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Volterra Integral Equations on Time Scales: Basic Qualitative and Quantitative Results with Applications to Initial Value Problems on Unbounded Domains*

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Abstract

This article introduces the basic qualitative and basic quantitative theory of Volterra integral equations on time scales and thus may be considered as a foundation for future advanced studies in the field. New sufficient conditions are introduced that guarantee: existence; uniqueness; approximation; boundedness and certain growth rates of solutions to both linear and nonlinear problems. The main techniques employed are contemporary components of nonlinear analysis, including: the fixed-point theorems of Banach and Schäfer; Picard iterations; inequality theory on time scales; and a novel definition of measuring distance in metric spaces and normed spaces. As an application of the new findings, we present some results concerning nonlinear initial value problems for dynamic, differential and difference equations on unbounded domains. We also present some suggestions concerning open problems and possible directions for further work.

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1. Introduction

Historically, two of the most important types of mathematical equations that have been used to mathematically describe various dynamic processes are: differential and integral equations; and difference and summation equations, which model phenomena, respectively: in continuous time; or in discrete time. Traditionally, researchers have used *either* differential and integral equations *or* difference and summation equations – but not a combination of the two areas – to describe dynamic models.

However, it is now becoming apparent that certain phenomena do not involve solely continuous aspects or solely discrete aspects. Rather, they feature elements of both the continuous and the discrete. These types of hybrid processes are seen, for example, in population dynamics where nonoverlapping generations [27, p.93] occur. Furthermore, neither difference equations nor differential equations give a good description of most population growth [23, p.306].

To effectively treat hybrid dynamical systems, a more modern and flexible mathematical framework is needed to accurately model continuous–discrete processes in a mutually consistent manner.

It appears to be advantageous to model certain processes by employing a suitable combination of both differential equations and difference equations at different stages in the process under consideration. For example, it seems desirable to use differential equations over periods of rapid, dynamic change; while a difference equation would be a suitable modelling tool over less volatile periods. Hence the dynamics of the model described through this hybrid approach would be accurately and more concisely reflected than by just using only one of the areas of differential equations or difference equations. In addition, there is a certain freedom for the modeller to be able to switch between "continuous modelling" and "discrete modelling" and back again, where appropriate [36].

An emerging area that has the potential to effectively manage the above situations is the field of *dynamic equations on time scales*. Created by Hilger in 1990 [20], this new and compelling area of mathematics is more general and versatile than the traditional theories of differential and difference equations, and "appears to be the way forward in the quest for accurate and flexible mathematical models" [35, Sec. 1]. In fact, the field of dynamic equations on time scales contains and extends the classical theory of differential, difference, integral and summation equations as special cases.

This paper considers equations of the type

$$\mathbf{x}(t) = \mathbf{f}(t) + \int_a^t \mathbf{k}(t, s, \mathbf{x}(s))\ \Delta s, \qquad t \in I_{\mathbb{T}} := I \cap \mathbb{T}; \tag{1.1}$$

$$\mathbf{x}(t) = \mathbf{f}(t) + \int_a^t \mathbf{k}(t, s, \mathbf{x}^{\sigma}(s))\ \Delta s, \qquad t \in I_{\mathbb{T}}; \tag{1.2}$$

where: $\mathbf{x} : I_{\mathbb{T}} \to \mathbb{R}^n$ is the unknown function; $\mathbf{f} : I_{\mathbb{T}} \to \mathbb{R}^n$ and $\mathbf{k} : I_{\mathbb{T}} \times I_{\mathbb{T}} \times \mathbb{R}^n \to \mathbb{R}^n$ are known, possibly nonlinear functions, $n \geq 1$; t is from a so-called "time scale" $\mathbb{T}$ (which is a nonempty closed subset of $\mathbb{R}$); the integral sign represents a very general type of operation, known as the "delta" integral; and I is an appropriate interval of $\mathbb{R}$. Equation (1.1) is known as a *Volterra integral equation on time scales*. Equation (1.2) is defined similarly, where $\mathbf{x}^\sigma := \mathbf{x} \circ \sigma$, with σ a particular function to be defined a little later.

If $\mathbb{T} = \mathbb{R}$, then (1.1) becomes the familiar Volterra integral equation

$$\mathbf{x}(t) = \mathbf{f}(t) + \int_a^t \mathbf{k}(t, s, \mathbf{x}(s))\, ds, \quad t \in I.$$

If $\mathbb{T} = \mathbb{Z}$, then (1.1) becomes the well-known Volterra summation equation

$$\mathbf{x}(t) = \mathbf{f}(t) + \sum_{s=a}^{t-1} \mathbf{k}(t, s, \mathbf{x}(s)), \quad t \in I_{\mathbb{Z}}.$$

There are many more time scales than just $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$ and hence there is a rich tapestry of equations contained as special cases of (1.1) and (1.2).

In the past 10 years, there has been interest in obtaining results for equations on time scales in which the general "delta" derivative $\mathbf{x}^\Delta$ appears. Several monographs and survey papers [1,6,7,25,35] contain detailed treatment of these types of equations, however, they do not discuss, in detail, the equally (or more) important case of equations that feature the delta integral, rather than the delta derivative. This is possibly due to the basic theory of integral equations on time scales lagging behind that of delta derivative equations on time scales. It is difficult to find any recent papers on the subject, except [37,38], where the theory of Fredholm-type equations on time scales is discussed.

Therefore, the main purpose of this article is to fill this gap in the literature by furnishing the basic theory of linear and nonlinear Volterra equations on time scales and thus we believe this article serves as a launching pad for future advanced studies in the area.

We believe that integral equations on time scales have an enormous potential for rich and diverse applications and thus they are most worthy of attention. Studies into the area will not only provide a deeper understanding of traditional integral and summation equations by uncovering the strange distinctions and interesting links between the two areas, but will also lead to new discoveries in those dynamic equations on time scales where the delta derivative is present.

If one subscribes to the view of Dieudonné [12, p.viii] on the relationship between differential equations and integral equations, then the importance of integral equations on time scales is clear: *there are no dynamic equations on time scales – there are only integral equations on time scales*. Our view here is based on two simple facts: in the investigation of dynamic equations on time scales, the analysis most often turns to that of a related integral equation on time scales; and the area of integral equations on time scales, by its general nature, will enjoy at least as many applications to science, engineering and technology as the field of dynamic equations on time scales.

This paper focuses on the qualitative and quantitative properties of solutions to Volterra integral equations on time scales. Some important questions that this work addresses are:

- Under what conditions do the integral equations (1.1), (1.2) actually have (possibly unique) solutions?
- If solutions do exist, then what are their nature; and how can we find them; or closely approximate them?

The main techniques that we employ to answer the above questions are from contemporary areas of nonlinear analysis, including: the fixed–point theorems of Banach and Schäfer; the method of Picard iterations; inequality theory; and a novel definition of measuring distance in metric spaces and normed spaces.

The results contained herein compliment those of Stefan Hilger's seminal paper of 1990 [20] and, more recently, those of Tisdell and Zaidi [35].

In addition, possible new directions for the novel results are also presented.

The style of our investigation has been shaped by the monographs [9, 11, 12] on Volterra equations, in which the interested reader will find ample historical perspectives on the formulation of Volterra equations (which we have omitted here for brevity).

There is also a large literature on the approach to continuous and discrete systems via Stieltjes integral equations, and we refer the reader to [18,19,22,24,30,31] for more information on these approaches.

2. Time Scales

To understand the notation used above, some preliminary definitions are needed, which we now present. For more detail see [6, Chap.1].

Definition 2.1. A time scale $\mathbb{T}$ is a nonempty closed subset of the real numbers $\mathbb{R}$.

Since a time scale may or may not be connected, the concept of the jump operator is useful to define the generalised derivative $\mathbf{x}^{\Delta}$ of a function $\mathbf{x}$.

Definition 2.2. The forward (backward) jump operator $\sigma(t)$ at t for $t < \sup \mathbb{T}$ (respectively $\rho(t)$ at t for $t > \inf \mathbb{T}$) is given by

$$\sigma(t) := \inf\{\tau > t : \tau \in \mathbb{T}\}, \quad (\rho(t) := \sup\{\tau < t : \tau \in \mathbb{T}\},) \quad \text{for all } t \in \mathbb{T}.$$

Define the graininess function $\mu : \mathbb{T} \to [0, \infty)$ as $\mu(t) := \sigma(t) - t$.

Throughout this work the assumption is made that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$.

Definition 2.3. The jump operators σ and ρ allow the classification of points in a time scale in the following way: If $\sigma(t) > t$, then the point t is called right-scattered; while if $\rho(t) < t$, then t is termed left-scattered. If $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then the point t is called right-dense; while if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then we say t is left-dense.

If $\mathbb{T}$ has a left-scattered maximum value m, then we define $\mathbb{T}^{\kappa} := \mathbb{T} - \{m\}$. Otherwise $\mathbb{T}^{\kappa} := \mathbb{T}$.

The following gives a formal $\varepsilon - \delta$ definition of the generalised delta derivative.

Definition 2.4. Fix $t \in \mathbb{T}^\kappa$ and let $\mathbf{x} : \mathbb{T} \to \mathbb{R}^n$. Define $\mathbf{x}^\Delta(t)$ to be the vector (if it exists) with the property that given $\varepsilon > 0$ there is a neighbourhood U of t with

$$|[x_i(\sigma(t)) - x_i(s)] - x_i^\Delta(t)[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s|, \quad \text{for all } s \in U \text{ and each } i = 1, \ldots, n.$$

Call $\mathbf{x}^\Delta(t)$ the delta derivative of $\mathbf{x}(t)$ and say that $\mathbf{x}$ is delta–differentiable.

Converse to the delta derivative, we now state the definition of the delta integral.

Definition 2.5. If $\mathbf{J}^\Delta(t) = \mathbf{j}(t)$, then define the (Cauchy) delta integral by

$$\int_a^t \mathbf{j}(s)\ \Delta s = \mathbf{J}(t) - \mathbf{J}(a).$$

If $\mathbb{T} = \mathbb{R}$, then $\int_a^t \mathbf{j}(s)\ \Delta s = \int_a^t \mathbf{j}(s)\ ds$, while if $\mathbb{T} = \mathbb{Z}$, then $\int_a^t \mathbf{j}(s)\Delta s = \sum_a^{t-1} \mathbf{j}(s)$. Once again, there are many more time scales than just $\mathbb{R}$ and $\mathbb{Z}$ and hence there are many more delta integrals. For a more general definition of the delta integral see [7].

The following theorem is a fundamental result of the time scales.

Theorem 2.6. [20] Assume that $\mathbf{j} : \mathbb{T} \to \mathbb{R}^n$ and let $t \in \mathbb{T}^\kappa$.

(i) If $\mathbf{j}$ is delta-differentiable at t, then $\mathbf{j}$ is continuous at t.

(ii) If $\mathbf{j}$ is continuous at t and t is right-scattered, then $\mathbf{j}$ is delta-differentiable at t with

$$\mathbf{j}^\Delta(t) = \frac{\mathbf{j}(\sigma(t)) - \mathbf{j}(t)}{\sigma(t) - t}.$$

(iii) If $\mathbf{j}$ is delta-differentiable and t is right-dense, then

$$\mathbf{j}^\Delta(t) = \lim_{s \to t} \frac{\mathbf{j}(t) - \mathbf{j}(s)}{t - s}.$$

(iv) If $\mathbf{j}$ is delta-differentiable at t, then $\mathbf{j}(\sigma(t)) = \mathbf{j}(t) + \mu(t)\mathbf{j}^\Delta(t)$.

For brevity, we will write $\mathbf{x}^\sigma$ to denote the composition $\mathbf{x} \circ \sigma$.
The following gives a generalised idea of continuity on time scales.

