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Positive Solutions of Nonlinear *m*-point BVP on Time Scales

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Abstract

In this paper, we are concerned with proving the existence of positive solutions of a nonlinear second order m-point boundary value problem (m-PBVP) on time scales. The proofs are based on the fixed point theorems concerning cones in a Banach space.

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1. Introduction

There are many well-known analogies in the concepts of difference equations and differential equations. The concept of dynamic equations on time scales can build bridges between differential and difference equations. This concept not only gives us unified approach to study the boundary value problems on discrete intervals with uniform step size and real intervals but also gives an extended approach to study on discrete case

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with nonuniform step size or combination of real and discrete intervals. Some basic definitions and theorems on time scales can be found in the books [3, 4].

The study of multipoint boundary value problems for linear second order ordinary differential equations started with Il'in and Mosiev [9]. Motivated by this article, Gupta [6] studied certain three point boundary value problems (TPBVP) for nonlinear differential equations. Since then, in the literature many authors focused on positive solutions of nonlinear differential equations and difference equations including two-point, three-point and m-point boundary conditions (see [1,2,7,10,13,14] and the references therein). However a few results have been obtained for multipoint BVP on time scales. Recently, Anderson [1], Sun and Li [15], Yaslan [16] have obtained some existence results for various kind of TPBVPs on time scales. Fixed point theorems, coincidence degree theory and nonlinear alternative of Leray–Schauder have been used to prove the existence results.

In 2004, Sun and Li [15] discussed the existence of single and multiple positive solutions to the TPBVP

$$\begin{gathered} u^{\Delta\nabla}(t) + a(t) f(u(t)) = 0, \quad t \in (0, T), \\ \beta u(0) + \gamma u^{\Delta}(0) = 0, \quad u(T) = \alpha u(\eta), \end{gathered}$$

where $\beta, \gamma \geq 0,\ \beta + \gamma > 0,\ \mu \in (0, \rho(T)),\ 0 < \alpha < T/\eta$ and $d = \beta(T - \alpha\eta) + \gamma(1 - \alpha) > 0$.

In 2004, He [8] dealt with the m-PBVP

$$\begin{gathered} u^{\Delta\nabla}(t) + q(t) f(u(t)) = 0, \quad t \in [0, T]_k, \\ u^{\Delta}(0) = \sum_{i=1}^{m-2} b_i u^{\Delta}(\eta_i), \quad u(T) = \sum_{i=1}^{m-2} a_i u(\eta_i), \end{gathered}$$

where $\eta_i \in (0, \rho(T))_k$ with $0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < \rho(T)$, and obtained existence of at least two positive solutions by imposing the growth conditions on f.

Motivated by the studies above, we are interested in the existence of positive solutions of the m-PBVP

$$u^{\Delta\nabla}(t) + h(t) f(t, u(t)) = 0, \ \ t \in [t_1, t_m]_k \subset \mathbb{T}, \tag{1.1}$$

$$u^{\Delta}(t_1) = 0, \ \ \alpha u(t_m) + \beta u^{\Delta}(t_m) = \sum_{i=2}^{m-1} u^{\Delta}(t_i), \ \ m \geq 3 \tag{1.2}$$

where $\mathbb{T}$ is a time scale, $0 \leq t_1 < t_2 < \cdots < t_{m-1} < t_m$ and α, β are constants such that $\alpha > 0, \beta > m - 2$.

The aim of this paper is to establish some simple criteria for the existence of single and multiple positive solutions of the m-PBVP (1.1)–(1.2). This paper is organized as follows: In Section 2 we first present the solution and some properties of the solution of the linear m-PBVP corresponding to (1.1)–(1.2). Consequently we define the Banach space, cone and the integral operator to prove the existence of the solution of (1.1)–(1.2).

Finally we state the fixed point theorems in order to prove the main results. In Section 3 we get the existence of at least one, two, three and odd number of multiple solutions for nonlinear m-PBVP (1.1)–(1.2).

2. Preliminaries and Fixed Point Theorems

In this section, we will give several fixed point theorems to prove existence of positive solutions of nonlinear m-PBVP (1.1)–(1.2). Also, to state the main results in this paper we employ the following lemmas. These lemmas are based on the linear dynamic equation

$$u^{\Delta\nabla}(t) + y(t) = 0, \tag{2.1}$$

coupled with the boundary conditions (1.2).

Lemma 2.1. Let $\alpha \neq 0$. If $y \in \mathcal{C}_{\mathrm{ld}}([t_1, t_m])$, then the m-PBVP (2.1)–(1.2) has a unique solution

$$\begin{aligned} u(t) &= \int_{t_1}^{t_m} \left(t_m + \frac{\beta}{\alpha} - s\right) y(s)\nabla s - \frac{1}{\alpha}\sum_{i=1}^{m-2}\int_{t_i}^{t_{i+1}} y(s)\nabla s \\ &\quad - \int_{t_1}^{t} (t-s)y(s)\nabla s. \end{aligned} \tag{2.2}$$

Proof. The proof is similar to [16, Lemma 2.1]. From (2.1) we have

$$u(t) = u(t_1) + u^{\Delta}(t_1)(t - t_1) - \int_{t_1}^{t} (t-s)y(s)\nabla s.$$

By applying the first boundary condition, we can obtain

$$u(t) = u(t_1) - \int_{t_1}^{t} (t-s)y(s)\nabla s. \tag{2.3}$$

Applying the second boundary condition leads us to have

$$\alpha u(t_1) - \alpha\int_{t_1}^{t_m} (t_m - s)y(s)\nabla s - \beta\int_{t_1}^{t_m} y(s)\nabla s = -\sum_{i=2}^{m-1}\int_{t_1}^{t_i} y(s)\nabla s,$$

and therefore

$$\begin{aligned} u(t_1) &= \int_{t_1}^{t_m} (t_m - s)y(s)\nabla s + \frac{\beta}{\alpha}\int_{t_1}^{t_m} y(s)\nabla s - \frac{1}{\alpha}\sum_{i=2}^{m-1}\int_{t_1}^{t_i} y(s)\nabla s \\ &= \int_{t_1}^{t_m} \left(t_m + \frac{\beta}{\alpha} - s\right) y(s)\nabla s - \frac{1}{\alpha}\sum_{i=2}^{m-1}\int_{t_1}^{t_i} y(s)\nabla s. \end{aligned} \tag{2.4}$$

