International Journal of Difference Equations (IJDE).
ISSN 0973-6069 Volume 2 Number 1 (2007), pp. 79–91
© Research India Publications
http://www.ripublication.com/ijde.htm

Heim–Lorek Supermodels

Nicole Garbers and Andreas Ruffing
Fakultät für Mathematik, Technische Universität München,
Boltzmannstraße 3, D-85747 Garching, Germany
E-mail: ngarbers@web.de, ruffing@ma.tum.de

Abstract

We first give a short review on supermodels in quantum mechanics where the main focus will be on selfsimilar supermodels, showing a point spectrum which consists of basic q-versions of the natural numbers. We want to relate these supermodels to discrete versions of Schrödinger operators which exhibit — at least partially — the same type of discrete point spectrum. To do so, the concept of strip discretizations is reviewed on basic linear grids. This type of discretization shows the typical point spectrum, consisting of basic q-versions of the natural numbers. Precisely the same type of spectrum is finally also presented in case of basic multigrid discretizations. We therefore obtain a unified discrete model of some Schrödinger equations which allow both, piecewise continuous solutions (Section 4) and sophisticated multigrid solutions (Section 5) — a scenario which plays already a great role in approaches established by A. Lorek and B. Heim.

AMS subject classification: 39A10, 39A13, 35Q40, 46N20.
Keywords: Schrödinger operator, quantum calculus, superpotential.

1. Schrödinger Equations and Superpotentials

Let us first review some basic facts which we had also stated in [7].

Supersymmetry is one of the most powerful tools being applied to problems of theoretical physics. In the last years, there were great achievements especially on the area of supersymmetry in quantum mechanics. For an excellent contribution to the topic see for instance the articles by M. Robnik [19] and M. Robnik and J. Liu [12]. The

Received January 23, 2007; Accepted June 12, 2007

stationary one-dimensional version of Schrödinger's equation (λ being a fixed value in $\mathbb{R}$)

$$-\psi''(x) + V(x)\psi(x) = \lambda\psi(x), \qquad x \in \mathbb{R} \tag{1.1}$$

can with some general success be factorized by using the concept of so-called superpotentials.

In Schrödinger theory, the following scenario is of particular interest: Given two Schrödinger equations with different potentials V_1 and V_2. Under some circumstances, it is possible to write them in the form

$$B^+B\varphi = \lambda\varphi, \qquad BB^+\psi = \mu\psi,$$

where B and B^+ are formally adjoint to each other, being defined on some common domain in $\mathcal{L}^2(\mathbb{R})$. Let us shortly review the method how to address the stated factorization problem.

The first step is the construction of a so-called **superpotential** W such that one can express the **partner potentials** V_1, V_2 as follows:

$$V_1 = \frac{1}{2}(W^2 - \sqrt{2}W'), \qquad V_2 = \frac{1}{2}(W^2 + \sqrt{2}W').$$

The superpotential W is fixed by assuming that the potential V_1 allows 0 as an eigenvalue, the corresponding eigenfunction φ being positive. This leads to the condition

$$-\varphi''(x) + \frac{1}{2}(W^2(x) - \sqrt{2}W'(x))\varphi(x) = 0, \qquad x \in \mathbb{R}.$$

A solution to this equation is given by

$$W(x) = -\sqrt{2}(\ln\varphi)'(x), \qquad x \in \mathbb{R}.$$

The aimed factorization is now achieved by the equality

$$H_1 = B^+B, \qquad H_2 = BB^+,$$

where the differential operators H_1, H_2 are specified by the **supersymmetric ladder operators**

$$B := \frac{1}{\sqrt{2}}\left(W + \sqrt{2}\frac{d}{dx}\right), \qquad B^+ := \frac{1}{\sqrt{2}}\left(W - \sqrt{2}\frac{d}{dx}\right).$$

To illustrate this formalism, let us consider the two potentials

$$V_1(x) = \frac{x^2}{4} - \frac{1}{2}, \qquad V_2(x) = \frac{x^2}{4} + \frac{1}{2}, \qquad x \in \mathbb{R}.$$

As for the superpotential W, we obtain just $W(x) = \sqrt{2}x$, leading to the well-understood conventional ladder operator formalism

$$H_1 = B^+B, \qquad H_2 = BB^+,$$

$$B = \frac{1}{\sqrt{2}}\left(x + \sqrt{2}\frac{d}{dx}\right), \qquad B^{+} = \frac{1}{\sqrt{2}}\left(x - \sqrt{2}\frac{d}{dx}\right).$$