Definition 2.7. Assume $\mathbf{j} : \mathbb{T} \to \mathbb{R}^n$. Define and denote $\mathbf{j} \in C_{\text{rd}}(\mathbb{T}; \mathbb{R}^n)$ as right-dense (rd) continuous if: $\mathbf{j}$ is continuous at every right-dense point $t \in \mathbb{T}$; and $\lim_{s \to t^-} \mathbf{j}(s)$ exists and is finite at every left-dense point $t \in \mathbb{T}$. For functions F of two (or more) variables,

$t \in \mathbb{T}$ and $u_i \in \mathbb{R}$, we say F is rd-continuous if it is rd-continuous in t and continuous in each u_i.

Of particular importance is the fact that every C_{rd} function is delta-integrable [6, Theorem 1.73].

Our methods will involve the "exponential function" on a time scale and we now provide a brief discussion about this special function.

Define the so-called set of regressive functions, $\mathcal{R}$, by

$$\mathcal{R} := \{p \in C_{rd}(\mathbb{T}) \ \text{ and } \ 1 + p(t)\mu(t) \neq 0, \ \forall t \in \mathbb{T}\}$$

and the set of positively regressive functions, $\mathcal{R}^+$, by

$$\mathcal{R}^+ := \{p \in C_{rd}(\mathbb{T}) \ \text{ and } \ 1 + p(t)\mu(t) > 0, \ \forall t \in \mathbb{T}\}.$$

For $p \in \mathcal{R}$ we define (see [6, Theorem 2.35]) the exponential function $e_p(\cdot, a)$ on the time scale $\mathbb{T}$ as the unique solution to the scalar IVP

$$x^{\Delta} = p(t)x, \quad x(a) = 1.$$

If $p \in \mathcal{R}^+$, then $e_p(t, a) > 0$ for all $t \in \mathbb{T}$, [6, Theorem 2.48].

More explicitly, the exponential function $e_p(\cdot, a)$ is given by

$$e_p(t,a) := \begin{cases} \exp\left(\displaystyle\int_a^t p(s)\, ds\right), & \text{for } t \in \mathbb{T}, \ \mu = 0; \\ \exp\left(\displaystyle\int_a^t \frac{\operatorname{Log}(1 + \mu(s)p(s))}{\mu(s)}\, \Delta s\right), & \text{for } t \in \mathbb{T}, \ \mu > 0; \end{cases}$$

where Log is the principal logarithm function.

A solution to (1.1) is a continuous function $\mathbf{x} : I_{\mathbb{T}} \to \mathbb{R}^n$ that satisfies (1.1). A solution to (1.2) is defined similarly.

Throughout this work, if $\mathbf{y}, \mathbf{z} \in \mathbb{R}^n$, then $\langle \mathbf{y}, \mathbf{z} \rangle$ denotes the usual Euclidean inner product on $\mathbb{R}^n$ and $\|\mathbf{z}\|$ denotes the Euclidean norm of $\mathbf{z}$ on $\mathbb{R}^n$.

For more on the basic theory and recent developments of time scales, see [1–3, 5–8, 13, 16, 17, 20, 21, 25, 29, 33, 34].

3. Linear Problems

In this section we examine the linear Volterra integral equation

$$\mathbf{x}(t) = \mathbf{f}(t) + \int_a^t \mathcal{B}(t,s)\mathbf{x}(s)\, \Delta s, \quad t \in I_{\mathbb{T}}, \tag{3.1}$$

where $\mathbf{f} : I_{\mathbb{T}} \to \mathbb{R}^n$ and $\mathcal{B} : I_{\mathbb{T}} \times I_{\mathbb{T}} \to \mathbb{R}^{n \times n}$ is a matrix-valued function.

Our first result concerns the existence, uniqueness and approximation of solutions $\mathbf{x}$ to (3.1) when $I_{\mathbb{T}} := [a, b]_{\mathbb{T}}$, that is, $I_{\mathbb{T}}$ is a compact interval of $\mathbb{T}$. Our methods involve Banach's fixed–point theorem and we now introduce the appropriate metric space setting.

Let $\beta > 0$ be a constant and let $\| \cdot \|$ denote the Euclidean norm on $\mathbb{R}^n$. When $I_{\mathbb{T}}$ is the compact interval $[a, b]_{\mathbb{T}}$, we will consider the space of continuous functions from $[a, b]_{\mathbb{T}}$ to $\mathbb{R}^n$ coupled with a suitable metric, either

$$d_\beta(\mathbf{x}, \mathbf{y}) := \sup_{t \in [a,b]_{\mathbb{T}}} \frac{\|\mathbf{x}(t) - \mathbf{y}(t)\|}{e_\beta(t, a)}; \quad \text{or} \quad d_0(\mathbf{x}, \mathbf{y}) := \sup_{t \in [a,b]_{\mathbb{T}}} \|\mathbf{x}(t) - \mathbf{y}(t)\|.$$

We will also consider $C([a, b]_{\mathbb{T}}; \mathbb{R}^n)$ coupled with a suitable norm, either

$$\|\mathbf{x}\|_\beta := \sup_{t \in [a,b]_{\mathbb{T}}} \frac{\|\mathbf{x}(t)\|}{e_\beta(t, a)}; \quad \text{or} \quad \|\mathbf{x}\|_0 := \sup_{t \in [a,b]_{\mathbb{T}}} \|\mathbf{x}(t)\|.$$

The above definitions of d_β and $\| \cdot \|_\beta$ come from [35]. Some important properties of d_β and $\| \cdot \|_\beta$ are now listed.

Lemma 3.1. If $\beta > 0$ is a constant, then:

(i) d_β is a metric;

(ii) $\| \cdot \|_\beta$ is a norm and is equivalent to the sup–norm $\| \cdot \|_0$;

(iii) $(C([a, b]_{\mathbb{T}}; \mathbb{R}^n), d_\beta)$ is a complete metric space;

(iv) $(C([a, b]_{\mathbb{T}}; \mathbb{R}^n), \| \cdot \|_\beta)$ is a Banach space.

Proof. See Tisdell and Zaidi [35, Lemma 3.3]. ■

Let (Y, d) be a complete metric space [10, Chap. 4] and $F : Y \to Y$. The map F is said to be contractive if there exists a positive constant $\alpha < 1$ such that

$$d(F(x), F(y)) \leq \alpha d(x, y), \quad \forall x, y \in Y.$$

The constant α is called the contraction constant of F.

For any given $y \in Y$ we define the sequence $\{F^i(y)\}$ recursively by: $F^0(y) := y$; and $F^{i+1}(y) := F(F^i(y))$.

Theorem 3.2. (Banach, [14, p. 10]) Let (Y, d) be a complete metric space and let $F : Y \to Y$ be contractive. Then F has a unique fixed-point u and $F^i(y) \to u$ for each $y \in Y$.

Remark 3.3. ([14, p. 10]) Furthermore to Banach's theorem, if we start at an arbitrary $y \in Y$ then Banach's theorem provides the following estimate on the "error" between the ith iteration $F^i y$ and the fixed point u, namely

$$d(F^i y, u) \leq \frac{\alpha^i}{1 - \alpha} d(y, Fy). \tag{3.2}$$

We are now ready to present our first major result of this section, which will be proved by using Banach's theorem.

Theorem 3.4. Consider the linear integral equation (3.1) with $I_{\mathbb{T}} := [a, b]_{\mathbb{T}}$. Let $\mathcal{B} : [a, b]_{\mathbb{T}}^2 \to \mathbb{R}^{n\times n}$ be continuous in its first variable and rd-continuous in its second variable and let $\mathbf{f} : [a, b]_{\mathbb{T}} \to \mathbb{R}^n$ be continuous. Then (3.1) has a unique solution. In addition, if a sequence of functions $\{\mathbf{x}_i\}$ is defined inductively by choosing any $\mathbf{x}_0 \in C([a, b]_{\mathbb{T}}; \mathbb{R}^n)$ and setting

$$\mathbf{x}_{i+1}(t) := \mathbf{f}(t) + \int_a^t \mathcal{B}(t, s)\mathbf{x}_i(s)\ \Delta s, \quad t \in [a, b]_{\mathbb{T}}, \tag{3.3}$$

then the sequence $\{\mathbf{x}_i\}$ converges uniformly on $[a, b]_{\mathbb{T}}$ to the unique solution $\mathbf{x}$ of (3.1).

Proof. Since $\mathcal{B}$ is continuous in its first variable and rd-continuous in its second variable, (3.3) is well defined. Let $L := \sup_{(t,s)\in[a,b]_{\mathbb{T}}^2} \|\mathcal{B}(t, s)\|$, with $\|\mathcal{B}\|$ a suitable matrix norm on the interval $[a, b]_{\mathbb{T}}^2$, and let $\beta := L\gamma$, where $\gamma > 1$ is an arbitrary constant. Consider the complete metric space $(C([a, b]_{\mathbb{T}}; \mathbb{R}^n), d_\beta)$ and let

$$\mathbf{F} : C([a, b]_{\mathbb{T}}; \mathbb{R}^n) \to C([a, b]_{\mathbb{T}}; \mathbb{R}^n)$$

be defined by

$$[\mathbf{F}\mathbf{x}](t) := \mathbf{f}(t) + \int_a^t \mathcal{B}(t, s)\mathbf{x}(s)\ \Delta s, \quad t \in [a, b]_{\mathbb{T}}. \tag{3.4}$$

It is easy to see that fixed-points of $\mathbf{F}$ will be solutions to (1.1). Thus, we want to prove that there exists a unique $\mathbf{x}$ such that $\mathbf{F}\mathbf{x} = \mathbf{x}$. To do this, we show that $\mathbf{F}$ is a contractive map with contraction constant $\alpha = 1/\gamma < 1$ and Banach's fixed point theorem will then apply. For any $\mathbf{u}, \mathbf{v} \in C([a, b]_{\mathbb{T}}; \mathbb{R}^n)$, consider

$$\begin{aligned}
d_\beta(\mathbf{F}\mathbf{u}, \mathbf{F}\mathbf{v}) &:= \sup_{t\in[a,b]_{\mathbb{T}}} \frac{\|[\mathbf{F}\mathbf{u}](t) - [\mathbf{F}\mathbf{v}](t)\|}{e_\beta(t, a)} \\
&\le \sup_{t\in[a,b]_{\mathbb{T}}} \frac{1}{e_\beta(t, a)} \int_a^t \|\mathcal{B}(t, s)\mathbf{u}(s)) - \mathcal{B}(t, s)\mathbf{v}(s))\|\ \Delta s \\
&\le L \sup_{t\in[a,b]_{\mathbb{T}}} \frac{1}{e_\beta(t, a)} \int_a^t \|\mathbf{u}(s) - \mathbf{v}(s)\|\ \Delta s, \\
&= L \sup_{t\in[a,b]_{\mathbb{T}}} \frac{1}{e_\beta(t, a)} \int_a^t e_\beta(s, a) \frac{\|\mathbf{u}(s) - \mathbf{v}(s)\|}{e_\beta(s, a)}\ \Delta s
\end{aligned}$$

$$
\begin{aligned}
&\le L d_\beta(\mathbf{u},\mathbf{v}) \sup_{t\in[a,b]_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \int_a^t e_\beta(s,a)\,\Delta s \\
&= L d_\beta(\mathbf{u},\mathbf{v}) \sup_{t\in[a,b]_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \left(\frac{e_\beta(t,a)-1}{\beta}\right) \\
&= \frac{d_\beta(\mathbf{u},\mathbf{v})}{\gamma} \sup_{t\in[a,b]_{\mathbb{T}}} \left[1-\frac{1}{e_\beta(t,a)}\right], \quad \text{since } \beta = L\gamma, \\
&= \frac{d_\beta(\mathbf{u},\mathbf{v})}{\gamma}\left[1-\frac{1}{e_\beta(b,a)}\right] < \frac{d_\beta(\mathbf{u},\mathbf{v})}{\gamma}.
\end{aligned}
$$

As $\gamma > 1$, we see that $\mathbf{F}$ is a contractive map and Banach's fixed-point theorem applies, yielding the existence of a unique fixed-point $\mathbf{x}$ of $\mathbf{F}$. In addition, from Banach's theorem, the sequence $\{\mathbf{x}_i\}$ defined in (3.3) converges uniformly in the norm $\|\cdot\|_\beta$ and thus the sequence $\{\mathbf{x}_i\}$ converges uniformly in the sup-norm $\|\cdot\|_0$ to that fixed-point $\mathbf{x}$. ■

We now supply a simple example to demonstrate the application of Theorem 3.4.