Hence substituting (2.4) into (2.3) we can get

$$\begin{aligned} u(t) &= \int_{t_1}^{t_m} \left(t_m + \frac{\beta}{\alpha} - s\right) y(s)\nabla s - \frac{1}{\alpha}\sum_{i=2}^{m-1}\int_{t_1}^{t_i} y(s)\nabla s \\ &\quad - \int_{t_1}^{t} (t-s)y(s)\nabla s. \end{aligned}$$

This completes the proof. ∎

Lemma 2.2. Let $\alpha > 0$, $\beta \geq m-2$. If $y \in \mathcal{C}_{\mathrm{ld}}([t_1, t_m], [0, \infty))$, then the solution m-PBVP (2.1)–(1.2) satisfies

$$u(t) \geq 0, \ \forall t \in [t_1, t_m].$$

Proof. From equation (2.3) we can find that $u^{\Delta}(t) = -\int_{t_1}^{t} y(s)\nabla s$. Hence $u(t)$ is decreasing on $[t_1, t_m]$. Therefore, if $u(t_m) \geq 0$, the result holds.

$$\begin{aligned} u(t_m) &= \int_{t_1}^{t_m} \left(t_m + \frac{\beta}{\alpha} - s\right) y(s)\nabla s - \frac{1}{\alpha}\sum_{i=2}^{m-1}\int_{t_1}^{t_i} y(s)\nabla s - \int_{t_1}^{t_m} (t_m - s)y(s)\nabla s \\ &= \frac{\beta}{\alpha}\int_{t_1}^{t_m} y(s)\nabla s - \frac{1}{\alpha}\sum_{i=2}^{m-1}\int_{t_1}^{t_i} y(s)\nabla s \\ &= \frac{\beta}{\alpha}\sum_{i=1}^{m-1}\int_{t_i}^{t_{i+1}} y(s)\nabla s - \frac{1}{\alpha}\sum_{i=2}^{m-1}\int_{t_1}^{t_i} y(s)\nabla s \\ &= \frac{\beta}{\alpha}\int_{t_{m-1}}^{t_m} y(s)\nabla s + \frac{1}{\alpha}\left\{\beta\left(\sum_{i=1}^{m-2}\int_{t_i}^{t_{i+1}} y(s)\nabla s\right) - \int_{t_1}^{t_2} y(s)\nabla s \right. \\ &\quad - \left(\int_{t_1}^{t_2} y(s)\nabla s + \int_{t_2}^{t_3} y(s)\nabla s\right) - \dots \\ &\quad \left. - \left(\int_{t_1}^{t_2} y(s)\nabla s + \cdots + \int_{t_{m-2}}^{t_{m-1}} y(s)\nabla s\right)\right\}. \end{aligned}$$

Arranging the preceding equation we get

$$u(t_m) = \sum_{i=1}^{m-1}\int_{t_i}^{t_{i+1}} \frac{\beta - (m-1-i)}{\alpha} y(s)\nabla s \geq 0.$$

The proof is complete. ∎

Let the Banach space $\mathcal{B} = \mathcal{C}_{\mathrm{ld}}[t_1, t_m]$ be endowed with the norm

$$\|u\| = \sup_{t\in[t_1,t_m]} |u(t)|,$$

and also $\mathcal{P} \subset \mathcal{B}$ be a cone defined by

$$\mathcal{P} = \{u \in \mathcal{B} : u(t) \geq 0,\ u \text{ is concave on } [t_1, t_m]_k, u^{\Delta}(t_1) = 0\}. \tag{2.5}$$

Lemma 2.3. If $u \in \mathcal{P}$, then $u(t) \geq \frac{t_m - t}{t_m}\|u\|$, for all $t \in [t_1, t_m]$.

Proof. Since u is concave, $u^{\Delta\nabla}(t) \leq 0$, for all $t \in [t_1, t_m]_k$ which implies $u^{\Delta}(t)$ is nonincreasing on $[t_1,\ t_m] \subset \mathbb{T}^k$. Then $u^{\Delta}(t) \leq u^{\Delta}(t_1) = 0$ and u is nonincreasing on $[t_1, t_m] \subset \mathbb{T}$. Hence $\|u\| = \sup_{t\in[t_1,t_m]} |u(t)| = |u(t_1)| = u(t_1)$. Define the auxiliary function

$$g(t) = u(t) - \frac{t_m - t}{t_m}\|u\|.$$

Then we have the following properties:

(i) $g^{\Delta\nabla}(t) = u^{\Delta\nabla}(t) \leq 0$ for all $t \in [t_1, t_m]$. Hence g is concave.