The key message is now that one can determine the point spectrum of H_1, H_2 completely by using the operators B, B^+. A further, more illustrative example is the so-called Rosen–Morse potential, in which a real parameter y occurs:

$$V_1(x, y) = y^2 - \frac{y(y+1)}{\cosh^2(x)}, \qquad x \in \mathbb{R}.$$

Assuming that the equation

$$-\varphi''(x) + V_1(x, y)\varphi(x) = 0$$

has a solution $\varphi \in \mathcal{L}^2(\mathbb{R})$, we are first led to the superpotential

$$W(x) = \sqrt{2}y \tanh(x), \qquad x \in \mathbb{R}$$

as well as to the potential

$$V_2(x, y) = y^2 - \frac{y(y-1)}{\cosh^2(x)}, \qquad x \in \mathbb{R}.$$

The difference of the two partner potentials is given by

$$V_2(x, y) - V_1(x, y-1) = \frac{2y-1}{2}, \qquad x \in \mathbb{R}.$$

Generalizing the observations made so far, the two potentials V_1, V_2 are called **form-invariant** if the following identity holds for different values of x, y_1, y_2, the expression $R(y_1)$ being a continuous function of y_1:

$$V_2(x, y_1) = V_1(x, y_2) + R(y_1). \tag{1.2}$$

According to the above construction, the pair B, B^+ specifies two different Schrödinger equations, which together are referred to as a **supersymmetric model** or just as a **supermodel**.

2. Selfsimilar Supermodels

In many important applications, it follows from the defining equation for form invariance

$$V_2(x, y_1) = V_1(x, y_2) + R(y_1) \tag{2.1}$$

that $y_2 = y_1 + h$ where h is a real constant. A completely new class of form invariant potentials has been proposed in [11], where potentials were constructed whose parameters are related to each other by

$$y_2 = q y_1, \qquad 0 < q < 1. \tag{2.2}$$

In order to make apparent what kind of possible point spectrum is generated by the property (2.2), we follow the basic outline in [10] where the superpotential is expanded as

$$W(x, y) = \sum_{j=0}^{\infty} g_j(x) y^j$$

for some suitable parameter $y \in \mathbb{R}$. The function R from (1.2) is assumed to be given by an analytic ansatz

$$R(y) = \sum_{j=0}^{\infty} R_j y^j. \tag{2.3}$$

Plugging this ansatz into formula (2.1) yields

$$\begin{aligned} R(y) &= V_2(x, y) - V_1(x, qy) \\ &= W^2(x, y) - W^2(x, qy) + \sqrt{2}((\partial_x W)(x, y) + (\partial_x W)(x, qy)). \end{aligned}$$

Inserting now the expansion (2.3) for the function R, one obtains by comparing the coefficients

$$R_n = \frac{1}{2} \sum_{i=1}^{\infty} (1 + q^{n-i})(1 - q^i) g_i g_{n-i} + \sqrt{2}(1 + q^n) g_n', \qquad n \in \mathbb{N}$$

and the value R_0 being given by $R_0 = g_0'$. With the abbreviations

$$r_n := \frac{R_n}{1 - q^n}, \qquad d_n := \frac{1 - q^n}{1 + q^n}, \qquad n \in \mathbb{N}$$

one is led to nonlinear integral equations, given by

$$g_n(x) = \frac{d_n}{\sqrt{2}} \int_a^x \left(2r_n - \sum_{i=1}^{n-1} g_i(t) g_{n-i}(t) \right) dt, \qquad x \in \mathbb{R}, \qquad n \in \mathbb{N},$$

where restrictions of the solutions of these equations are put by the conditions

$$R_0 = 0, \qquad g_0(x) = 0, \qquad r_n = z\delta_{n1}, \qquad n \in \mathbb{N},$$

δ_{n1} denoting the Kronecker symbol and z being a positive parameter. This nonlinear integral equation allows now the solutions

$$R(y) = R_1 y = R, \qquad g_n(x) = \frac{1}{\sqrt{2}} \beta_n x^{2n-1}, \qquad x \in \mathbb{R}, \qquad n \in \mathbb{N},$$

where the coefficients β_n are fixed by the recurrence formula

$$\beta_1 = \frac{2R_1}{1 + q}, \qquad \beta_0 = 0, \qquad \beta_n = -\frac{d_n}{2n - 1} \sum_{i=1}^{n-1} \beta_i \beta_{n-i}, \qquad n \in \mathbb{N} \setminus \{1\}.$$

The superpotential now reads

$$W(x, y) = \sum_{j=1}^{\infty} \beta_j y^j \left(\frac{x}{\sqrt{2}}\right)^{2j-1}, \qquad x \in \mathbb{R}.$$