Example 3.5. We claim that the scalar, linear integral equation

$$x(t) = 2t + \int_a^t [t-\sigma(s)]\,x(s)\,\Delta s, \quad t \in [a,b]_{\mathbb{T}}; \tag{3.5}$$

has a unique solution for arbitrary $\mathbb{T}$.

Proof. Comparing (3.5) with (3.1) it is easy to see that $\mathcal{B}(t,s) := t-\sigma(s)$ is continuous in t and rd-continuous in s (because σ is, in general, rd-continuous) and $f(t) := 2t$ is continuous. Hence, all of the conditions of Theorem 3.4 are satisfied and thus the conclusions of Theorem 3.4 apply to (3.5). Furthermore, if we take the delta derivative of both sides of (3.5) twice, then we obtain the following linear, second-order dynamic equation

$$x^{\Delta\Delta} - x = 0; \quad \text{with} \quad x(a) = 2a, \quad x^\Delta(a) = 2. \tag{3.6}$$

If $-\mu \in \mathcal{R}$, then it is interesting to note that we may uniquely solve (3.6) in terms of hyperbolic functions on time scales, see [6, p.80] for the details. ■

We now present some basic qualitative results concerning the boundedness of solutions to (3.1) when $I_{\mathbb{T}} := [a,\infty)_{\mathbb{T}}$.

Theorem 3.6. Consider the linear integral equation (3.1) with $I_{\mathbb{T}} := [a,\infty)_{\mathbb{T}}$. Let $\mathbf{f} : [a,\infty)_{\mathbb{T}} \to \mathbb{R}^n$ and let $\mathcal{B} : [a,b]_{\mathbb{T}}^2 \to \mathbb{R}^{n\times n}$ both be rd-continuous. If there exist constants $0 \le m < 1$ and $M \ge 0$ such that

$$\|\mathbf{f}(t)\| \le M, \quad \forall t \in [a,\infty)_{\mathbb{T}}; \tag{3.7}$$

$$\int_a^t \|\mathcal{B}(t,s)\|\,\Delta s \le m, \quad \forall t \in [a,\infty)_{\mathbb{T}}, \tag{3.8}$$

then all solutions to (3.1) are bounded on $[a, \infty)_{\mathbb{T}}$.

Proof. Choose an $R > 0$ such that $M + mR < R$. We claim that $\|\mathbf{x}(t)\| < R$ for all $t \in [a, \infty)_{\mathbb{T}}$. If this is not the case, then there exists a t_1 such that

$$\begin{aligned} \|\mathbf{x}(t)\| &< R, \quad \forall t \in [a, t_1)_{\mathbb{T}}; \quad \text{and} \\ \|\mathbf{x}(t_1)\| &\geq R. \end{aligned}$$

Hence we see

$$\begin{aligned} R \leq \|\mathbf{x}(t_1)\| &= \|\mathbf{f}(t_1) + \int_a^{t_1} \mathcal{B}(t_1, s)\mathbf{x}(s)\, \Delta s\| \\ &\leq M + mR \\ &< R \end{aligned}$$

and we reach a contradiction. Thus $\mathbf{x}$ is bounded on $[a, \infty)_{\mathbb{T}}$. ■

Our final result of this section will require the following lemma, known as Gronwall's lemma on time scales [6, Corollary 6.7].

Lemma 3.7. (Gronwall) Let $y \in C_{\mathrm{rd}}([a, \infty)_{\mathbb{T}})$; let $p \in \mathcal{R}^+$ with $p \geq 0$ on $[a, \infty)_{\mathbb{T}}$; and let A be a constant. We have

$$y(t) \leq A + \int_a^t p(s)y(s)\, \Delta s, \quad \forall t \in [a, \infty)_{\mathbb{T}};$$

implies

$$y(t) \leq A e_p(t, a), \quad t \in [a, \infty)_{\mathbb{T}}.$$

Proof. See [6, Chap.6]. ■

The following result examines the boundedness and limiting behaviour of solutions to (3.1) on $[a, \infty)_{\mathbb{T}}$. The following "circle minus" identity will be useful:

$$B \ominus \alpha := \frac{B - \alpha}{1 + \mu(t)\alpha}$$

and we refer the reader to [6, Chap.2] for more information on the "circle minus" operator.

Theorem 3.8. Consider the linear integral equation (3.1) with $I_{\mathbb{T}} := [a, \infty)_{\mathbb{T}}$. Let $\mathbf{f} : [a, \infty)_{\mathbb{T}} \to \mathbb{R}^n$ and let $\mathcal{B} : [a, \infty)_{\mathbb{T}}^2 \to \mathbb{R}^{n \times n}$ be rd-continuous. If there exist positive constants α, A and B such that

$$\begin{aligned} \|\mathbf{f}(t)\| &\leq \frac{A}{e_\alpha(t, a)}, \quad \forall t \in [a, \infty)_{\mathbb{T}}; \\ \|\mathcal{B}(t, s)\| &\leq B e_\alpha(s, t), \quad \forall (t, s) \in [a, \infty)_{\mathbb{T}}^2, \end{aligned}$$

then all solutions $\mathbf{x}$ to (3.1) satisfy

$$\|\mathbf{x}(t)\| \leq Ae_{B\ominus\alpha}(t,a), \quad t \in [a,\infty)_{\mathbb{T}}.$$

In addition, if $B - \alpha < 0$, then $\|\mathbf{x}(t)\| \to 0$ as $t \to \infty$.

Proof. From (3.1) we have

$$\|\mathbf{x}(t)\| \leq \frac{A}{e_\alpha(t,a)} + \int_a^t Be_\alpha(s,t)\|\mathbf{x}(s)\| \,\Delta s, \quad t \in [a,\infty)_{\mathbb{T}};$$

so that

$$e_\alpha(t,a)\|\mathbf{x}(t)\| \leq A + \int_a^t Be_\alpha(s,a)\|\mathbf{x}(s)\| \,\Delta s, \quad t \in [a,\infty)_{\mathbb{T}}; \tag{3.9}$$

where we have used the identity

$$e_\alpha(s,t) = \frac{e_\alpha(s,a)}{e_\alpha(t,a)}.$$

Gronwall's lemma for time scales applied to (3.9) furnishes the estimate

$$e_\alpha(t,a)\|\mathbf{x}(t)\| \leq Ae_B(t,a), \quad t \in [a,\infty)_{\mathbb{T}};$$

with a rearrangement giving our first conclusion by use of the identity [6, p. 62]

$$\frac{e_B(t,a)}{e_\alpha(t,a)} = e_{B\ominus\alpha}(t,a).$$

Our second conclusion follows from

$$e^{\Delta}_{B\ominus\alpha}(t,a) = [B \ominus \alpha]e_{B\ominus\alpha}(t,a)$$

and from $B - \alpha < 0$. See that $B \ominus \alpha < 0$ and so $e_{B\ominus\alpha}(t,a)$ decreases monotonically towards zero as $t \to \infty$. ■

As a corollary and application of Theorem 3.8, we present the following example concerning the "renewal summation equation"

$$x(t) = f(t) + \sum_{s=0}^{t-1} C(t-s)x(s), \quad t \in [0,\infty)_{\mathbb{N}} \tag{3.10}$$

where f and C are scalar-valued functions.

Example 3.9. Consider the renewal summation equation (3.10) with $f : [0,\infty)_{\mathbb{N}} \to \mathbb{R}$ and $C : [0,\infty)_{\mathbb{N}} \to \mathbb{R}$. If there exist positive constants A, B and α such that

$$|f(t)| \leq \frac{A}{(1+\alpha)^t}, \quad \forall t \in [0,\infty)_{\mathbb{N}};$$

$$|C(q)| \leq \frac{B}{(1+\alpha)^q}, \quad \forall q \in [0,\infty)_{\mathbb{N}},$$

then all solutions x to (3.10) satisfy

$$|x(t)| \le A \left[\frac{1+B}{1+\alpha}\right]^t, \quad t \in [0, \infty)_{\mathbb{N}}.$$

In addition, if $B - \alpha < 0$, then $|x(t)| \to 0$ as $t \to \infty$.

Proof. For the special time scale $\mathbb{T} = \mathbb{N} \cup \{0\}$, we have $\mu \equiv 1$ and the exponential function is

$$e_\alpha(t, 0) = (1+\alpha)^t, \quad t \in [0, \infty)_{\mathbb{N}}.$$

The proof of the claim follows similar steps in the proof of Theorem 3.8 and so is omitted for brevity. ■

4. Nonlinear Problems on Unbounded Intervals

In this section we examine (1.1) and (1.2), where

$$I_{\mathbb{T}} = [a, \infty)_{\mathbb{T}} := [a, \infty) \cap \mathbb{T}, \quad a \in \mathbb{T},$$

and $\mathbb{T}$ is unbounded above, so that we are interested in solutions to these problems that are defined on the unbounded interval $[a, \infty)_{\mathbb{T}}$. We apply our results to some initial value problems with unbounded domains.

4.1. Integral Equations

Consider the (possibly nonlinear) integral equations (1.1) and (1.2) with $I_{\mathbb{T}} := [a, \infty)_{\mathbb{T}}$. We now construct the appropriate metric space for our analysis. Let $\beta > 0$ be a constant and let $\|\cdot\|$ denote the Euclidean norm on $\mathbb{R}^n$. We will consider the space of continuous functions $C([a, \infty)_{\mathbb{T}}; \mathbb{R}^n)$ such that

$$\sup_{t \in [a,\infty)_{\mathbb{T}}} \frac{\|\mathbf{x}(t)\|}{e_\beta(t, a)} < \infty;$$

and denote this special space by $C_\beta([a, \infty)_{\mathbb{T}}; \mathbb{R}^n)$. We couple the linear space $C_\beta([a, \infty)_{\mathbb{T}}; \mathbb{R}^n)$ with a suitable metric, namely

$$d_\beta^\infty(\mathbf{x}, \mathbf{y}) := \sup_{t \in [a,\infty)_{\mathbb{T}}} \frac{\|\mathbf{x}(t) - \mathbf{y}(t)\|}{e_\beta(t, a)}.$$

We will also consider $C_\beta([a, \infty)_{\mathbb{T}}; \mathbb{R}^n)$ coupled with a suitable norm, expressly

$$\|\mathbf{x}\|_\beta^\infty := \sup_{t \in [a,\infty)_{\mathbb{T}}} \frac{\|\mathbf{x}(t)\|}{e_\beta(t, a)}.$$

The above definitions of d_β^∞ and $\|\cdot\|_\beta^\infty$ are generalisations of Bielecki's metric and norm [4], [14, pp.25–26], [15, pp.153–155], [32, p.44] in the time scale environment and complement those introduced in [35].

Some important properties of d_β^∞ and $\|\cdot\|_\beta^\infty$ are now listed.

Lemma 4.1. If $\beta > 0$ is a constant, then:

(i) d_β^∞ is a metric;

(ii) $\|\cdot\|_\beta^\infty$ is a norm;

(iii) $(C_\beta([a,\infty)_{\mathbb{T}};\mathbb{R}^n), \|\cdot\|_\beta^\infty)$ is a Banach space;

(iv) $(C_\beta([a,\infty)_{\mathbb{T}};\mathbb{R}^n), d_\beta^\infty)$ is a complete metric space.

Proof. We only prove part (iii), as the other parts are easily verified.