(ii) $g(t_1) = u(t_1) - \frac{t_m - t_1}{t_m}\|u\| = u(t_1) - \frac{t_m - t_1}{t_m}u(t_1) = \frac{t_1}{t_m}u(t_1) \geq 0.$

(iii) $g(t_m) = u(t_m) \geq 0.$

Therefore concluding the properties (i), (ii), and (iii) of g, we guarantee that $g(t) \geq 0$, for all $t \in [t_1, t_m]$, which implies $u(t) \geq \frac{t_m - t}{t_m}\|u\|$, for all $t \in [t_1, t_m]$. ■

It is clear that the solutions of m-PBVP (1.1)–(1.2) are the fixed points of the integral operator

$$\begin{aligned}(Au)(t) \;=\; & \int_{t_1}^{t_m}\left(t_m + \frac{\beta}{\alpha} - s\right)h(s)f(s,u(s))\nabla s - \frac{1}{\alpha}\sum_{i=2}^{m-1}\int_{t_1}^{t_i} h(s)f(s,u(s))\nabla s \\ & - \int_{t_1}^{t}(t-s)h(s)f(s,u(s))\nabla s. \end{aligned} \tag{2.6}$$

In order to establish the main results of this paper easily, we now state the fixed point theorems which we apply to prove Theorems 3.1–3.5.

Theorem 2.4. (Schauder Fixed Point Theorem [11]) Let E be a Banach space, and let $A : E \to E$ be a completely continuous operator. Assume $K \subset E$ is a bounded, closed and convex set. If $A(K) \subset K$, then A has a fixed point in K.

Theorem 2.5. (Krasnoselskii Fixed Point Theorem [5]) Let E be a Banach space, and let $K \subset E$ be a cone. Assume Ω_1 and Ω_2 are open, bounded subsets of E with $0 \in \Omega_1, \overline{\Omega_1} \subset \Omega_2$, and let

$$A : K \cap (\overline{\Omega_2} \setminus \Omega_1) \to K$$

be a completely continuous operator such that either

(i) $\|Au\| \leq \|u\|$ for $u \in K \cap \partial\Omega_1$, $\|Au\| \geq \|u\|$ for $u \in K \cap \partial\Omega_2$; or

(ii) $\|Au\| \geq \|u\|$ for $u \in K \cap \partial\Omega_1$, $\|Au\| \leq \|u\|$ for $u \in K \cap \partial\Omega_2$

hold. Then A has a fixed point in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Theorem 2.6. (Avery–Henderson Fixed Point Theorem [2]) Let $\mathcal{P}$ be a cone in a real Banach space E. Set

$$\mathcal{P}(\phi, r) = \{u \in \mathcal{P} : \phi(u) < r\}.$$

If μ and ϕ are increasing, nonnegative, continuous functionals on $\mathcal{P}$, let θ be a nonnegative continuous functional on $\mathcal{P}$ with $\theta(0) = 0$ such that for some positive constants r and W,

$$\phi(u) \leq \theta(u) \leq \mu(u) \text{ and } \|u\| \leq M\phi(u)$$

for all $u \in \overline{\mathcal{P}(\phi, r)}$. Suppose that there exist positive numbers $p < q < r$ such that

$$\theta(\lambda u) \leq \lambda\theta(u) \text{ for all } 0 \leq \lambda \leq 1 \text{ and } u \in \partial\mathcal{P}(\theta, q).$$

If $A : \overline{\mathcal{P}(\phi, r)} \to \mathcal{P}$ is a completely continuous operator satisfying

(i) $\phi(Au) > r$ for all $u \in \partial\mathcal{P}(\phi, r)$,

(ii) $\theta(Au) < q$ for all $u \in \partial\mathcal{P}(\theta, q)$,

(iii) $\mathcal{P}(\mu, q) \neq \emptyset$ and $\mu(Au) > p$ for all $u \in \partial\mathcal{P}(\mu, p)$,

then A has at least two fixed points u_1 and u_2 such that

$$p < \mu(u_1) \text{ with } \theta(u_1) < q \text{ and } q < \theta(u_2) \text{ with } \phi(u_2) < r.$$

Theorem 2.7. (Legget–Williams Fixed Point Theorem [12]) Let $\mathcal{P}$ be a cone in a real Banach space E. Set

$$\begin{aligned} \mathcal{P}_r : &= \{x \in \mathcal{P} : \|x\| < r\}, \\ \mathcal{P}(\psi, a, b) : &= \{x \in \mathcal{P} : a \leq \psi(x), \|x\| \leq b\}. \end{aligned}$$

Suppose $A : \overline{\mathcal{P}_r} \to \overline{\mathcal{P}_r}$ is a completely continuous operator and ψ is a nonnegative, continuous, concave functional on $\mathcal{P}$ with $\psi(u) \leq \|u\|$ for all $u \in \overline{\mathcal{P}_r}$. Assume there exist $0 < p < q < l \leq r$ such that the following conditions hold:

(i) $\{u \in \mathcal{P}(\psi, q, l) : \psi(u) > q\} \neq \emptyset$ and $\psi(Au) > q$ for all $u \in \mathcal{P}(\psi, q, l)$,

(ii) $\|Au\| < p$ for all $\|u\| \leq p$,

(iii) $\psi(Au) > q$ for $u \in \mathcal{P}(\psi, q, r)$ with $\|Au\| > l$.

Then A has at least three positive solutions u_1, u_2 and u_3 in $\overline{\mathcal{P}_r}$ satisfying

$$\|u_1\| < p, \ \psi(u_2) > q, \ p < \|u_3\| \text{ with } \psi(u_3) < q.$$

3. Main Results

In this section we will prove the existence of at least one, two, three and odd number of multiple positive solution of m-PBVP (1.1)–(1.2). In the following four theorems we will make use of Schauder, Krasnoselskii, Avery–Henderson and Legget–Williams fixed point theorems respectively. In the proof of Theorem 3.5 we also employ Legget–Williams' fixed point theorem. For the rest of this work assume that the following conditions hold:

(H1) $h \in \mathcal{C}_{\text{ld}}([t_1, t_m], [0, \infty))$ and $\exists t_0 \in [t_{m-2}, t_{m-1}]$ such that $h(t_0) > 0$.

(H2) $f : [t_1, t_m] \times [0, \infty) \to [0, \infty)$ is continuous such that $f(t, \cdot) > 0$ on any subset of $\mathbb{T}$ containing t_0.