The formal groundstate, belonging to V_1, is given by the formula

$$\psi_0(x, y) = Ce^{-\sum_{j=1}^{\infty} \frac{\beta_j}{2j} y^j (\frac{x}{\sqrt{2}})^{2j}}, \qquad x \in \mathbb{R}.$$

Direct calculation leads to the formula

$$W(x, y_2) = \sqrt{q}W(\sqrt{q}x, y_1), \qquad x \in \mathbb{R}$$

showing some selfsimilar property. This type of superpotential is therefore also referred to by the name **selfsimilar superpotential**. Applying a generalized version of the ladder operator formalism, one is led to the energies of the operator H_1, being given by

$$\lambda_n = R \sum_{j=0}^{n-1} q^j = R\frac{1-q^n}{1-q}, \qquad n \in \mathbb{N}_0. \tag{2.4}$$

Let us now arrive at an interesting isospectrality scenario in context of the q-discretization, where we are interested in discrete versions of Schrödinger operators which allow (2.4) as their (at least partial) point spectrum.

3. Using Heim–Lorek Discretizations

A. Lorek investigated in her PhD thesis from 1995 several quantum mechanical toy models in which the phase space is discrete and which link in an elegant way continuous solutions and discrete solutions of q-difference equations in context of quantum mechanics, [14]. This is a fancy propagation of original ideas by B. Heim on discretized versions of fundamental equations in mathematical physics, see [8]. We speak therefore in the sequel of the **Heim–Lorek discretization** of Schrödinger operators: We address Schrödinger q-difference equations where we study piecewise continuous solutions to these equations, having support on some kind of strip structures which are generated by the symmetries of the lattice $\{+q^n, -q^n | n \in \mathbb{Z}\}$ and also sophisticated related discrete solutions to these equations. The concept of strip discretizations is reviewed on basic linear grids: A brief review on related results by N. Garbers and A. Ruffing, see [7], will happen in **Section 4**. Already these results reveal a new and challenging aspect of isospectrality. In **Section 5**, we will address the concept of basic multigrid discretizations which shed new light on applications of basic linear grids within numerical analysis. Again, it is isospectrality which turns out to play a key role in understanding the related spectra. The main surprise is that there exist in parallel piecewise continuous solutions (Section 4) and multigrid discrete solutions to the same Schrödinger q-difference equations (Section 5). This perfectly matches the original scenario of the Heim–Lorek discretization.

4. Strip Discretizations and Isospectrality

Throughout this section, we refer to the recently obtained results by N. Garbers and A. Ruffing on strip discretizations, see [7]. As stated above, we will generalize these results in Section 5 to multigrid discretizations.

Let us refer throughout the sequel to a parameter $0 < q < 1$, as it was motivated by the investigation of selfsimilar superpotentials in the previous section.

The following result reveals that the discrete Schrödinger equation with an oscillator potential on Ω shows similar properties than its classical analog does.

Definition 4.1. [Strip Discretization] Let $\Omega \subseteq \mathbb{R} \setminus \{0\}$ be a nonempty closed set with Lebesgue measure $\mu(\Omega) > 0$ as well as

$$\forall x \in \Omega : \qquad qx \in \Omega, \qquad q^{-1}x \in \Omega, \qquad -x \in \Omega.$$

We call the time scale Ω a homogeneous strip discretization or just **strip discretization** of the configuration space. The Hilbert space of the strip discretization is introduced by the requirement

$$\mathcal{L}^2(\Omega) := \{f \in \mathcal{L}^2(\mathbb{R}) \mid f = f \circ \chi_\Omega\},$$

and the scalar product of two functions $f, g \in \mathcal{L}^2(\Omega)$ is introduced by

$$(f, g)_\Omega := \int_{-\infty}^{\infty} f(x)\overline{g(x)}\chi_\Omega(x)dx = \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx, \tag{4.1}$$

using the **characteristic function** χ_Ω of the time scale Ω. By construction, it is clear that $\mathcal{L}^2(\Omega)$ is a Hilbert space over $\mathbb{C}$, being a proper subspace of the square-integrable functions themselves, i.e., of $\mathcal{L}^2(\mathbb{R})$. In order to proceed, let us first review some facts on the Schrödinger equation with quadratic potential, given by

$$-\psi''(x) + x^2\psi(x) = \lambda\psi(x), \qquad x \in \mathbb{R}.$$

The following structure is one the most familiar facts within mathematical physics: Let the sequence of functions $(\psi_n)_{n\in\mathbb{N}_0}$ be recursively given by the requirement

$$\psi_{n+1}(x) := -\psi_n'(x) + x\psi_n(x), \qquad x \in \mathbb{R}, \qquad n \in \mathbb{N}_0,$$

where $\psi_0 : \mathbb{R} \to \mathbb{R}, x \mapsto \psi_0(x) := e^{-\frac{1}{2}x^2}$. We then have $\psi_n \in \mathcal{L}^2(\mathbb{R}) \cap C^2(\mathbb{R})$ for $n \in \mathbb{N}_0$ and moreover

$$-\psi_n''(x) + x^2\psi_n(x) = (2n+1)\psi_n(x), \qquad x \in \mathbb{R}, \qquad n \in \mathbb{N}_0.$$