(iii) Our argument follows that of [12, pp.2–3] and so we only sketch the details. Let $\{\mathbf{x}_m(t)\} \subset C_\beta([a,\infty)_{\mathbb{T}};\mathbb{R}^n)$ be a Cauchy sequence, that is, for every $\varepsilon > 0$ there exists a positive integer N_ε such that

$$\|\mathbf{x}_j - \mathbf{x}_m\|_\beta^\infty < \varepsilon, \quad \text{whenever } m, j > N_\varepsilon. \tag{4.1}$$

We show that there exists a $\mathbf{x} \in C_\beta([a,\infty)_{\mathbb{T}};\mathbb{R}^n)$ such that

$$\lim_{m\to\infty} \left(\|\mathbf{x} - \mathbf{x}_m\|_\beta^\infty\right) = 0.$$

From (4.1) we see that

$$\|\mathbf{x}_j(t) - \mathbf{x}_m(t)\| < \varepsilon e_\beta(t,a), \quad \text{for } m, j > N_\varepsilon, \text{ and } t \in [a,\infty)_{\mathbb{T}}. \tag{4.2}$$

From (4.2) we derive the existence of a mapping $\mathbf{x}(t)$ such that $\mathbf{x}(t) = \lim_{m\to\infty} \mathbf{x}_m(t)$ for any fixed $t \in [a,\infty)_{\mathbb{T}}$. Moreover, it follows from (4.2) that $\mathbf{x}(t) = \lim_{m\to\infty} \mathbf{x}_m(t)$ uniformly on any finite interval of $[a,\infty)_{\mathbb{T}}$. Hence $\mathbf{x}(t)$ is a continuous mapping from $[a,\infty)_{\mathbb{T}}$ to $\mathbb{R}^n$.

Finally we show that $\mathbf{x} \in C_\beta([a,\infty)_{\mathbb{T}};\mathbb{R}^n)$. We keep m fixed and let $j \to \infty$ in (4.2). We then obtain

$$\mathbf{x} - \mathbf{x}_m \in C_\beta([a,\infty)_{\mathbb{T}};\mathbb{R}^n), \quad \text{for } m > N_\varepsilon.$$

See that $\mathbf{x} = (\mathbf{x} - \mathbf{x}_m) + \mathbf{x}_m$ and because both terms in the right-hand side belong to $C_\beta([a,\infty)_{\mathbb{T}};\mathbb{R}^n)$ it follows that $\mathbf{x} \in C_\beta([a,\infty)_{\mathbb{T}};\mathbb{R}^n)$. Thus, $(C_\beta([a,\infty)_{\mathbb{T}};\mathbb{R}^n), \|\cdot\|_\beta^\infty)$ is a Banach space. ■

We now present our first major result of this section concerning the existence, uniqueness, approximation and growth of solutions to the nonlinear problem (1.1).

Theorem 4.2. Consider the integral equation (1.1) with $I_{\mathbb{T}} := [a,\infty)_{\mathbb{T}}$. Let $\mathbf{k} : [a,\infty)_{\mathbb{T}}^2 \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous in its first and third variables and rd-continuous in its second variable; let $\mathbf{f} : [a, \infty)_{\mathbb{T}} \to \mathbb{R}^n$ be continuous; and let $\gamma > 1$, L and β be positive constants with $\beta := L\gamma$. If

$$\|\mathbf{k}(t, s, \mathbf{p}) - \mathbf{k}(t, s, \mathbf{q})\| \leq L\|\mathbf{p} - \mathbf{q}\|, \quad \forall (t, s) \in [a, \infty)_{\mathbb{T}}^2, \ (\mathbf{p}, \mathbf{q}) \in \mathbb{R}^{2n}; \tag{4.3}$$

$$m := \sup_{t \in [a,\infty)_{\mathbb{T}}} \frac{1}{e_\beta(t, a)} \left\| \mathbf{f}(t) + \int_a^t \mathbf{k}(t, s, \mathbf{0}) \, \Delta s \right\| < \infty, \tag{4.4}$$

then the integral equation (1.1) has a unique solution $\mathbf{x} \in C_\beta([a, \infty)_{\mathbb{T}}; \mathbb{R}^n)$. In addition, if a sequence of functions $\{\mathbf{x}_i\}$ is defined inductively by choosing any $\mathbf{x}_0 \in C_\beta([a, \infty)_{\mathbb{T}}; \mathbb{R}^n)$ and setting

$$\mathbf{x}_{i+1}(t) := \mathbf{f}(t) + \int_a^t \mathbf{k}(t, s, \mathbf{x}_i(s)) \, \Delta s, \quad t \in [a, \infty)_{\mathbb{T}}, \tag{4.5}$$

then $\lim_{i \to \infty} \mathbf{x}_i = \mathbf{x}$ on $[a, \infty)_{\mathbb{T}}$ in $C_\beta([a, \infty)_{\mathbb{T}}; \mathbb{R}^n)$, with the convergence being uniform on compact subsets of $[a, \infty)_{\mathbb{T}}$.

Proof. Since $\mathbf{k}$ is rd-continuous function with respect to the integrated variable s, (4.5) is well defined. Let $L > 0$ be the constant defined in (4.3) and let $\beta := L\gamma$, where $\gamma > 1$.

Consider the following equivalent formulation of (1.1), namely

$$\begin{aligned} \mathbf{x}(t) = &\left(\mathbf{f}(t) + \int_a^t \mathbf{k}(t, s, \mathbf{0}) \, \Delta s \right) \\ &+ \int_a^t (\mathbf{k}(t, s, \mathbf{x}(s)) - \mathbf{k}(t, s, \mathbf{0})) \, \Delta s, \quad t \in [a, \infty)_{\mathbb{T}}. \end{aligned} \tag{4.6}$$

We will show that (4.6) has a unique solution and thus (1.1) must also have a unique solution. With this in mind, consider the complete metric space $(C_\beta([a, \infty)_{\mathbb{T}}; \mathbb{R}^n), d_\beta^\infty)$ and let $\mathbf{F}$ be defined by

$$\begin{aligned} [\mathbf{F}\mathbf{x}](t) := &\left(\mathbf{f}(t) + \int_a^t \mathbf{k}(t, s, \mathbf{0}) \, \Delta s \right) \\ &+ \int_a^t (\mathbf{k}(t, s, \mathbf{x}(s)) - \mathbf{k}(t, s, \mathbf{0})) \, \Delta s, \quad t \in [a, \infty)_{\mathbb{T}}. \end{aligned} \tag{4.7}$$

Fixed-points of $\mathbf{F}$ will be solutions to (4.6). Thus, we want to prove that there exists a unique $\mathbf{x}$ such that $\mathbf{F}\mathbf{x} = \mathbf{x}$. To do this we show that the conditions of Banach's theorem are satisfied.

We show that $\mathbf{F} : C_\beta([a, \infty)_{\mathbb{T}}; \mathbb{R}^n) \to C_\beta([a, \infty)_{\mathbb{T}}; \mathbb{R}^n)$. Let $\mathbf{x} \in C_\beta([a, \infty)_{\mathbb{T}}; \mathbb{R}^n)$.

Taking norms in (4.7) we obtain

$$\begin{aligned}
\|\mathbf{Fx}\|_\beta^\infty := \sup_{t\in[a,\infty)_\mathbb{T}} \frac{1}{e_\beta(t,a)} \bigg\| &\mathbf{f}(t) + \int_a^t \mathbf{k}(t,s,\mathbf{0})\ \Delta s \\
&+ \int_a^t \mathbf{k}(t,s,\mathbf{x}(s)) - \mathbf{k}(t,s,\mathbf{0})\ \Delta s \bigg\| \\
&\leq m + \sup_{t\in[a,\infty)_\mathbb{T}} \frac{1}{e_\beta(t,a)} \int_a^t \|\mathbf{k}(t,s,\mathbf{x}(s)) - \mathbf{k}(t,s,\mathbf{0})\|\ \Delta s \\
&\leq m + \sup_{t\in[a,\infty)_\mathbb{T}} \frac{1}{e_\beta(t,a)} L \int_a^t \|\mathbf{x}(s)\|\ \Delta s, \quad \text{from (4.3)} \\
&= m + \sup_{t\in[a,\infty)_\mathbb{T}} \frac{1}{e_\beta(t,a)} L \int_a^t e_\beta(s,a) \frac{\|\mathbf{x}(s)\|}{e_\beta(s,a)}\ \Delta s \\
&\leq m + L\|\mathbf{x}\|_\beta^\infty \sup_{t\in[a,\infty)_\mathbb{T}} \frac{1}{e_\beta(t,a)} \int_a^t e_\beta(s,a)\ \Delta s \\
&= m + L\|\mathbf{x}\|_\beta^\infty \sup_{t\in[a,\infty)_\mathbb{T}} \frac{1}{e_\beta(t,a)} \left[\frac{e_\beta(t,a)-1}{\beta}\right] \\
&= m + \frac{\|\mathbf{x}\|_\beta^\infty}{\gamma} \sup_{t\in[a,\infty)_\mathbb{T}} \left[1 - \frac{1}{e_\beta(t,a)}\right] \\
&= m + \frac{\|\mathbf{x}\|_\beta^\infty}{\gamma} < \infty.
\end{aligned}$$

Hence we see that $\mathbf{F} : C_\beta([a,\infty)_\mathbb{T};\mathbb{R}^n) \to C_\beta([a,\infty)_\mathbb{T};\mathbb{R}^n)$.

We show that $\mathbf{F}$ is a contractive map with contraction constant $\alpha = 1/\gamma < 1$ and then Banach's fixed point theorem will apply. For any $\mathbf{u}, \mathbf{v} \in C_\beta([a,\infty)_\mathbb{T};\mathbb{R}^n)$, consider

$$\begin{aligned}
d_\beta^\infty(\mathbf{Fu},\mathbf{Fv}) &:= \sup_{t\in[a,\infty)_\mathbb{T}} \frac{\|[\mathbf{Fu}](t) - [\mathbf{Fv}](t)\|}{e_\beta(t,a)} \\
&\leq \sup_{t\in[a,\infty)_\mathbb{T}} \frac{1}{e_\beta(t,a)} \int_a^t \|\mathbf{k}(t,s,\mathbf{u}(s)) - \mathbf{k}(t,s,\mathbf{v}(s))\|\ \Delta s \\
&\leq \sup_{t\in[a,\infty)_\mathbb{T}} \frac{1}{e_\beta(t,a)} \int_a^t L\|\mathbf{u}(s) - \mathbf{v}(s)\|\ \Delta s, \quad \text{from (4.3)} \\
&= L \sup_{t\in[a,\infty)_\mathbb{T}} \frac{1}{e_\beta(t,a)} \int_a^t e_\beta(s,a) \frac{\|\mathbf{u}(s)-\mathbf{v}(s)\|}{e_\beta(s,a)}\ \Delta s \\
&\leq L d_\beta^\infty(\mathbf{u},\mathbf{v}) \sup_{t\in[a,\infty)_\mathbb{T}} \frac{1}{e_\beta(t,a)} \int_a^t e_\beta(s,a)\ \Delta s
\end{aligned}$$

$$
\begin{aligned}
&= L d_\beta^\infty(\mathbf{u}, \mathbf{v}) \sup_{t \in [a,\infty)_\mathbb{T}} \frac{1}{e_\beta(t,a)} \left(\frac{e_\beta(t,a) - 1}{\beta} \right) \\
&= \frac{d_\beta^\infty(\mathbf{u}, \mathbf{v})}{\gamma} \sup_{t \in [a,\infty)_\mathbb{T}} \left[1 - \frac{1}{e_\beta(t,a)} \right], \quad \text{since } \beta = L\gamma, \\
&= \frac{d_\beta^\infty(\mathbf{u}, \mathbf{v})}{\gamma}.
\end{aligned}
$$

As $\gamma > 1$, we see that $\mathbf{F}$ is a contractive map and so Banach's fixed-point theorem applies, yielding the existence of a unique fixed-point $\mathbf{x}$ of $\mathbf{F}$. In addition, from Banach's theorem, the sequence $\{\mathbf{x}_i\}$ defined in (4.5) converges uniformly in the norm $\| \cdot \|_\beta$ on compact subsets of $[a,\infty)_\mathbb{T}$ (and thus the sequence $\{\mathbf{x}_i\}$ converges uniformly in the sup-norm $\| \cdot \|_0$ on compact subsets) to that fixed-point $\mathbf{x}$. ■

We now present a simple example to illustrate Theorem 4.2.