The sufficient conditions for existence of at least one positive solution of m-PBVP (1.1)–(1.2) are stated in Theorem 3.1 and Theorem 3.2. For this purpose we make use of the Krasnoselskii fixed point theorem (Theorem 2.5) and the Schauder fixed point theorem (Theorem 2.4), respectively.

Theorem 3.1. Let $\alpha > 0$, $\beta \geq m - 2$. Assume that (H1) and (H2) are satisfied and there exist numbers $0 < r < R < \infty$ such that

$$f(s, u) \leq \frac{1}{k_1} u \ \text{ if } \ 0 \leq u \leq r$$

and

$$f(s, u) \geq \frac{t_m}{k_2(t_m - t_{m-1})} u \ \text{ if } \ R \leq u \leq \infty,$$

where

$$k_1 = \sum_{i=1}^{m-1} \int_{t_i}^{t_{i+1}} \left(t_m + \frac{\beta - (m - 1 - i)}{\alpha} - s \right) h(s) \nabla s$$

and

$$k_2 = \int_{t_{m-2}}^{t_{m-1}} \left(t_m + \frac{\beta - 1}{\alpha} - s \right) h(s) \nabla s.$$

Then m-PBVP (1.1)–(1.2) has at least one positive solution.

Proof. Define the cone as in equation (2.5). From Lemmas 2.2 and 2.3 and the conditions (H1) and (H2) we can obtain $A(\mathcal{P}) \subset \mathcal{P}$. Also it is easy to obtain that $A : \mathcal{P} \to \mathcal{P}$ is completely continuous.

If $u \in \mathcal{P}$ with $\|u\| = r$, then from Lemma 2.2 we have

$$\begin{aligned}\|Au\| = \sup_{t\in[t_1,t_m]} \Bigg| \int_{t_1}^{t_m} \left(t_m + \frac{\beta}{\alpha} - s\right) h(s) f(s, u(s))\nabla s - \frac{1}{\alpha}\sum_{i=2}^{m-1}\int_{t_1}^{t_i} h(s) f(s, u(s))\nabla s \\ - \int_{t_1}^{t} (t-s) h(s) f(s,u(s))\nabla s \Bigg| \\ = \int_{t_1}^{t_m} \left(t_m + \frac{\beta}{\alpha} - s\right) h(s) f(s, u(s))\nabla s - \frac{1}{\alpha}\sum_{i=2}^{m-1}\int_{t_1}^{t_i} h(s) f(s, u(s))\nabla s.\end{aligned}$$

By using the similar calculations and Lemma 2.2 we get

$$\begin{aligned}\|Au\| &= \sum_{i=1}^{m-1}\int_{t_i}^{t_{i+1}} \left(t_m + \frac{\beta-(m-1-i)}{\alpha} - s\right) h(s) f(s, u(s))\nabla s \\ &\le \sum_{i=1}^{m-1}\int_{t_i}^{t_{i+1}} \left(t_m + \frac{\beta-(m-1-i)}{\alpha} - s\right) h(s) \frac{1}{k_1} u(s)\nabla s \\ &\le \frac{1}{k_1}\|u\| \sum_{i=1}^{m-1}\int_{t_i}^{t_{i+1}} \left(t_m + \frac{\beta-(m-1-i)}{\alpha} - s\right) h(s)\nabla s \\ &= \|u\|.\end{aligned}$$

Therefore we have

$$\|Au\| \le \|u\| \text{ for all } u \in \mathcal{P}\cap\partial\Omega_1, \tag{3.1}$$

where $\Omega_1 = \{u \in \mathcal{C}_{\mathrm{ld}}([t_1,t_m],\mathbb{R}) : \|u\| < r\}$.

Let us now define $\Omega_2 = \left\{u \in \mathcal{C}_{\mathrm{ld}}([t_1,t_m],\mathbb{R}) : \|u\| < \dfrac{t_m}{t_m - t_{m-1}} R\right\}$.

$$\begin{aligned}\|Au\| &= \sum_{i=1}^{m-1}\int_{t_i}^{t_{i+1}} \left(t_m + \frac{\beta-(m-1-i)}{\alpha} - s\right) h(s) f(s, u(s))\nabla s \\ &\ge \sum_{i=1}^{m-2}\int_{t_i}^{t_{i+1}} \left(t_m + \frac{\beta-(m-1-i)}{\alpha} - s\right) h(s) f(s, u(s))\nabla s \\ &\ge \int_{t_{m-2}}^{t_{m-1}} \left(t_m + \frac{\beta-1}{\alpha} - s\right) h(s) f(s, u(s))\nabla s \\ &\ge \frac{t_m}{k_2(t_m - t_{m-1})}\int_{t_{m-2}}^{t_{m-1}} \left(t_m + \frac{\beta-1}{\alpha} - s\right) h(s) u(s)\nabla s\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{t_m}{k_2(t_m - t_{m-1})} \int_{t_{m-2}}^{t_{m-1}} \left(t_m + \frac{\beta - 1}{\alpha} - s \right) h(s) \frac{t_m - t_{m-1}}{t_m} \|u\| \nabla s \\
&= \frac{1}{k_2} \|u\| \int_{t_{m-2}}^{t_{m-1}} \left(t_m + \frac{\beta - 1}{\alpha} - s \right) h(s) \nabla s \\
&= \|u\|,
\end{aligned}
$$

where we used Lemma 2.3 in the fifth line of the preceding inequality. Therefore we have

$$
\|Au\| \geq \|u\|, \quad u \in \mathcal{P} \cap \partial\Omega_2. \tag{3.2}
$$

The equations (3.1) and (3.2) imply that the first case of Knasnoselskii's fixed point theorem (Theorem 2.5) is satisfied. Then A has a fixed point $u \in \mathcal{P} \cap (\overline{\Omega_2} \setminus \Omega_1)$. Hence the m-PBVP (1.1)–(1.2) has at least one positive solution. ■

Existence of at least one positive solution is also proved using Schauder fixed point theorem (Theorem 2.4). Then we have the following result.