This result reflects the conventional **ladder operator formalism**. We now develop a result in discrete Schrödinger theory on strip structures which turns out to be a q-analog of the just described continuous situation. Let us therefore state in a next step some more useful tools for the strip discretization approach.

Definition 4.2. Let $\Omega \subseteq \mathbb{R} \setminus \{0\}$ be a nonempty closed set with the property $\mu(\Omega) > 0$ as well as

$$\forall x \in \Omega : \qquad qx \in \Omega, \qquad q^{-1}x \in \Omega, \qquad -x \in \Omega. \tag{4.2}$$

Let for any $f : \Omega \to \mathbb{R}$ the right-shift resp. left-shift operation be defined by

$$(Rf)(x) := f(qx), \qquad (Lf)(x) := f(q^{-1}x), \qquad x \in \Omega.$$

Respectively, the right-hand resp. left-hand q-difference operation shall for any $f : \Omega \to \mathbb{R}$ be given by

$$(D_q f)(x) := \frac{f(qx) - f(x)}{qx - x}, \qquad (D_{q^{-1}} f)(x) := \frac{f(q^{-1}x) - f(x)}{q^{-1}x - x}, \qquad x \in \Omega.$$

Let moreover $\alpha > 0$ and let

$$g : \Omega \to \mathbb{R}^+, \qquad x \mapsto g(x) := \frac{\sqrt{\varphi(qx)} - \sqrt{\varphi(x)}}{\sqrt{\varphi(x)}(q-1)x} = \frac{\sqrt{1 + \alpha(1-q)x^2} - 1}{qx - x}, \tag{4.3}$$

where the positive even continuous function $\varphi : \Omega \to \mathbb{R}^+$ is chosen as a solution to the q-difference equation

$$\varphi(qx) = (1 + \alpha(1-q)x^2)\varphi(x), \qquad x \in \Omega. \tag{4.4}$$

We are now able to define **discrete ladder operators** on strip structures.

The **creation operation** $A_q^\dagger$ resp. **annihilation operation** A_q are introduced by their actions on any $\psi : \Omega \to \mathbb{R}$ as follows:

$$A_q^\dagger \psi = (-D_q + g(X)R)\psi, \qquad A_q \psi = q^{-1}(LD_q + Lg(X))\psi.$$

We refer to the **discrete Schrödinger equation** with an oscillator potential on Ω by

$$q^{-1}(-D_q + g(X)R)(LD_q + Lg(X))\psi = \lambda\psi. \tag{4.5}$$

The following result reveals that the discrete Schrödinger equation with an oscillator potential on Ω shows similar properties than its classical analog does.

Theorem 4.3. Let the time scale Ω be a strip discretization in the sense of Definition 4.1 and let the function φ be specified like in Definition 4.2, satisfying the q-difference equation (4.4) on Ω:

$$\varphi(qx) = (1 + \alpha(1-q)x^2)\varphi(x), \qquad \varphi(x) = \varphi(-x) > 0, \qquad x \in \Omega.$$

For $n \in \mathbb{N}_0$, the functions $\psi_n : \Omega \to \mathbb{R}$, given by $\psi_n(x) := ((A_q^\dagger)^n \sqrt{\varphi})(x)$, $x \in \Omega$ are well defined in $\mathcal{L}^2(\Omega)$ and solve the q-Schrödinger equation (4.5) in the following sense:

$$q^{-1}(-D_q + g(X)R)(LD_q + Lg(X))\psi_n = \frac{\alpha}{q}\frac{q^n - 1}{q - 1}\psi_n. \tag{4.6}$$