Example 4.3. Consider the scalar integral equation

$$x(t) = 2t + 2 \int_a^t \left[x(s)^2 + 7 \right]^{1/2} \Delta s, \quad t \in [a,\infty)_\mathbb{T}.$$

We claim that this integral equation has a unique solution for arbitrary $\mathbb{T}$.

Proof. We will use Theorem 4.2 and make use of the fact that $k(t,s,p) := 2(p^2+7)^{1/2}$ has a bounded partial derivative with respect to p everywhere. Consider

$$
\begin{aligned}
|k(t,s,p) - k(t,s,q)| &= \left| 2\left[p^2+7\right]^{1/2} - 2\left[q^2+7\right]^{1/2} \right| \\
&\le \sup_{r \in \mathbb{R}} \left| \frac{2r}{[r^2+7]^{1/2}} \right| \cdot |p-q|, \quad \text{by the mean value theorem} \\
&\le 2|p-q|
\end{aligned}
$$

so that (4.3) holds with $L = 2$. For a choice of, say, $\gamma = 2$ we then have $\beta = 4$ and it is not difficult to verify (4.4) holds. The result now follows from Theorem 4.2. ■

We now present a result concerning the existence, uniqueness and approximation of solutions to (1.2). Although (1.1) and (1.2) appear to be of a similar nature, we shall see that there are interesting distinctions between the two.

Theorem 4.4. Consider the delta intergal equation (1.2) with $I_\mathbb{T} := [a,\infty)_\mathbb{T}$. Let $\mathbf{k} : [a,\infty)_\mathbb{T}^2 \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous in its first and third variables and rd-continuous in its second variable, let $\mathbf{f} : [a,\infty)_\mathbb{T} \to \mathbb{R}^n$ be continuous and let $\gamma > 1$, β and L be positive constants with $\beta := L\gamma$. If

$$\|\mathbf{k}(t,s,\mathbf{p}) - \mathbf{k}(t,s,\mathbf{q})\| \le L\|\mathbf{p}-\mathbf{q}\|, \quad \forall (t,s) \in [a,\infty)^2_{\mathbb{T}},\ (\mathbf{p},\mathbf{q}) \in \mathbb{R}^{2n}; \tag{4.8}$$

$$L \sup_{t\in[a,\infty)_{\mathbb{T}}} \mu(t) < 1 - \frac{1}{\gamma}; \tag{4.9}$$

$$\sup_{t\in[a,\infty)_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \left\| \mathbf{f}(t) + \int_a^t \mathbf{k}(t,s,\mathbf{0})\ \Delta s \right\| < \infty, \tag{4.10}$$

then (1.2) has a unique solution $\mathbf{x} \in C_\beta([a,\infty)_{\mathbb{T}} : \mathbb{R}^n)$. In addition, if a sequence of functions $\{\mathbf{x}_i\}$ is defined inductively by choosing any $\mathbf{x}_0 \in C_\beta([a,\infty)_{\mathbb{T}}; \mathbb{R}^n)$ and setting

$$\mathbf{x}_{i+1}(t) := \mathbf{f}(t) + \int_a^t \mathbf{k}(t,s,\mathbf{x}_i^\sigma(s))\ \Delta s, \quad t \in [a,\infty)_{\mathbb{T}}, \tag{4.11}$$

then $\lim_{i\to\infty} \mathbf{x}_i = \mathbf{x}$ in $C_\beta([a,\infty)_{\mathbb{T}}, \mathbb{R}^n)$ on $[a,\infty)_{\mathbb{T}}$, with the convergence being uniform on compact subsets of $[a,\infty)_{\mathbb{T}}$.

Proof. Since $\mathbf{k}$ is rd-continuous with respect to the integrated variable s, (4.11) is well defined. Let $L > 0$ be the constant defined in (4.8) and let $\beta := L\gamma$, where $\gamma > 1$. Consider the following equivalent formulation of (1.2), namely

$$\mathbf{x}(t) = \left(\mathbf{f}(t) + \int_a^t \mathbf{k}(t,s,\mathbf{0})\ \Delta s \right) + \int_a^t \big(\mathbf{k}(t,s,\mathbf{x}^\sigma(s)) - \mathbf{k}(t,s,\mathbf{0}) \big)\ \Delta s, \quad t \in [a,\infty)_{\mathbb{T}}. \tag{4.12}$$

We will show that (4.12) has a unique solution and thus (1.2) must also have a unique solution.

With this in mind, consider the complete metric space $(C_\beta([a,\infty)_{\mathbb{T}}; \mathbb{R}^n), d_\beta^\infty)$ and let $\mathbf{F}$ be defined by

$$[\mathbf{F}\mathbf{x}](t) := \left(\mathbf{f}(t) + \int_a^t \mathbf{k}(t,s,\mathbf{0})\ \Delta s \right) + \int_a^t \big(\mathbf{k}(t,s,\mathbf{x}^\sigma(s)) - \mathbf{k}(t,s,\mathbf{0}) \big)\ \Delta s, \quad t \in [a,\infty)_{\mathbb{T}}. \tag{4.13}$$

Fixed-points of $\mathbf{F}$ will be solutions to (4.12). Thus, we want to prove that there exists a unique $\mathbf{x}$ such that $\mathbf{F}\mathbf{x} = \mathbf{x}$. To do this we show that the conditions of Banach's theorem are satisfied.

We show that $\mathbf{F} : C_\beta([a,\infty)_{\mathbb{T}}; \mathbb{R}^n) \to C_\beta([a,\infty)_{\mathbb{T}}; \mathbb{R}^n)$. Let $\mathbf{x} \in C_\beta([a,\infty)_{\mathbb{T}}; \mathbb{R}^n)$.

Taking norms in (4.13) we obtain

$$\begin{aligned}
\|\mathbf{F}\mathbf{x}\|_\beta^\infty &= \sup_{t\in[a,\infty)_\mathbb{T}} \frac{1}{e_\beta(t,a)} \left\| \mathbf{f}(t) + \int_a^t \mathbf{k}(t,s,\mathbf{0})\,\Delta s \right. \\
&\quad \left. + \int_a^t \mathbf{k}(t,s,\mathbf{x}^\sigma(s)) - \mathbf{k}(t,s,\mathbf{0})\,\Delta s \right\| \\
&\le m + \sup_{t\in[a,\infty)_\mathbb{T}} \frac{1}{e_\beta(t,a)} \int_a^t \|\mathbf{k}(t,s,\mathbf{x}^\sigma(s)) - \mathbf{k}(t,s,\mathbf{0})\|\,\Delta s \\
&\le m + \sup_{t\in[a,\infty)_\mathbb{T}} \frac{1}{e_\beta(t,a)} L \int_a^t \|\mathbf{x}^\sigma(s)\|\,\Delta s, \quad \text{from (4.3)} \\
&= m + \sup_{t\in[a,\infty)_\mathbb{T}} \frac{1}{e_\beta(t,a)} L \int_a^t e_\beta^\sigma(s,a) \frac{\|\mathbf{x}^\sigma(s)\|}{e_\beta^\sigma(s,a)}\,\Delta s \\
&= m + \sup_{t\in[a,\infty)_\mathbb{T}} \frac{1}{e_\beta(t,a)} L \int_a^t [1+\mu(s)\beta] e_\beta(s,a) \frac{\|\mathbf{x}^\sigma(s)\|}{e_\beta^\sigma(s,a)}\,\Delta s, \\
&\qquad \text{c.f. Theorem 2.6 (iv)} \\
&\le m + L\left(1 + \sup_{t\in[a,\infty)_\mathbb{T}} \mu(t)\beta\right) \|\mathbf{x}\|_\beta^\infty \sup_{t\in[a,\infty)_\mathbb{T}} \frac{1}{e_\beta(t,a)} \left[\frac{e_\beta(t,a)-1}{\beta}\right] \\
&< \infty.
\end{aligned}$$

Hence we see that $\mathbf{F} : C_\beta([a,\infty)_\mathbb{T};\mathbb{R}^n) \to C_\beta([a,\infty)_\mathbb{T};\mathbb{R}^n)$.

We show that $\mathbf{F}$ is a contractive map and Banach's fixed point theorem will then apply. For any $\mathbf{u}, \mathbf{v} \in C_\beta([a,\infty)_\mathbb{T};\mathbb{R}^n)$, in a similar fashion to the above working, we obtain

$$\begin{aligned}
d_\beta^\infty(\mathbf{F}\mathbf{u}, \mathbf{F}\mathbf{v}) &\le \sup_{t\in[a,\infty)_\mathbb{T}} \frac{1}{e_\beta(t,a)} \int_a^t \|\mathbf{k}(t,s,\mathbf{u}^\sigma(s)) - \mathbf{k}(t,s,\mathbf{v}^\sigma(s))\|\,\Delta s \\
&\le L\left(1 + \sup_{t\in[a,\infty)_\mathbb{T}} \mu(t)\beta\right) d_\beta(\mathbf{u},\mathbf{v}) \sup_{t\in[a,\infty)_\mathbb{T}} \frac{1}{e_\beta(t,a)} \int_a^t e_\beta(s,a)\,\Delta s \\
&= L\left(1 + \sup_{t\in[a,\infty)_\mathbb{T}} \mu(t)\beta\right) d_\beta(\mathbf{u},\mathbf{v}) \sup_{t\in[a,\infty)_\mathbb{T}} \frac{1}{e_\beta(t,a)} \left(\frac{e_\beta(t,a)-1}{\beta}\right) \\
&\le \left(\frac{1}{\gamma} + L \sup_{t\in[a,\infty)_\mathbb{T}} \mu(t)\right) d_\beta(\mathbf{u},\mathbf{v}) \sup_{t\in[a,\infty)_\mathbb{T}} \left[1 - \frac{1}{e_\beta(t,a)}\right], \\
&\qquad \text{since } \beta = L\gamma, \\
&< d_\beta^\infty(\mathbf{u},\mathbf{v}).
\end{aligned}$$

We see that $\mathbf{F}$ is a contractive map and Banach's fixed point theorem applies, yielding the existence of a unique fixed-point $\mathbf{x}$ of $\mathbf{F}$. ■

Corollary 4.5. If the conditions of Theorem 4.2 (or Theorem 4.4) hold with (4.4) (or (4.10)) replaced by

$$\mathbf{f} \in C_\beta([a,\infty)_{\mathbb{T}}; \mathbb{R}^n); \quad \text{and} \quad \mathbf{k}(t,s,\mathbf{0}) = \mathbf{0}, \quad \forall (t,s) \in [a,\infty)^2_{\mathbb{T}}, \tag{4.14}$$

then the conclusions of Theorem 4.2 (or Theorem 4.4) hold.

Proof. It is easy to see that if (4.14) holds, then (4.4) (or (4.10)) holds and the result follows from Theorem 4.2 (or Theorem 4.4). ■

Remark 4.6. We note that the theorems of this section not only provide existence, uniqueness and approximation results for solutions to (1.1) and (1.2), but they also provide some information about the behaviour of solutions $\mathbf{x}$ on the entire interval $[a,\infty)_{\mathbb{T}}$. This is due to our solutions $\mathbf{x}$ being in the space $C_\beta([a,\infty)_{\mathbb{T}}; \mathbb{R}^n)$, so that

$$\|\mathbf{x}(t)\| \le K e_\beta(t,a), \quad t \in [a,\infty)_{\mathbb{T}};$$

for some constant K. This kind of information is usually useful in applications [11, p.37].

4.2. Applications to Initial Value Problems

We now apply our new results of the previous section to initial value problems for dynamic equations on time scales, where the solutions $\mathbf{x}$ have unbounded domains $[a,\infty)_{\mathbb{T}}$. In particular, we will obtain existence, uniqueness, approximation and growth behaviour of solutions to dynamic equations on the time scale half-line $[a,\infty)_{\mathbb{T}}$.