Theorem 3.2. Let $\alpha > 0,\ \beta \geq m - 2$. Assume that (H1) and (H2) are satisfied. If k_1 satisfies

$$
\max_{\|u\| \leq r} \mid f(s, u) \mid \leq \frac{1}{k_1} u \ \text{ for } \ t \in [t_1, t_m],
$$

where

$$
k_1 = \sum_{i=1}^{m-1} \int_{t_i}^{t_{i+1}} \left(t_m + \frac{\beta - (m - 1 - i)}{\alpha} - s \right) h(s) \nabla s,
$$

then the m-PBVP (1.1)–(1.2) has at least one positive solution.

Proof. Let $\mathcal{P}_r := \{u \in \mathcal{P} : \|u\| < r\}$. Note that $\mathcal{P}_r$ is a closed, bounded and convex subset of $\mathcal{B}$ to which the Schauder fixed point theorem is applicable.

Define $A : \mathcal{P}_r \to \mathcal{B}$ as in (2.6) for $t \in [t_1, t_m]$. It can be shown that $A : \mathcal{P}_r \to \mathcal{B}$ is continuous. Claim that $A : \mathcal{P}_r \to \mathcal{P}_r$. Let $u \in \mathcal{P}_r$. By using similar methods as the ones used in the proof of Theorem 3.1, we have $\|Au\| \leq \|u\| < r$ which implies $Au \in \mathcal{P}_r$. The compactness of the operator $A : \mathcal{P}_r \to \mathcal{P}_r$ follows from the Arzela–Ascoli theorem. Hence A has a fixed point in $\mathcal{P}_r$. ■

Now we will give the sufficient conditions to have at least two positive solutions for m-PBVP (1.1)–(1.2). The Avery–Henderson fixed point theorem (Theorem 2.6) will be used to prove the result.

Theorem 3.3. Let $\alpha > 0,\ \beta > m - 2$. Assume that (H1) and (H2) are satisfied. Suppose that there exist positive numbers $p < q < r$ such that the function f satisfies the following conditions:

(i) $f(s, u) \geq rW$ for $s \in [t_1, t_{m-1}]$ and $u \in \left[r, \dfrac{rt_m}{t_m - t_{m-1}} \right]$,

(ii) $f(s,u) < qw$ for $s \in [t_1, t_m]$ and $u \in \left[0, \dfrac{qt_m}{t_m - t_{m-1}}\right]$,

(iii) $f(s,u) \geq pW$ for $s \in [t_1, t_{m-1}]$ and $u \in \left[p\dfrac{t_m - t_{m-1}}{t_m}, p\right]$,

for some positive constants w and W. Then the m-PBVP (1.1)–(1.2) has at least two positive solutions.

Proof. Define the cone as in equation (2.5). From Lemmas 2.2 and 2.3 and the conditions (H1) and (H2) we can obtain $A(\mathcal{P}) \subset \mathcal{P}$. Also it is easy to obtain that $A : \mathcal{P} \to \mathcal{P}$ is completely continuous.

Let the nonnegative, increasing, continuous functionals ϕ, θ and μ be defined on the cone $\mathcal{P}$ by

$$\phi(u) = u(t_{m-1}), \quad \theta(u) = u(t_{m-1}), \quad \mu(u) = u(t_1).$$

So by the definition of u we have $\phi(u) = \theta(u) \leq \mu(u)$, for all $u \in \mathcal{P}$. In addition from Lemma 2.3 we have $\phi(u) = u(t_{m-1}) \geq \dfrac{t_m - t_{m-1}}{t_m}\|u\|$, for all $u \in \mathcal{P}$. Therefore

$$\|u\| \leq \frac{t_m}{t_m - t_{m-1}}\phi(u). \tag{3.3}$$

If $u \in \partial\mathcal{P}(\phi, r) = \{u \in \mathcal{P} : \phi(u) = r\}$, then from inequality (3.3) we can obtain

$$r = u(t_{m-1}) \leq u(s) \leq \|u\| \leq r\frac{t_m}{t_m - t_{m-1}}, \quad \forall s \in [t_1, t_{m-1}].$$

Define

$$W = \frac{\alpha}{\beta - 1}\left(\int_{t_{m-2}}^{t_{m-1}} h(s)\nabla s\right)^{-1}. \tag{3.4}$$

We will show that the condition (i) of Theorem 2.6 is satisfied. From the assumption (i) and the positivity of $t_m + \dfrac{\beta}{\alpha} - s$ and $t_m - t_{m-1}$, we can obtain

$$\begin{aligned}
\phi(Au) &= (Au)(t_{m-1}) \\
&= \int_{t_1}^{t_m}\left(t_m + \frac{\beta}{\alpha} - s\right)h(s)f(s,u(s))\nabla s - \frac{1}{\alpha}\sum_{i=2}^{m-1}\int_{t_1}^{t_i} h(s)f(s,u(s))\nabla s \\
&\quad - \int_{t_1}^{t_{m-1}}(t_{m-1} - s)h(s)f(s,u(s))\nabla s \\
&= \int_{t_{m-1}}^{t_m}\left(t_m + \frac{\beta}{\alpha} - s\right)h(s)f(s,u(s))\nabla s \\
&\quad + \sum_{i=1}^{m-2}\int_{t_i}^{t_{i+1}}\left(t_m - t_{m-1} + \frac{\beta - (m-1-i)}{\alpha}\right)h(s)f(s,u(s))\nabla s
\end{aligned}$$

$$
\begin{aligned}
&\geq \sum_{i=1}^{m-2}\int_{t_i}^{t_{i+1}}\left(t_m - t_{m-1} + \frac{\beta-(m-1-i)}{\alpha}\right)h(s)f(s,u(s))\nabla s\\
&> \sum_{i=1}^{m-2}\int_{t_i}^{t_{i+1}}\left(\frac{\beta-(m-1-i)}{\alpha}\right)h(s)f(s,u(s))\nabla s\\
&\geq \frac{\beta-1}{\alpha}\int_{t_{m-2}}^{t_{m-1}} h(s)f(s,u(s))\nabla s\\
&\geq \frac{\beta-1}{\alpha} rW \int_{t_{m-2}}^{t_{m-1}} h(s)\nabla s = r.
\end{aligned}
$$