Moreover, the linear maps $A_q, A_q^\dagger$ act as ladder operators on the functions $(\psi_n)_{n\in\mathbb{N}_0}$ in the following sense ($n \in \mathbb{N}_0$, $\psi_{-1} := 0$):

$$A_q^\dagger \psi_n = \psi_{n+1}, \qquad A_q\psi_n = \frac{\alpha}{q}\frac{q^n-1}{q-1}\psi_{n-1}, \qquad \psi_n(x) = H_n^q(x)\psi_0(x), \qquad x \in \Omega, \tag{4.7}$$

where for $n \in \mathbb{N}_0$, the functions $H_n^q : \Omega \to \mathbb{R}$ are given by

$$H_{n+1}^q(x) - \alpha q^n x H_n^q(x) + \alpha\frac{q^n-1}{q-1}H_{n-1}^q(x) = 0, \qquad H_0^q(x) = 1, \qquad H_1^q(x) = \alpha x.$$

$$(A_q^\dagger\psi_m, \psi_n)_\Omega = (\psi_m, A_q\psi_n)_\Omega, \qquad m, n \in \mathbb{N}_0,$$

and the functions $(\psi_n)_{n\in\mathbb{N}_0}$ constitute an orthonormal system in $\mathcal{L}^2(\Omega)$.

Proof. Let us for $\varphi \in C(\mathbb{R})$ first consider the equation

$$\varphi(qx)x^n = (1 + \alpha(1-q)x^2)\varphi(x)x^n, \qquad x \in \Omega, \qquad n \in \mathbb{N}_0,$$

which directly follows from (4.4). Using standard arguments, one can show that the function φ fulfilling (4.4) is in $\mathcal{L}^1(\mathbb{R})$. This implies

$$\int_{-\infty}^{\infty} \varphi(qx)x^n\chi_\Omega(x)dx = \int_{-\infty}^{\infty} (1 + \alpha(1-q)x^2)\varphi(x)x^n\chi_\Omega(x)dx, \qquad n \in \mathbb{N}_0.$$

Using the substitution rule on the left-hand side, this directly implies

$$\int_{-\infty}^{\infty} \varphi(t)t^n q^{-n}\chi_\Omega(q^{-1}t)q^{-1}dt = \int_{-\infty}^{\infty} (1 + \alpha(1-q)x^2)\varphi(x)x^n\chi_\Omega(x)dx, \qquad n \in \mathbb{N}_0. \tag{4.8}$$

Because of (4.2) we have $\chi_\Omega(q^{-1}t) = \chi_\Omega(t)$ for any $t \in \mathbb{R}$ and therefore, (4.8) is equivalent to

$$\int_{-\infty}^{\infty} \varphi(t)t^n q^{-n}\chi_\Omega(t)q^{-1}dt = \int_{-\infty}^{\infty} (1 + \alpha(1-q)x^2)\varphi(x)x^n\chi_\Omega(x)dx, \qquad n \in \mathbb{N}_0.$$

Using the abbreviation $\mu_n(\Omega) := \int_\Omega x^n\varphi(x)dx$ for $n \in \mathbb{N}_0$ we obtain the following result:

$$\mu_{2n+2}(\Omega) = \frac{q^{-2n-1}-1}{\alpha(1-q)}\mu_{2n}(\Omega), \qquad \mu_{2n+1}(\Omega) = 0, \qquad n \in \mathbb{N}_0. \tag{4.9}$$

We have shown earlier that any probability density ψ which generates moments of type (4.9) yields an orthogonality measure to the polynomials $(H_n^q)_{n\in\mathbb{N}_0}$ which are for $k \in \mathbb{N}$ fixed through the recurrence relation

$$H_{k+1}^q(x) - \alpha q^k x H_k^q(x) + \alpha \frac{q^k - 1}{q - 1} H_{k-1}^q(x) = 0, \qquad H_0^q(x) = 1, \qquad H_1^q(x) = \alpha x,$$

the variable x being chosen in a suitable integration support. As a consequence of this general result, we obtain the orthogonality relation

$$\int_{-\infty}^{\infty} H_m^q(x) H_n^q(x) \varphi(x) \chi_\Omega(x) dx = v_n(\Omega)\delta_{mn}, \qquad m, n \in \mathbb{N}_0,$$

with a sequence of positive numbers $(v_n(\Omega))_{n\in\mathbb{N}_0}$. Direct calculations and induction show

$$\psi_n(x) := ((A_q^\dagger)^n \sqrt{\varphi} \circ \chi_\Omega)(x) = (H_n^q(X)\sqrt{\varphi} \circ \chi_\Omega)(x), \qquad x \in \mathbb{R}, \qquad n \in \mathbb{N}_0.$$