Consider either of the dynamic equations

$$\mathbf{x}^\Delta = \mathbf{g}(t,\mathbf{x}), \quad t \in I_{\mathbb{T}} := [a,\infty)_{\mathbb{T}}; \tag{4.15}$$

$$\mathbf{x}^\Delta = \mathcal{M}(t)\mathbf{x} + \mathbf{h}(t,\mathbf{x}), \quad t \in I_{\mathbb{T}}; \tag{4.16}$$

subject to the initial condition

$$\mathbf{x}(a) = \mathbf{A}. \tag{4.17}$$

Above, $\mathbf{g} : [a,\infty)_{\mathbb{T}} \times \mathbb{R}^n \to \mathbb{R}^n$ and $\mathbf{h} : [a,\infty)_{\mathbb{T}} \times \mathbb{R}^n \to \mathbb{R}^n$ may be nonlinear functions, $n \ge 1$; $\mathbf{x}^\Delta$ is the generalised delta derivative of $\mathbf{x}$; $\mathcal{M} : [a,\infty)_{\mathbb{T}} \to \mathbb{R}^{n\times n}$ is a matrix-valued function and $\mathbf{A}$ is a given constant in $\mathbb{R}^n$. The above problems are known as initial value problems (IVPs) on time scales.

We remark that when $\mathbb{T} = \mathbb{R}$, then (4.15) and (4.16) become, respectively, the well-known ordinary differential equations

$$\mathbf{x}' = \mathbf{g}(t,\mathbf{x}), \quad t \in I := [a,\infty);$$
$$\mathbf{x}' = \mathcal{M}(t)\mathbf{x} + \mathbf{h}(t,\mathbf{x}), \quad t \in I;$$

whereas if $\mathbb{T} = \mathbb{Z}$, then (4.15) and (4.16) become, respectively, the familiar difference equations

$$\Delta\mathbf{x} = \mathbf{g}(t,\mathbf{x}), \quad t \in I := [a,\infty)_{\mathbb{Z}};$$
$$\Delta\mathbf{x} = \mathcal{M}(t)\mathbf{x} + \mathbf{h}(t,\mathbf{x}), \quad t \in I_{\mathbb{Z}};$$

where the Δ operator is the usual forward difference, that is, $\Delta p(t) := p(t+1) - p(t)$.

Our first result concerns the nonlinear problem (4.15), (4.17).

Theorem 4.7. Consider the dynamic IVP (4.15), (4.17). Let $\mathbf{g} : [a,\infty)_{\mathbb{T}} \times \mathbb{R}^n \to \mathbb{R}^n$ be rd-continuous and let $\gamma > 1$, L and β be positive constants with $\beta := L\gamma$. If

$$\|\mathbf{g}(s,\mathbf{p}) - \mathbf{g}(s,\mathbf{q})\| \le L\|\mathbf{p}-\mathbf{q}\|, \forall s \in [a,\infty)_{\mathbb{T}},\ (\mathbf{p},\mathbf{q}) \in \mathbb{R}^{2n}; \tag{4.18}$$

$$\sup_{t\in[a,\infty)_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \|\mathbf{A} + \int_a^t \mathbf{g}(s,\mathbf{0})\,\Delta s\| < \infty, \tag{4.19}$$

then the dynamic IVP (4.15), (4.17) has a unique solution $\mathbf{x}$. In addition, if a sequence of functions $\{\mathbf{x}_i\}$ is defined inductively by choosing any $\mathbf{x}_0 \in C_\beta([a,\infty)_{\mathbb{T}}; \mathbb{R}^n)$ and setting

$$\mathbf{x}_{i+1}(t) := \mathbf{A} + \int_a^t \mathbf{g}(s,\mathbf{x}_i(s))\,\Delta s, \quad t \in [a,\infty)_{\mathbb{T}}, \tag{4.20}$$

then $\lim_{i\to\infty} \mathbf{x}_i = \mathbf{x}$ on $[a,\infty)_{\mathbb{T}}$ with the convergence being uniform on compact subsets of $[a,\infty)_{\mathbb{T}}$. Furthermore, $\mathbf{x}^\Delta \in C_{\mathrm{rd}}([a,\infty)_{\mathbb{T}}; \mathbb{R}^n)$.

Proof. Since $\mathbf{g}$ is an rd-continuous function, (4.20) is well defined. Let $L > 0$ be the constant defined in (4.18) and let $\beta := L\gamma$, where $\gamma > 1$. Consider the complete metric space $(C_\beta([a,\infty)_{\mathbb{T}}; \mathbb{R}^n), d_\beta^\infty)$ and consider the equivalent reformulation of (4.15), (4.17), namely

$$\mathbf{x}(t) = \mathbf{A} + \int_a^t \mathbf{g}(s,\mathbf{x}(s))\,\Delta s, \quad t \in [a,\infty)_{\mathbb{T}}. \tag{4.21}$$

See that the integral equation (4.21) satisfies all of the conditions of Theorem 4.2 and so (4.21) has a unique solution. Since (4.21) has a solution if and only if (4.15) and (4.17) have a solution we are finished, with the claim that $\mathbf{x}^\Delta \in C_{\mathrm{rd}}([a,\infty)_{\mathbb{T}}; \mathbb{R}^n)$ following from [35, Lemma 2.3]. ■

Hilger's existence and uniqueness result [20, Theorem 5.7] for (4.15), (4.17) does not employ a condition of type (4.19). On the other hand, in view of Remark 4.6 our result provides more information about solutions than Hilger's result does. In particular, our result gives a nice rate of growth of solutions on the entire interval $[a,\infty)_{\mathbb{T}}$ and we believe that this idea has many potential applications concerning boundedness, asymptotic behaviour and stability of solutions.

Example 4.8. We claim that the scalar dynamic equation

$$x^\Delta = t + \left[x^2 + 10\right]^{1/3}, \quad t \in [a,\infty)_{\mathbb{T}}; \tag{4.22}$$

subject to the initial condition $x(a) = A$, has a unique solution on $[a,\infty)_{\mathbb{T}}$.

Proof. Using a similar argument to that in the proof of Example 4.3 it is not difficult to show that all of the conditions of Theorem 4.7 hold. ■

Consider the dynamic IVP (4.16), (4.17). If $\mathcal{M}$ is regressive in the matrix sense, that is $I + \mu(t)\mathcal{M}(t)$ is invertible for all $t \in \mathbb{T}$, then we can solve the matrix dynamic equation

$$\mathcal{X}^{\Delta} = \mathcal{M}(t)\mathcal{X}, \tag{4.23}$$

for any initial condition and, furthermore, $\det \mathcal{X} \neq 0$. For more detail we refer the reader to [6, Chap. 5].

The following result concerns the semilinear problem (4.16), (4.17).

Theorem 4.9. Consider the dynamic IVP (4.16), (4.17). Let $\mathbf{h} : [a,\infty)_{\mathbb{T}} \times \mathbb{R}^n \to \mathbb{R}^n$ be rd-continuous; let $\mathcal{M} : [a,\infty)_{\mathbb{T}} \to \mathbb{R}^{n\times n}$ be rd-continuous with $\mathcal{M} \in \mathcal{R}$; and let $\gamma > 1$, L and β be positive constants with $\beta := L\gamma$. If

$$\begin{aligned}&\|\mathcal{X}(t)\mathcal{X}^{-1}(\sigma(s))\left[\mathbf{h}(s,\mathbf{p}) - \mathbf{h}(s,\mathbf{q})\right]\| \le L\|\mathbf{p}-\mathbf{q}\|,\\ &\qquad \forall (t,s) \in [a,\infty)^2_{\mathbb{T}},\ (\mathbf{p},\mathbf{q}) \in \mathbb{R}^{2n};\end{aligned} \tag{4.24}$$

$$\sup_{t\in[a,\infty)_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \left\| \mathcal{X}(t)\mathcal{X}^{-1}(a)\mathbf{A} + \int_a^t \mathcal{X}(t)\mathcal{X}^{-1}(\sigma(s))\mathbf{h}(s,\mathbf{0})\,\Delta s \right\| < \infty, \tag{4.25}$$

then the dynamic IVP (4.16), (4.17) has a unique solution $\mathbf{x}$. In addition, if a sequence of functions $\{\mathbf{x}_i\}$ is defined inductively by choosing any $\mathbf{x}_0 \in C_\beta([a,\infty)_{\mathbb{T}}; \mathbb{R}^n)$ and setting

$$\mathbf{x}_{i+1}(t) = \mathcal{X}(t)\mathcal{X}^{-1}(a)\mathbf{A} + \int_a^t \mathcal{X}(t)\mathcal{X}^{-1}(\sigma(s))\mathbf{h}(s,\mathbf{x}_i(s))\,\Delta s, \quad t \in [a,\infty)_{\mathbb{T}}, \tag{4.26}$$

then $\lim_{i\to\infty} \mathbf{x}_i = \mathbf{x}$ on $[a,\infty)_{\mathbb{T}}$, with uniform convergence on compact subsets of $[a,\infty)_{\mathbb{T}}$. Furthermore, $\mathbf{x}^{\Delta} \in C_{\mathrm{rd}}([a,\infty)_{\mathbb{T}}; \mathbb{R}^n)$.

Proof. Since $\mathbf{h}$ is an rd-continuous function, (4.26) is well defined. Let $L > 0$ be the constant defined in (4.24) and let $\beta := L\gamma$, where $\gamma > 1$. Consider the complete metric space $(C_\beta([a,\infty)_{\mathbb{T}}; \mathbb{R}^n), d_\beta^\infty)$ and consider the equivalent reformulation of (4.16), (4.17), namely

$$\mathbf{x}(t) = \mathcal{X}(t)\mathcal{X}^{-1}(a)\mathbf{A} + \int_a^t \mathcal{X}(t)\mathcal{X}^{-1}(\sigma(s))\mathbf{h}(s,\mathbf{x}(s))\,\Delta s, \quad t \in [a,\infty)_{\mathbb{T}}, \tag{4.27}$$

where $\mathcal{X}$ is derived from (4.23). See that the delta integral equation (4.27) satisfies all of the conditions of Theorem 4.2 and so (4.27) has a unique solution. The claim that $\mathbf{x}^{\Delta} \in C_{\mathrm{rd}}([a,\infty)_{\mathbb{T}}; \mathbb{R}^n)$ follows from [35, Lemma 3.3]. ∎

5. Nonlinear Problems on Compact Intervals

In this section we obtain some new results concerning the existence, uniqueness and approximation of solutions to the integral equations (1.1) and (1.2) with

$$I_{\mathbb{T}} = [a,b]_{\mathbb{T}} := [a,b] \cap \mathbb{T}, \quad a, b \in \mathbb{T},$$

a compact interval in $\mathbb{T}$.

5.1. Contractive Mapping Approach

Motivated by the results for the linear case (3.1) in Theorem 3.4, we now discuss the nonlinear case.

Theorem 5.1. Consider the integral equation (1.1) with $I_{\mathbb{T}} := [a,b]_{\mathbb{T}}$. Let $\mathbf{k} : [a,b]^2_{\mathbb{T}} \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous in its first and third variables and rd-continuous in its second variable, let $\mathbf{f} : [a,b]_{\mathbb{T}} \to \mathbb{R}^n$ be continuous, and let L be a positive constant. If

$$\|\mathbf{k}(t,s,\mathbf{p}) - \mathbf{k}(t,s,\mathbf{q})\| \le L\|\mathbf{p}-\mathbf{q}\|, \quad \forall (t,s) \in [a,b]^2_{\mathbb{T}},\ (\mathbf{p},\mathbf{q}) \in \mathbb{R}^{2n}, \tag{5.1}$$

then (1.1) has a unique solution. In addition, if a sequence of functions $\{\mathbf{x}_i\}$ is defined inductively by choosing any $\mathbf{x}_0 \in C([a,b]_{\mathbb{T}}; \mathbb{R}^n)$ and setting

$$\mathbf{x}_{i+1}(t) := \mathbf{f}(t) + \int_a^t \mathbf{k}(t,s,\mathbf{x}_i(s))\,\Delta s, \quad t \in [a,b]_{\mathbb{T}}, \tag{5.2}$$

then the sequence $\{\mathbf{x}_i\}$ converges uniformly on $[a,b]_{\mathbb{T}}$ to the unique solution $\mathbf{x}$ of (1.1).