Thus the condition (i) of Theorem 2.6 holds. Next we will show the condition (ii) of Theorem 2.6 is satisfied. If $u \in \partial\mathcal{P}(\theta, q)$, then from equation (3.3) we have $0 \leq u(s) \leq \|u\| \leq q\dfrac{t_m}{t_m - t_{m-1}}$, for all $s \in [t_1, t_m]$. Define

$$
w = \left(\int_{t_1}^{t_m}\left(t_m + \frac{\beta}{\alpha} - s\right)h(s)\nabla s\right)^{-1}. \tag{3.5}
$$

Then

$$
\begin{aligned}
\theta(Au) &= (Au)(t_{m-1})\\
&= \int_{t_1}^{t_m}\left(t_m + \frac{\beta}{\alpha} - s\right)h(s)f(s,u(s))\nabla s - \frac{1}{\alpha}\sum_{i=2}^{m-1}\int_{t_1}^{t_i} h(s)f(s,u(s))\nabla s\\
&\quad - \int_{t_1}^{t_{m-1}}(t-s)h(s)f(s,u(s))\nabla s\\
&\leq \int_{t_1}^{t_m}\left(t_m + \frac{\beta}{\alpha} - s\right)h(s)f(s,u(s))\nabla s\\
&< qw\int_{t_1}^{t_m}\left(t_m + \frac{\beta}{\alpha} - s\right)h(s)\nabla s = q.
\end{aligned}
$$

So condition (ii) of Theorem 2.6 holds.

Since $0 \in \mathcal{P}$ and $p > 0$, $\mathcal{P}(\mu, p) \neq \emptyset$. If $u \in \partial\mathcal{P}(\mu, p)$, from Lemma 2.3 we get

$$
p\cdot\frac{t_m - t_{m-1}}{t_m} \leq u(t_{m-1}) \leq u(s) \leq \|u\| = p.
$$

Then from the assumption (iii), we have

$$
\begin{aligned}
\mu(Au) &= (Au)(t_1)\\
&= \int_{t_1}^{t_m}\left(t_m + \frac{\beta}{\alpha} - s\right)h(s)f(s,u(s))\nabla s - \frac{1}{\alpha}\sum_{i=2}^{m-1}\int_{t_1}^{t_i} h(s)f(s,u(s))\nabla s
\end{aligned}
$$

$$
\begin{aligned}
&> \frac{\beta}{\alpha}\int_{t_1}^{t_m} h(s)f(s,u(s))\nabla s - \frac{1}{\alpha}\sum_{i=2}^{m-1}\int_{t_1}^{t_i} h(s)f(s,u(s))\nabla s \\
&= \sum_{i=1}^{m-1}\frac{\beta-(m-1-i)}{\alpha}\int_{t_i}^{t_{i+1}} h(s)f(s,u(s))\nabla s \\
&> \frac{\beta-1}{\alpha}\int_{t_{m-2}}^{t_{m-1}} h(s)f(s,u(s))\nabla s \\
&\geq \frac{\beta-1}{\alpha} pW \int_{t_{m-2}}^{t_{m-1}} h(s)\nabla s = p.
\end{aligned}
$$

Therefore the condition (iii) of Theorem 2.6 holds. Since all the conditions of Theorem 2.6 are satisfied, the m-PBVP (1.1)–(1.2) has at least two positive solutions $u_1$and u_2 satisfying

$$u_1(t_1) > p \ \text{ with } \ u_1(t_{m-1}) < q \ \text{ and } \ u_2(t_{m-1}) > q \ \text{ with } \ u_2(t_{m-1}) < r.$$

This completes the proof. ■

Theorem 3.4. Assume that the conditions (H1) and (H2) are satisfied and $\alpha > 0,\ \beta > m-2$. Suppose that there exist constants $0 < p < q < q\dfrac{t_m}{t_m - t_{m-1}} \leq r$ such that

(i) $f(s,u) \leq rw$ for $s \in [t_1, t_m]$ and $u \in [0, r]$,

(ii) $f(s,u) \geq qW$ for $s \in [t_1, t_{m-1}]$ and $u \in \left[q, \dfrac{qt_m}{t_m - t_{m-1}}\right]$,

(iii) $f(s,u) < pw$ for $s \in [t_1, t_m]$ and $u \in [0, p]$.