Let us from now on — without any restriction — refer to the special parameter choice $\alpha = 1$. The functions $(\psi_n)_{n\in\mathbb{N}_0}$ constitute an orthonormal system in $\mathcal{L}^2(\Omega)$. Let us show next that the ladder property (4.7) is fulfilled. The first equation in (4.7) is trivial due to the definition of the functions $(\psi_n)_{n\in\mathbb{N}_0}$. We remember that the function g is specified like in Definition 4.2. We obtain in the sense of the multiplication operator notation

$$(LD_q + Lg(X))(X^n\psi_0) = LD_q X^n \psi_0 + Lg(X)X^n\psi_0, \qquad n \in \mathbb{N}_0$$

which yields

$$(LD_q+Lg(X))(X^n\psi_0) = L\left(\frac{q^n - 1}{q - 1} X^{n-1} R\psi_0 + X^n D_q \psi_0\right) + Lg(X)X^n\psi_0, \quad n \in \mathbb{N}_0.$$

This may be rewritten as

$$(LD_q + Lg(X))(X^n\psi_0) = \frac{q^n - 1}{q - 1} q^{-n+1} X^{n-1}\psi_0 + LX^n(D_q\psi_0 + g\psi_0), \quad n \in \mathbb{N}_0.$$

Using now however the formulas in (4.3) for the function g, we obtain $(D_q\psi_0 + g\psi_0) = 0$ and therefore

$$(LD_q + Lg(X))(X^n\psi_0) = \frac{q^n - 1}{q - 1} q^{-n+1} X^{n-1}\psi_0, \qquad n \in \mathbb{N}.$$

For $m \in \mathbb{N}_0$, the first $m + 1$ polynomials of the sequence $(H_n^q(X))_{n\in\mathbb{N}_0}$ can uniquely be generated by linear combinations of the first $m+1$ monomials of the sequence $(X^n)_{n\in\mathbb{N}_0}$. We therefore conclude

$$A_q\psi_n = \sum_{j=0}^{n-1} c_j^n \psi_j, \qquad n \in \mathbb{N}$$

with uniquely defined real numbers c_j^n where $j = 0, \dots, n-1$ with $n \in \mathbb{N}$. Applying again standard substitution techniques to the scalar product integral (4.1), we can derive for any functions $f, g \in \mathcal{L}^2(\Omega)$ which are both in the algebraic span of the functions $(\psi_n)_{n\in\mathbb{N}_0}$ the following relation:

$$(A_q^\dagger f, g)_\Omega = (f, A_q g)_\Omega.$$

In particular, this result implies

$$(A_q^\dagger \psi_m, \psi_n)_\Omega = (\psi_m, A_q \psi_n)_\Omega, \qquad m, n \in \mathbb{N}_0.$$

Using the first equation in (4.7) and because of the fact that the functions $(\psi_n)_{n\in\mathbb{N}_0}$ constitute an orthogonal system in $\mathcal{L}^2(\Omega)$, the second relation in (4.7) follows from standard methods of calculating the norms of the functions $(\psi_n)_{n\in\mathbb{N}_0}$.

Equation (4.6) now follows immediately from the first two relations in (4.7). Taking all the steps of the proof together, this finally confirms the statements of Theorem 4.3.■

5. Basic Multigrid Discretizations and Isospectrality

In this last section, we concentrate our interest to isospectrality results in context of some kind of special type of multigrid discretization. We will generalize the results from Section 4 to purely discrete versions of Heim–Lorek discretizations. To do so, we first provide the following.

Definition 5.1. [Basic Multigrid Discretizations] Let for a fixed value $n \in \mathbb{N}$ the pairwise different real numbers $c_1, \dots, c_n$ be given and consider a finite set $M \subseteq \mathbb{Z} \setminus \{0\}$ such that all

$$\Omega_m^j := \{+c_j q^{\frac{k}{m}}, -c_j q^{\frac{k}{m}} | k \in \mathbb{Z}\},$$

where $m \in M$ and $j \in \{1, \dots, n\}$, are pairwise disjoint. Let moreover

$$\Omega := \bigcup_{j\in\{1,\dots,n\}} \bigcup_{m\in M} \Omega_m^j.$$

The Hilbert space of the **basic multigrid discretization** is denoted by $\mathcal{L}^2(\Omega)$ and introduced as the set

$$\left\{ f : \Omega \to \mathbb{C} \;\middle|\; \sum_{m\in M} \sum_{j=1}^{n} \sum_{k=-\infty}^{\infty} \sum_{\sigma\in\{1,-1\}} |c_j| (q^{\frac{k}{m}} - q^{\frac{k+1}{m}}) f(\sigma c_j q^{\frac{k}{m}}) \overline{f(\sigma c_j q^{\frac{k}{m}})} < \infty \right\}.$$

