the sequence $\{\beta_n\}$ converges in $C(\mathbb{T})$ to the maximal solution in $[\alpha_0, \beta_0]$ of problem (1.1)–(1.3).

Example 3.2. Let $\mathbb{T}$ be a bounded time scale such that the graininess function μ is constant. Let $A \in \mathbb{R}$, $q \in \mathbb{R}$, $0 < D\,(1 + \mu\, q) < q^2$, $0 < B < 1$ and $\eta \in \mathbb{T}$ be fixed.

Denoting by $[x]$ the integer part of a real number x, we consider the nonlinear functional problem

$$(P)\begin{cases} -u^{\Delta\Delta}(t) = -\dfrac{q^2}{1+\mu\, q} u^\sigma(t) + D\,[u(\eta)] & \text{for all } t \in \mathbb{T}, \\ u(a) = A, & \\ u(\sigma^2(b)) = B\,[u(\eta)]. & \end{cases}$$

It is not difficult to verify that problem (P) is of the form (1.1)–(1.3) for the particular case of

$$\phi(x) = x,$$

$$f(t, x, v) = -\frac{q^2}{1+\mu\, q} x + D\,[v(\eta)],$$

$$B_1(x, v) = -x + A,$$

and

$$B_2(x, v) = x - B\,[v(\eta)].$$

One can verify that the conditions of Theorem 2.2 are fulfilled. As a consequence, for N large enough, problem (P) has extremal solutions in $S \cap [-N, N]$.

In this situation problem (P_v), considered in the proof of Theorem 2.2, follows the form

$$(P_v) \begin{cases} -u^{\Delta\Delta}(t) &= -\dfrac{q^2}{1+\mu\, q} u^{\sigma}(t) + D\,[v(\eta)] \quad \text{for all } t \in \mathbb{T}, \\ u(a) &= A, \\ u(\sigma^2(b)) &= B\,[v(\eta)]. \end{cases}$$

The unique solution of this problem, for $v \in C(\mathbb{T})$ fixed, is given by the following expression (see [6, Theorem 3.45]):

$$\begin{aligned} u_v(t) = {} & \left(A - D\,\frac{1+q\mu}{q^2}[v(\eta)]\right) e_q(t,a) \\ & + C_v\, e_q(t,a) \int_a^t e_q^2(a,s)\,\Delta s + D\,\frac{1+q\mu}{q^2}[v(\eta)], \end{aligned}$$

with

$$C_v = \frac{-A\, e_q(\sigma^2(b),a) + [v(\eta)]\left(D\,\dfrac{1+q\mu}{q^2}\left(e_q(\sigma^2(b),a)-1\right) + B\right)}{e_q(\sigma^2(b),a) \displaystyle\int_a^{\sigma^2(b)} e_q^2(a,s)\,\Delta s}.$$

So, starting at $\beta_0 > 0$ large enough, we have that the sequence $\{\beta_n\}$ satisfies the following recursive equation

$$\begin{aligned} \beta_{n+1}(t) = {} & \left(A - D\,\frac{1+q\mu}{q^2}[\beta_n(\eta)]\right) e_q(t,a) \\ & + C_n\, e_q(t,a) \int_a^t e_q^2(a,s)\,\Delta s + D\,\frac{1+q\mu}{q^2}[\beta_n(\eta)], \end{aligned}$$

with

$$C_n = \frac{-A\, e_q(\sigma^2(b),a) + [\beta_n(\eta)]\left(D\,\dfrac{1+q\mu}{q^2}\left(e_q(\sigma^2(b),a)-1\right) + B\right)}{e_q(\sigma^2(b),a) \displaystyle\int_a^{\sigma^2(b)} e_q^2(a,s)\,\Delta s}.$$

Since the integer part of a real number is a right-continuous function, we have, from Lemma 3.1, that the sequence $\{\beta_n\}$ converges in $C(\mathbb{T})$ to the maximal solution u under β_0 of problem (P), and it is given by the expression

$$u(t) = \left(A - D\,\frac{1+q\mu}{q^2}[\tau]\right) e_q(t,a) + C\, e_q(t,a) \int_a^t e_q^2(a,s)\,\Delta s + D\,\frac{1+q\mu}{q^2}[\tau],$$

with

$$C = \frac{-A\, e_q(\sigma^2(b), a) + [\tau] \left(D \, \frac{1 + q\mu}{q^2} \left(e_q(\sigma^2(b), a) - 1 \right) + B \right)}{e_q(\sigma^2(b), a) \displaystyle\int_a^{\sigma^2(b)} e_q^2(a, s)\, \Delta s}.$$

Here τ is the maximal solution under β_0 of the equation

$$x = \left(A - D\, \frac{1 + q\mu}{q^2} [x] \right) e_q(\eta, a) + C_x\, e_q(\eta, a) \int_a^{\eta} e_q^2(a, s)\, \Delta s + D\, \frac{1 + q\mu}{q^2} [x],$$

with

$$C_x = \frac{-A\, e_q(\sigma^2(b), a) + [x] \left(D \, \frac{1 + q\mu}{q^2} \left(e_q(\sigma^2(b), a) - 1 \right) + B \right)}{e_q(\sigma^2(b), a) \displaystyle\int_a^{\sigma^2(b)} e_q^2(a, s)\, \Delta s}.$$

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