Then the m-PBVP (1.1)–(1.2) has at least three positive solutions u_1, u_2 and u_3 satisfying

$$u_1(t_1) < p,\ u_2(t_{m-1}) > q,\ u_3(t_1) > p \ \text{ with } \ u_3(t_{m-1}) < q.$$

Proof. We will show that the conditions of Legget–Williams fixed point theorem (Theorem 2.7) are satisfied. For this purpose we first define the nonnegative, continuous, concave functional $\psi : \mathcal{P} \to [0,\ \infty)$ to be $\psi(u) := u(t_{m-1})$, the cone $\mathcal{P}$ as is (2.5), W as in (3.4) and w as is (3.5). Then $\psi(u) \leq \|u\|$ for all $u \in \mathcal{P}$. If $u \in \overline{\mathcal{P}_r}$, then $\|u\| \leq r$ and from the assumption (i) $f(s,u) \leq rw$. So we have

$$
\begin{aligned}
\|Au\| &= \int_{t_1}^{t_m}\left(t_m + \frac{\beta}{\alpha} - s\right) h(s)f(s,u(s))\nabla s - \frac{1}{\alpha}\sum_{i=2}^{m-1}\int_{t_1}^{t_i} h(s)f(s,u(s))\nabla s \\
&\leq rw\int_{t_1}^{t_m}\left(t_m + \frac{\beta}{\alpha} - s\right) h(s)\nabla s \\
&= r.
\end{aligned}
$$

Thus $A : \overline{\mathcal{P}_r} \to \overline{\mathcal{P}_r}$. Since

$$\frac{qt_m}{t_m - t_{m-1}} \in \mathcal{P}\left(\psi, q, \frac{qt_m}{t_m - t_{m-1}}\right)$$

and

$$\psi\left(\frac{qt_m}{t_m - t_{m-1}}\right) = \frac{qt_m}{t_m - t_{m-1}} > q, \quad \left\{u \in \mathcal{P}\left(\psi, q, \frac{qt_m}{t_m - t_{m-1}}\right) : \psi(u) > q\right\} \neq \emptyset.$$

For $u \in \mathcal{P}\left(\psi, q, \frac{qt_m}{t_m - t_{m-1}}\right)$, we have $q \leq u(t_{m-1}) \leq u(s) \leq \frac{qt_m}{t_m - t_{m-1}}$ for $s \in [t_1, t_{m-1}]$. The use of the assumption (ii), the nonnegativity of $t_m + \frac{\beta}{\alpha} - s$ and the positivity of $t_m - t_{m-1}$ establishes

$$\begin{aligned}
\psi(Au) &= (Au)(t_{m-1}) \\
&= \int_{t_1}^{t_m} \left(t_m + \frac{\beta}{\alpha} - s\right) h(s) f(s, u(s)) \nabla s - \frac{1}{\alpha} \sum_{i=2}^{m-1} \int_{t_1}^{t_i} h(s) f(s, u(s)) \nabla s \\
&\quad - \int_{t_1}^{t_{m-1}} (t_{m-1} - s) h(s) f(s, u(s)) \nabla s \\
&= \int_{t_{m-1}}^{t_m} \left(t_m + \frac{\beta}{\alpha} - s\right) h(s) f(s, u(s)) \nabla s \\
&\quad + \sum_{i=1}^{m-2} \int_{t_i}^{t_{i+1}} \left(t_m - t_{m-1} + \frac{\beta - (m-1-i)}{\alpha}\right) h(s) f(s, u(s)) \nabla s \\
&\geq \sum_{i=1}^{m-2} \int_{t_i}^{t_{i+1}} \left(t_m - t_{m-1} + \frac{\beta - (m-1-i)}{\alpha}\right) h(s) f(s, u(s)) \nabla s \\
&> \sum_{i=1}^{m-2} \int_{t_i}^{t_{i+1}} \left(\frac{\beta - (m-1-i)}{\alpha}\right) h(s) f(s, u(s)) \nabla s \\
&\geq \frac{\beta - 1}{\alpha} \int_{t_{m-2}}^{t_{m-1}} h(s) f(s, u(s)) \nabla s \\
&\geq \frac{\beta - 1}{\alpha} qW \int_{t_{m-2}}^{t_{m-1}} h(s) \nabla s = q.
\end{aligned}$$

Therefore the condition (i) of Legget–Williams fixed point theorem (Theorem 2.7) is satisfied.

If $\|u\| \leq p$, then $f(s,u) < pw$, for all $s \in [t_1, t_m]$ is obtained from condition (iii). Then

$$\begin{aligned}\|Au\| &= \int_{t_1}^{t_m} \left(t_m + \frac{\beta}{\alpha} - s\right) h(s) f(s,u(s)) \nabla s - \frac{1}{\alpha} \sum_{i=2}^{m-1} \int_{t_1}^{t_i} h(s) f(s,u(s)) \nabla s \\ &< \int_{t_1}^{t_m} \left(t_m + \frac{\beta}{\alpha} - s\right) h(s) f(s,u(s)) \nabla s \\ &< pw \int_{t_1}^{t_m} \left(t_m + \frac{\beta}{\alpha} - s\right) h(s) \nabla s = p.\end{aligned}$$

Therefore the condition (ii) of Legget–Williams fixed point theorem (Theorem 2.7) is satisfied.

Finally for condition (iii) of Theorem 2.7 we suppose that $u \in \mathcal{P}(\psi, q, r)$ with $\|Au\| > \dfrac{q t_m}{t_m - t_{m-1}}$. Using Lemma 2.3 we deduce

$$\psi(Au) = Au(t_{m-1}) \geq \frac{t_m - t_{m-1}}{t_m} \|Au\| > q,$$

which implies that condition (iii) of Theorem 2.7 is satisfied.

Since all the conditions of Legget–Williams fixed point theorem are satisfied, the m-PBVP (1.1)-(1.2) has at least three positive solutions u_1, u_2 and u_3 in $\overline{\mathcal{P}_r}$ satisfying $\|u_1\| < p,\ \psi(u_2) > q,\ p < \|u_3\|$ with $\psi(u_3) < q$. ■

If we generalize the conditions of Theorem 3.4, we can establish the existence of an odd number of positive solutions of the m-PBVP (1.1)–(1.2).