For any $f : \Omega \to \mathbb{C}$, the shift operators shall — as previously — be defined by

$$(Rf)(x) := f(qx), \qquad (Lf)(x) := f(q^{-1}x), \qquad x \in \Omega.$$

And again, the q-difference operations shall for any $f : \Omega \to \mathbb{C}$ be given by

$$(D_q f)(x) := \frac{f(qx) - f(x)}{qx - x}, \qquad (D_{q^{-1}} f)(x) := \frac{f(q^{-1}x) - f(x)}{q^{-1}x - x}, \qquad x \in \Omega.$$

Definition 5.2. [Discrete Ladder Operators and Basic Multigrids] Let $\alpha > 0$ and let Ω be a basic multigrid discretization in the sense of the last definition. Let moreover

$$g : \Omega \to \mathbb{R}^+, \qquad x \mapsto g(x) := \frac{\sqrt{\varphi(qx)} - \sqrt{\varphi(x)}}{\sqrt{\varphi(x)}(q-1)x} = \frac{\sqrt{1 + \alpha(1-q)x^2} - 1}{qx - x},$$

where the positive even continuous function $\varphi : \Omega \to \mathbb{R}^+$ is chosen as a solution to the q-difference equation

$$\varphi(qx) = (1 + \alpha(1-q)x^2)\varphi(x), \qquad \varphi(x) = \varphi(-x) > 0, \qquad x \in \Omega.$$

The **creation operation** $A^\dagger$ resp. **annihilation operation** A are again introduced by their actions on any $\psi : \Omega \to \mathbb{R}$ as follows:

$$A^\dagger \psi = (-D_q + g(X)R)\psi, \qquad A\psi = q^{-1}(LD_q + Lg(X))\psi.$$

Like in the strip discretization case, we refer to the **discrete Schrödinger equation** with an oscillator potential on Ω by

$$q^{-1}(-D_q + g(X)R)(LD_q + Lg(X))\psi = \lambda\psi. \tag{5.1}$$

Evaluating the moments of the density under consideration, we see — and this is a certain kind of surprise — that the density generates — up to a constant factor — the same moments that the corresponding density in the strip discretization case generates. Using a similar argumentation than in the strip discretization case from the last subsection, this directly leads to the basic multigrid analogon of Theorem 4.3.

Theorem 5.3. [Solutions of the Schrödinger Oscillator Equation] Let Ω be a basic multigrid discretization in the sense of Definition 5.1 and let the function φ be specified like in Definition 5.2, satisfying the q-difference equation (4.4) on Ω. Under these assertions, for $n \in \mathbb{N}_0$, the functions $\psi_n : \Omega \to \mathbb{R}$, given by $\psi_n(x) := ((A^\dagger)^n \sqrt{\varphi})(x)$, $x \in \Omega$ are well defined in $\mathcal{L}^2(\Omega)$ and solve the q-Schrödinger equation (5.1)

$$q^{-1}(-D_q + g(X)R)(LD_q + Lg(X))\psi_n = \alpha \frac{q^n - 1}{q - 1}\psi_n, \qquad n \in \mathbb{N}_0.$$

In analogy to the strip discretization case, the linear maps $A, A^\dagger$ act as ladder operators on the functions $(\psi_n)_{n \in \mathbb{N}_0}$ in the following sense ($n \in \mathbb{N}_0$, $\psi_{-1} := 0$):

$$A^\dagger \psi_n = \psi_{n+1}, \qquad A\psi_n = \frac{q^n - 1}{q - 1}\psi_{n-1}, \qquad \psi_n(x) = H_n^q(x)\psi_0(x),$$

where for $n \in \mathbb{N}_0$, the functions $H_n^q : \Omega \to \mathbb{R}$ are given by

$$H_{n+1}^q(x) - \alpha q^n x H_n^q(x) + \alpha \frac{q^n - 1}{q - 1} H_{n-1}^q(x) = 0, \qquad H_0^q(x) = 1, \qquad H_1^q(x) = \alpha x,$$

the recurrence relations applying again for $x \in \Omega$ and $n \in \mathbb{N}_0$, where we have chosen $\psi_{-1} := 0$, $H_{-1}^q := 0$. In the sense of the Hilbert space scalar product, we have also here the general observation

$$(A^\dagger \psi_m, \psi_n)_\Omega = (\psi_m, A\psi_n)_\Omega, \qquad m, n \in \mathbb{N}_0,$$

the functions $(\psi_n)_{n \in \mathbb{N}_0}$ constituting an orthonormal system in $\mathcal{L}^2(\Omega)$.

References

[1] M. Arik and D. D. Coon, Hilbert spaces of analytic functions and generalized coherent states, *J. Mathematical Phys.*, 17(4):524–527, 1976.