Proof. Since $\mathbf{k}$ is rd-continuous with respect to the integrated variable s, (5.2) is well defined. Let $L > 0$ be the constant defined in (5.1) and let $\beta := L\gamma$, where $\gamma > 1$ is an arbitrary constant. Consider the complete metric space $(C([a,b]_{\mathbb{T}}; \mathbb{R}^n), d_\beta)$ and let

$$\mathbf{F} : C([a,b]_{\mathbb{T}}; \mathbb{R}^n) \to C([a,b]_{\mathbb{T}}; \mathbb{R}^n)$$

be defined by

$$[\mathbf{F}\mathbf{x}](t) := \mathbf{f}(t) + \int_a^t \mathbf{k}(t,s,\mathbf{x}(s))\,\Delta s, \quad t \in [a,b]_{\mathbb{T}}. \tag{5.3}$$

It is easy to see that fixed-points of $\mathbf{F}$ will be solutions to (1.1). Thus, we want to prove that there exists a unique $\mathbf{x}$ such that $\mathbf{F}\mathbf{x} = \mathbf{x}$. To do this, we show that $\mathbf{F}$ is a contractive map with contraction constant $\alpha = 1/\gamma < 1$ and Banach's fixed point theorem will then apply. For any $\mathbf{u}, \mathbf{v} \in C([a,b]_{\mathbb{T}}; \mathbb{R}^n)$, consider

$$\begin{aligned}
d_\beta(\mathbf{F}\mathbf{u}, \mathbf{F}\mathbf{v}) &:= \sup_{t\in[a,b]_{\mathbb{T}}} \frac{\|[\mathbf{F}\mathbf{u}](t) - [\mathbf{F}\mathbf{v}](t)\|}{e_\beta(t,a)} \\
&\le \sup_{t\in[a,b]_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \int_a^t \|\mathbf{k}(t,s,\mathbf{u}(s)) - \mathbf{k}(t,s,\mathbf{v}(s))\|\,\Delta s \\
&\le \sup_{t\in[a,b]_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \int_a^t L\|\mathbf{u}(s) - \mathbf{v}(s)\|\,\Delta s, \quad \text{from (5.1)} \\
&< \frac{d_\beta(\mathbf{u},\mathbf{v})}{\gamma},
\end{aligned}$$

where the argument parallels that of the proof of Theorem 3.4. As $\gamma > 1$ we see that $\mathbf{F}$ is a contractive map and Banach's fixed point theorem applies, yielding the existence of a unique fixed-point $\mathbf{x}$ of $\mathbf{F}$. In addition, from Banach's theorem, the sequence $\{\mathbf{x}_i\}$ defined in (4.26) converges uniformly in the norm $\|\cdot\|_\beta$ and thus the sequence $\{\mathbf{x}_i\}$ converges uniformly in the sup-norm $\|\cdot\|_0$ to that fixed-point $\mathbf{x}$. ■

We now present a result on the dependence of solutions to (1.1) on initial values. Essentially, this theorem gives conditions under which two solutions to (1.1) will not stray too far apart.

Theorem 5.2. Consider the dynamic integral equation (1.1) with $I_{\mathbb{T}} := [a, b]_{\mathbb{T}}$ and

$$\mathbf{y}(t) = \mathbf{h}(t) + \int_a^t \mathbf{k}(t, s, \mathbf{y}(s))\,\Delta s, \quad t \in [a, b]_{\mathbb{T}}. \tag{5.4}$$

Let (1.1) and (5.4) satisfy the conditions of Theorem 5.1. Then the respective solutions of (1.1) and (5.4) satisfy

$$\|\mathbf{x}(t) - \mathbf{y}(t)\| \le \delta e_L(t, a), \quad t \in [a, b]_{\mathbb{T}};$$

where $\delta := \max\limits_{t\in[a,b]_{\mathbb{T}}} \|\mathbf{f}(t) - \mathbf{h}(t)\|$.

Proof. Consider

$$\begin{aligned}\|\mathbf{x}(t) - \mathbf{y}(t)\| &\le \|\mathbf{f}(t) - \mathbf{h}(\mathbf{t})\| + \int_a^t \|\mathbf{k}(t, s, \mathbf{x}(s)) - \mathbf{k}(t, s, \mathbf{y}(s))\|\,\Delta s, \quad t \in [a, b]_{\mathbb{T}}\\ &\le \delta + L\int_a^t \|\mathbf{x}(s) - \mathbf{y}(s)\|\,\Delta s\end{aligned}$$

and the result now follows from Gronwall's lemma for time scales, Lemma 3.7. ■

Our attention now shifts to the integral equation (1.2). Although (1.2) appears to be almost the same as (1.1), there are genuine differences between the two in terms of mathematical theory, as the following theorem illustrates.

Theorem 5.3. Consider the dynamic integral equation (1.2) with $I_{\mathbb{T}} := [a, b]_{\mathbb{T}}$. Let $\mathbf{k} : [a, b]_{\mathbb{T}}^2 \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous in its first and third variables and rd-continuous in its second variable, let $\mathbf{f} : [a, b]_{\mathbb{T}} \to \mathbb{R}^n$ be continuous and let $L > 0$ be a constant. If

$$\|\mathbf{k}(t, s, \mathbf{p}) - \mathbf{k}(t, s, \mathbf{q})\| \le L\|\mathbf{p} - \mathbf{q}\|, \forall (t, s) \in [a, b]_{\mathbb{T}}^2,\ (\mathbf{p}, \mathbf{q}) \in \mathbb{R}^{2n}; \tag{5.5}$$

$$L \sup_{t\in[a,b]_{\mathbb{T}}} \mu(t) < 1, \tag{5.6}$$

then the dynamic IVP (1.2) has a unique solution. In addition, if a sequence of functions $\{\mathbf{x}_i\}$ is defined inductively by choosing any $\mathbf{x}_0 \in C([a, b]_{\mathbb{T}}; \mathbb{R}^n)$ and setting

$$\mathbf{x}_{i+1}(t) = \mathbf{f}(t) + \int_a^t \mathbf{k}(t, s, \mathbf{x}_i^\sigma(s))\,\Delta s, \quad t \in [a, b]_{\mathbb{T}}, \tag{5.7}$$

then the sequence $\{\mathbf{x}_i\}$ converges uniformly on $[a, b]_{\mathbb{T}}$ to the unique solution $\mathbf{x}$ of (1.2).

Proof. Since $\mathbf{k}$ is rd-continuous with respect to the integrated variable s, (5.7) is well defined. Let $L > 0$ be the constant defined in (5.5) and let $\beta := L\gamma$, where $\gamma > 1$ is a constant chosen such that $L|\mu|_0 = 1 - 1/\gamma$. Consider the complete metric space $(C([a, b]_{\mathbb{T}}; \mathbb{R}^n), d_\beta)$. Let

$$\mathbf{F} : C([a, b]_{\mathbb{T}}; \mathbb{R}^n) \to C([a, b]_{\mathbb{T}}; \mathbb{R}^n)$$

be defined by

$$[\mathbf{F}\mathbf{x}](t) := \mathbf{f}(t) + \int_a^t \mathbf{k}(t, s, \mathbf{x}^\sigma(s))\, \Delta s, \quad t \in [a, b]_{\mathbb{T}}. \tag{5.8}$$

Fixed-points of $\mathbf{F}$ will be solutions to the dynamic IVP (1.2). Thus, we want to prove that there exists a unique $\mathbf{x}$ such that $\mathbf{F}\mathbf{x} = \mathbf{x}$. To do this, we show that $\mathbf{F}$ is a contractive map and Banach's fixed point theorem will then apply. For $\mathbf{u}, \mathbf{v} \in C([a, b]_{\mathbb{T}}; \mathbb{R}^n)$, consider

$$\begin{aligned}
d_\beta(\mathbf{F}\mathbf{u}, \mathbf{F}\mathbf{v}) &\le \sup_{t\in[a,b]_{\mathbb{T}}} \frac{1}{e_\beta(t, a)} \int_a^t \|\mathbf{f}(t, s, \mathbf{u}^\sigma(s)) - \mathbf{f}(t, s, \mathbf{v}^\sigma(s))\|\, \Delta s \\
&\le \sup_{t\in[a,b]_{\mathbb{T}}} \frac{1}{e_\beta(t, a)} \int_a^t L\|\mathbf{u}^\sigma(s) - \mathbf{v}^\sigma(s)\|\, \Delta s, \quad \text{from (5.5)} \\
&= L \sup_{t\in[a,b]_{\mathbb{T}}} \frac{1}{e_\beta(t, a)} \int_a^t e_\beta(\sigma(s), a) \frac{\|\mathbf{u}^\sigma(s) - \mathbf{v}^\sigma(s)\|}{e_\beta(\sigma(s), a)}\, \Delta s \\
&\le L d_\beta(\mathbf{u}, \mathbf{v}) \sup_{t\in[a,b]_{\mathbb{T}}} \frac{1}{e_\beta(t, a)} \int_a^t e_\beta(\sigma(s), a)\, \Delta s \\
&= L d_\beta(\mathbf{u}, \mathbf{v}) \sup_{t\in[a,b]_{\mathbb{T}}} \frac{1}{e_\beta(t, a)} \int_a^t (1 + \beta\mu(s)) e_\beta(s, a)\, \Delta s, \\
&\qquad \text{c.f. Theorem 2.6 (iv);} \\
&\le L(1 + \beta|\mu|_0) d_\beta(\mathbf{u}, \mathbf{v}) \sup_{t\in[a,b]_{\mathbb{T}}} \frac{1}{e_\beta(t, a)} \int_a^t e_\beta(s, a)\, \Delta s \\
&= L(1 + \beta|\mu|_0) d_\beta(\mathbf{u}, \mathbf{v}) \sup_{t\in[a,b]_{\mathbb{T}}} \frac{1}{e_\beta(t, a)} \left(\frac{e_\beta(t, a) - 1}{\beta}\right) \\
&= \left(\frac{1}{\gamma} + L|\mu|_0\right) d_\beta(\mathbf{u}, \mathbf{v}) \sup_{t\in[a,b]_{\mathbb{T}}} \left[1 - \frac{1}{e_\beta(t, a)}\right], \quad \text{since } \beta = L\gamma, \\
&= d_\beta(\mathbf{u}, \mathbf{v}) \left[1 - \frac{1}{e_\beta(b, a)}\right] < d_\beta(\mathbf{u}, \mathbf{v}).
\end{aligned}$$

We see that $\mathbf{F}$ is a contractive map and Banach's fixed point theorem applies, yielding the existence of a unique fixed-point $\mathbf{x}$ of $\mathbf{F}$. In addition, from Banach's theorem, the sequence $\{\mathbf{x}_i\}$ defined in (4.11) converges uniformly in the norm $\|\cdot\|_\beta$ and thus the sequence $\{\mathbf{x}_i\}$ converges uniformly in the sup-norm $\|\cdot\|_0$ to that fixed-point $\mathbf{x}$. This completes the proof. ■

Theorem 5.3 gives existence and uniqueness of solutions to (1.2) for those time scales where the points are not spaced “too far apart”, whereas Theorem 5.1 does not involve any such restriction.

As an application of of Theorem 5.3, we present the following example.

Example 5.4. Consider the scalar integral equation

$$x(t) = A + B(t-a) + \int_a^t [t - \sigma(s)] g(s, x^\sigma(s))\, \Delta s, \quad t \in [a,b]_{\mathbb{T}}, \tag{5.9}$$

where A and B are constants and $g : [a,b]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ is rd-continuous. We claim that if g satisfies the Lipschitz condition (4.18) on $[a,b]_{\mathbb{T}} \times \mathbb{R}$ and (5.6) holds, then (5.9) has a unique solution.