Theorem 3.5. Assume that the conditions (H1) and (H2) are satisfied and $\alpha > 0,\ \beta > m-2$. Suppose that there exist constants

$$0 < p_1 < q_1 < q_1 \frac{t_m}{t_m - t_{m-1}} \leq p_2 < q_2 < q_2 \frac{t_m}{t_m - t_{m-1}} \leq p_3 < \dots \leq p_n$$

such that

(i) $f(s,u) < p_i w$ for $s \in [t_1, t_m]$ and $u \in [0, p_i]$,

(ii) $f(s,u) \geq q_i W$ for $s \in [t_1, t_{m-1}]$ and $u \in \left[q_i, \dfrac{q_i t_m}{t_m - t_{m-1}}\right]$,

Then the m-PBVP (1.1)–(1.2) has at least $2n-1$ positive solutions.

Proof. When $n = 1$, it is immediate from (i) that $A : \mathcal{P}_{p_1} \to \mathcal{P}_{p_1} \subset \mathcal{P}_{p_1}$, which means that A has at least one fixed point u_1 by Theorem 3.2. When $n = 2$, by Theorem 3.4, we can obtain at least three positive solutions u_1, u_2, u_3 satisfying

$$u_1(t_1) < p_2, u_2(t_{m-1}) > q_1, u_3(t_1) > p_2 \text{ with } u_3(t_{m-1}) < q_1.$$

Following this way, we finish the proof by induction. ■

We can illustrate our last result in the following example.

Example 3.6. Let $\mathbb{T} = \left[0, \frac{2}{5}\right] \cup \left\{\frac{3}{5}\right\} \cup \left[\frac{4}{5}, 1\right]$. We consider the following four-point BVP:

$$u^{\Delta\nabla}(t) + \frac{4000u^4}{u^4 + 4004} = 0, \quad t \in \left[\frac{1}{5}, \frac{4}{5}\right] \subset \mathbb{T}, \tag{3.6}$$

$$u^{\Delta}\left(\frac{1}{5}\right) = 0, \quad \frac{1}{2}u\left(\frac{4}{5}\right) + 3u^{\Delta}\left(\frac{4}{5}\right) = u^{\Delta}\left(\frac{2}{5}\right) + u^{\Delta}\left(\frac{3}{5}\right). \tag{3.7}$$

Taking $t_i = \frac{i}{5}$, $(i = 1, 2, 3, 4)$, $\alpha = \frac{1}{2}$, $\beta = 3$ and $h(t) = 1$ we have $w = \frac{50}{187}$ and $W = \frac{5}{4}$. If we take

$$p_1 = \frac{1}{4},\ q_1 = 2,\ p_2 = 15000,\ q_2 = 16000,\ p_3 = 65000, q_3 = 66000,\ p_4 = 265000$$

we can obtain

$$0 < p_1 < q_1 < 4q_1 \le p_2 < q_2 < 4q_2 \le p_3 < q_3 < 4q_3 \le p_4.$$

The use of Theorem 3.5 implies four-PBVP (3.6)–(3.7) has at least 7 positive solutions.

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References

[1] Douglas R. Anderson, Solutions to second-order three-point problems on time scales, *J. Difference Equ. Appl.*, 8(8):673–688, 2002.

[2] Richard I. Avery and Johnny Henderson, Two positive fixed points of nonlinear operators on ordered Banach spaces, *Comm. Appl. Nonlinear Anal.*, 8(1):27–36, 2001.

[3] Martin Bohner and Allan Peterson, *Dynamic equations on time scales*, Birkhäuser Boston Inc., Boston, MA, 2001, An introduction with applications.

[4] Martin Bohner and Allan Peterson, *Advances in dynamic equations on time scales*, Birkhäuser, Boston, 2003.

[5] Da Jun Guo and V. Lakshmikantham, *Nonlinear problems in abstract cones*, volume 5 of *Notes and Reports in Mathematics in Science and Engineering*, Academic Press Inc., Boston, MA, 1988.

[6] Chaitan P. Gupta, Solvability of a three-point nonlinear boundary value problem for a second order ordinary differential equation, *J. Math. Anal. Appl.*, 168(2):540–551, 1992.

[7] Chaitan P. Gupta, A generalized multi-point boundary value problem for second order ordinary differential equations, *Appl. Math. Comput.*, 89(1-3):133–146, 1998. Differential equations and computational simulations, II (Mississippi State, MS, 1995).

[8] Zhimin He, Existence of two solutions of m-point boundary value problem for second order dynamic equations on time scales, *J. Math. Anal. Appl.*, 296(1):97–109, 2004.

[9] V.A. Il′in and E.I. Moiseev, A nonlocal boundary value problem of the first kind for the Sturm-Liouville operator in differential and difference interpretations, *Differentsial′nye Uravneniya*, 23(7):1198–1207, 1987.

[10] Weihua Jiang and Yanping Guo, Multiple positive solutions for second-order m-point boundary value problems, *J. Math. Anal. Appl.*, 327(1):415–424, 2007.

[11] M.A. Krasnosel′skiĭ, *Positive solutions of operator equations*, Translated from the Russian by Richard E. Flaherty; edited by Leo F. Boron. P. Noordhoff Ltd. Groningen, 1964.

[12] Richard W. Leggett and Lynn R. Williams, Multiple positive fixed points of nonlinear operators on ordered Banach spaces, *Indiana Univ. Math. J.*, 28(4):673–688, 1979.

[13] Ruyun Ma, Nodal solutions for a second-order m-point boundary value problem, *Czechoslovak Math. J.*, 56(131)(4):1243–1263, 2006.

[14] Ruyun Ma and Haiyan Wang, Positive solutions of nonlinear three-point boundary-value problems, *J. Math. Anal. Appl.*, 279(1):216–227, 2003.

[15] Hong-Rui Sun and Wan-Tong Li, Positive solutions for nonlinear three-point boundary value problems on time scales, *J. Math. Anal. Appl.*, 299(2):508–524, 2004.

[16] İsmail Yaslan, Existence of positive solutions for nonlinear three-point problems on time scales, *J. Comput. Appl. Math.*, 206(2):888–897, 2007.