[2] R. Askey and S.K. Suslov, The q-harmonic oscillator and an analogue of the Charlier polynomials, *J. Phys. A*, 26(15):L693–L698, 1993.

[3] R. Askey and S.K. Suslov, The q-harmonic oscillator and the Al-Salam and Carlitz polynomials, *Lett. Math. Phys.*, 29(2):123–132, 1993.

[4] Christian Berg and Andreas Ruffing, Generalized q-Hermite polynomials, *Comm. Math. Phys.*, 223(1):29–46, 2001.

[5] Jacob Stordal Christiansen, The moment problem associated with the q-Laguerre polynomials, *Constr. Approx.*, 19(1):1–22, 2003.

[6] V.V. Dodonov, "Nonclassical" states in quantum optics: a "squeezed" review of the first 75 years, *J. Opt. B Quantum Semiclass. Opt.*, 4(1):R1–R33, 2002.

[7] Nicole Garbers and Andreas Ruffing, Using supermodels in quantum optics, *Adv. Difference Equ.*, pages Art. ID 72768, 14, 2006.

[8] Burkhard Heim, *Elementarstrukturen der Materie*, Resch Verlag, Innsbruck, Austria, 1984.

[9] Mourad E.H. Ismail and Mizan Rahman, The q-Laguerre polynomials and related moment problems, *J. Math. Anal. Appl.*, 218(1):155–174, 1998.

[10] H. Kalka and G. Soff, *Supersymmetrie*, Edition Teubner, Stuttgart, Germany, 1997.

[11] A. Khare and U.P. Sukhatme, New shape-invariant potentials in supersymmetric quantum mechanics, *J. Phys. A*, 26(18):L901–L904, 1993.

[12] J. Liu and Marko Robnik, Supersymmetric quantum mechanics based on higher excited states II: a few new examples of isospectral partner potentials, 1997. Preprint CAMTP.

[13] X.-M. Liu and C. Quesne, Even and odd q-deformed charge coherent states and their nonclassical properties, *Phys. Lett. A*, 317(3-4):210–222, 2003.

[14] A. Lorek, *Dissertation*, PhD thesis, Universität München, 1995.

[15] K.A. Penson and A.I. Solomon, New generalized coherent states, *J. Math. Phys.*, 40(5):2354–2363, 1999.

[16] C. Quesne, Completeness of photon-added squeezed vacuum and one-photon states and of photon-added coherent states on a circle, *Phys. Lett. A*, 288(5-6):241–250, 2001.

[17] C. Quesne, New q-deformed coherent states with an explicitly known resolution of unity, *J. Phys. A*, 35(43):9213–9226, 2002.

[18] C. Quesne, K.A. Penson, and V.M. Tkachuk, Maths-type q-deformed coherent states for $q > 1$, *Phys. Lett. A*, 313(1-2):29–36, 2003.

[19] Marko Robnik, Supersymmetric quantum mechanics based on higher excited states, *J. Phys. A*, 30(4):1287–1294, 1997.

[20] Marko Robnik and Valery G. Romanovski, Two-dimensional hydrogen atom in a strong magnetic field, *J. Phys. A*, 36(29):7923–7951, 2003.

[21] Andreas Ruffing and Moritz Simon, Analytic properties of a special q-exponential function, *Progr. Theoret. Phys. Suppl.*, (150):401–410, 2003. Let's face chaos through nonlinear dynamics (Maribor, 2002).

[22] Barry Simon, The classical moment problem as a self-adjoint finite difference operator, *Adv. Math.*, 137(1):82–203, 1998.

[23] Moritz Simon and Andreas Ruffing, Power series techniques for a special Schrödinger operator and related difference equations, *Adv. Difference Equ.*, (2):109–118, 2005.

[24] V.P. Spiridonov, The factorization method, self-similar potentials and quantum algebras, In *Special functions 2000: current perspective and future directions (Tempe, AZ)*, volume 30 of *NATO Sci. Ser. II Math. Phys. Chem.*, pages 335–364. Kluwer Acad. Publ., Dordrecht, 2001.

[25] V.P. Spiridonov and A.S. Zhedanov, Generalized eigenvalue problem and a new family of rational functions biorthogonal on elliptic grids, In *Special functions 2000: current perspective and future directions (Tempe, AZ)*, volume 30 of *NATO Sci. Ser. II Math. Phys. Chem.*, pages 365–388. Kluwer Acad. Publ., Dordrecht, 2001.

[26] Sergei K. Suslov, *An introduction to basic Fourier series*, volume 9 of *Developments in Mathematics*, Kluwer Academic Publishers, Dordrecht, 2003. With a foreword by Mizan Rahman.