Proof. It is not difficult to show that all of the conditions of Theorem 5.3 all hold and the claim follows from there. ■

It is interesting to note that (5.9) arises in the analysis of the IVP

$$x^{\Delta\Delta} = g(t, x^\sigma), \quad t \in [a,b]_{\mathbb{T}}^{\kappa^2} := [a,b]_{\mathbb{T}}^{\kappa\kappa}; \tag{5.10}$$

$$x(a) = A, \quad x^\Delta(a) = B \tag{5.11}$$

as the two problems are equivalent formulations. Thus, Example 5.4 contains a new existence and uniqueness result for solutions to the above IVP. This appears to be interesting because (5.10), (5.11) does not reduce to a “nice” system of first-order IVPs and so the approach in, say, [35, Sec. 5] does not apply. To see this, introduce the standard substitutions $u = x$ and $v = x^\Delta$ so (5.10) becomes

$$u^\Delta = v, \quad v^\Delta = f(t, u^\sigma). \tag{5.12}$$

Notice that one of the above equations features a σ, while the other does not. This kind of “mixed” situation would seem to cause problems in the general analysis of (5.12).

In view of Remark 3.3, our approach in the proof of Theorems 5.1, 5.3 and 3.4 can be used to evaluate the rate of convergence of a sequence of Picard iterates to our solutions in the following way. Let $\mathbf{x}$ be the unique solution and choose $\mathbf{x}_0 \in C([a,b]_{\mathbb{T}}; \mathbb{R}^n)$. Then for $\beta := L\gamma$ with $\gamma > 1$ we have, from (3.2),

$$d_\beta(\mathbf{F}^i \mathbf{x}_0, \mathbf{x}) \le \frac{\gamma^{-i}}{1-\gamma^{-1}} d_\beta(\mathbf{x}_0, \mathbf{F}\mathbf{x}_0)$$

and so

$$\|\mathbf{F}^i \mathbf{x}_0 - \mathbf{x}\|_0 \le e_\beta(b,a) \frac{\gamma^{-i}}{1-\gamma^{-1}} \|\mathbf{x}_0 - \mathbf{F}\mathbf{x}_0\|_0. \tag{5.13}$$

If we choose $\gamma := i/L[b-a]$, then we obtain a nice evaluation of the rate of convergence in (5.13), namely

$$\|\mathbf{F}^i\mathbf{x}_0 - \mathbf{x}\|_0 \le e_{\frac{i}{b-a}}(b, a)\left(\frac{L[b-a]}{i}\right)^i \frac{i}{i - L[b-a]}\|\mathbf{x}_0 - \mathbf{F}\mathbf{x}_0\|_0.$$

5.2. Topological Degree Approach

Our focus now shifts from concerns of existence, uniqueness and approximation of solutions, to the lone question of existence of solutions to the integral equations (1.1) and (1.2).

Our interest is in placing very mild assumptions of **k** so that our problems admit at least one solution. This situation may arise when **k** does not satisfy the conditions of previous sections. Fixed-point theorems based on the ideas of topological degree [26, Chap.4] will be the main tools that we use. In particular, Schäfer's fixed point theorem will be employed, rather than Banach's fixed point theorem.

On one hand, Schäfer's theorem is very wide ranging, but on the other hand, it does not provide as much information as Banach's theorem. No explicit knowledge of topological degree theory is needed to verify the conditions of Schäfer's theorem.

Theorem 5.5. (Schäfer, [26, Theorem 4.4.12]) Let X be a normed space with $H : X \to X$ a compact mapping. If the set

$$S := \{u \in X : \ u = \lambda H u \quad \text{for some } \lambda \in [0, 1)\}$$

is bounded, then H has at least one fixed-point.

Recall that a mapping between normed spaces is compact if it is continuous and carries bounded sets into relatively compact sets.

Lemma 5.6. Consider the Banach space $(C([a, b]_{\mathbb{T}}; \mathbb{R}^n), \|\cdot\|_0)$ and consider the map $\mathbf{F} : C([a, b]_{\mathbb{T}}; \mathbb{R}^n) \to C([a, b]_{\mathbb{T}}; \mathbb{R}^n)$ defined by

$$[\mathbf{F}\mathbf{x}](t) := \mathbf{f}(t) + \int_a^t \mathbf{k}(t, s, \mathbf{x}(s))\, \Delta s, \quad t \in [a, b]_{\mathbb{T}}. \tag{5.14}$$

If $\mathbf{k} : [a, b]_{\mathbb{T}}^2 \times \mathbb{R}^n \to \mathbb{R}^n$ and $\mathbf{f} : [a, b]_{\mathbb{T}} \to \mathbb{R}^n$ are continuous, then $\mathbf{F}$ is a compact map.

Proof. Our argument follows that of [14, pp.52–53] and so is only sketched. We show that the conditions of the Arzela–Ascoli theorem [28, p.3] are satisfied. That is, given $\{\mathbf{x}_i\}$ with $\|\mathbf{x}_i\|_0 \le r$ we show that the sequence $\mathbf{v}_i := \mathbf{F}(\mathbf{x}_i)$ is bounded and equicontinuous.

(a): We claim that $\{\mathbf{v}_i\}$ is bounded. Let

$$M := \sup\{\|\mathbf{k}(t, s, \mathbf{p})\| : (t, s) \in [a, b]_{\mathbb{T}}^2, \ \|\mathbf{p}\| \le r\} < \infty.$$

We have

$$\begin{aligned}\|\mathbf{v}_i\|_0 &:= \sup_{t\in[a,b]_{\mathbb{T}}} \|\mathbf{v}_i(t)\| \\ &\le \sup_{t\in[a,b]_{\mathbb{T}}} \left(\int_a^t \|\mathbf{k}(t,s,\mathbf{x}_i(s))\| \,\Delta s + \|\mathbf{f}(t)\| \right) \\ &\le [b-a]M + \sup_{t\in[a,b]_{\mathbb{T}}} \|\mathbf{f}(t)\|\end{aligned}$$

and hence $\{\mathbf{v}_i\}$ is bounded.

(b): We claim that $\{\mathbf{v}_i\}$ is equicontinuous. Let $\varepsilon > 0$ be given and for $t_1, t_2 \in [a,b]_{\mathbb{T}}$ consider

$$\begin{aligned}\|\mathbf{v}_i(t_1) - \mathbf{v}_i(t_2)\| &\le \int_{\min\{t_1,t_2\}}^{\max\{t_1,t_2\}} \|\mathbf{k}(t,s,\mathbf{x}_i(s)\| \,\Delta s \\ &\le M|t_1 - t_2| \\ &< \varepsilon, \quad \text{whenever } |t_1 - t_2| < \delta(\varepsilon) := \varepsilon/M\end{aligned}$$

and hence $\{\mathbf{v}_i\}$ is equicontinuous.

The result now follows from the Arzela–Ascoli theorem [28, p.3]. ■

The following existence theorem permits linear growth of $\|\mathbf{k}(t,s,\mathbf{p})\|$ in $\|\mathbf{p}\|$.

Theorem 5.7. Consider the dynamic integral equation (1.1) with $I_{\mathbb{T}} := [a,b]_{\mathbb{T}}$. Let $\mathbf{k} : [a,b]_{\mathbb{T}}^2 \times \mathbb{R}^n \to \mathbb{R}^n$ and $\mathbf{f} : [a,b]_{\mathbb{T}} \to \mathbb{R}^n$ be continuous and $L > 0$, $N \ge 0$ be constants. If

$$\|\mathbf{k}(t,s,\mathbf{p})\| \le L\|\mathbf{p}\| + N, \quad \forall (t,s) \in [a,b]_{\mathbb{T}}^2,\ \mathbf{p} \in \mathbb{R}^n, \tag{5.15}$$

then (1.1) has at least one solution.

Proof. We will use Schäfer's theorem. Let $L > 0$ be the constant defined in (5.15). Consider the normed space

$$(C([a,b]_{\mathbb{T}}; \mathbb{R}^n), \|\cdot\|_L)$$

with the family of equations

$$\mathbf{x} = \lambda \mathbf{F}\mathbf{x}, \quad \lambda \in [0,1), \tag{5.16}$$

where

$$\mathbf{F} : C([a,b]_{\mathbb{T}}; \mathbb{R}^n) \to C([a,b]_{\mathbb{T}}; \mathbb{R}^n)$$

is defined in (5.14). Note that $\mathbf{F}$ is a compact map from Lemma 5.6. Fixed points of $\mathbf{F}$ will be solutions to the dynamic IVP (1.1).

For a fixed $\lambda \in [0, 1)$ let $\mathbf{x} \in C([a, b]_{\mathbb{T}}; \mathbb{R}^n)$ be a solution to (5.16). We then have

$$\begin{aligned}
\|\mathbf{x}\|_L &\le \sup_{t\in[a,b]_{\mathbb{T}}} \frac{1}{e_L(t,a)}\left(\|\lambda \mathbf{f}(t)\| + \int_a^t \|\lambda \mathbf{k}(t,s,\mathbf{x}(s))\|\ \Delta s\right)\\
&\le \sup_{t\in[a,b]_{\mathbb{T}}} \frac{1}{e_L(t,a)}\left(\|\mathbf{f}(t)\| + \int_a^t (L\|\mathbf{x}(s)\| + N)\ \Delta s\right), \quad \text{from (5.15)}\\
&\le \sup_{t\in[a,b]_{\mathbb{T}}} \frac{1}{e_L(t,a)}\left(\int_a^t e_L(s,a)\frac{L\|\mathbf{x}(s)\|}{e_L(s,a)}\ \Delta s + N[t-a] + \|\mathbf{f}(t)\|\right)\\
&\le L\|\mathbf{x}\|_L \sup_{t\in[a,b]_{\mathbb{T}}} \frac{1}{e_L(t,a)}\int_a^t e_L(s,a)\ \Delta s + N[b-a] + \|\mathbf{f}\|_L\\
&= \|\mathbf{x}\|_L\left[1 - \frac{1}{e_L(b,a)}\right] + N[b-a] + \|\mathbf{f}\|_L
\end{aligned}$$

and a rearrangement leads to

$$\|\mathbf{x}\|_L \le e_L(b,a)(N[b-a] + \|\mathbf{f}\|_L).$$

Thus, the set of possible solutions to the family (5.16) is bounded *a priori*, with the bound being independent of λ. Schäfer's theorem now applies to $\mathbf{F}$, yielding the existence of at least one fixed-point. Hence the dynamic IVP (1.1) also has at least one solution. ■

If we had used the sup-norm $\|\cdot\|_0$ in the proof of Theorem 5.7 rather than the norm $\|\cdot\|_L$, then we would have needed an additional assumption in Theorem 5.7, namely, $L[b-a] < 1$.

Example 5.8. We claim that the integral equation

$$x(t) = t^2 + \int_a^t \frac{|x(s)|}{t^2 + s^2 + 1}\ \Delta s, \quad t \in [a,b]_{\mathbb{T}}; \tag{5.17}$$

has at least one solution.

Proof. It is not difficult to show (5.17) satisfies (5.15) for $L = 1$ and $N = 0$. The result now follows from Theorem 5.7. ■

The following is an important special case of Theorem 5.7.

Theorem 5.9. Consider the dynamic IVP (1.1) with $I_{\mathbb{T}} := [a, b]_{\mathbb{T}}$. Let $\mathbf{k} : [a, b]_{\mathbb{T}}^2 \times \mathbb{R}^n \to \mathbb{R}^n$ and $\mathbf{f} : [a, b]_{\mathbb{T}} \to \mathbb{R}^n$ be both continuous. If there exists a nonnegative constant N such that

$$\|\mathbf{k}(t,s,\mathbf{p})\| \le N, \quad \forall (t,s) \in [a,b]_{\mathbb{T}}^2,\ \mathbf{p} \in \mathbb{R}^n, \tag{5.18}$$

then the dynamic IVP (1.1) has at least one solution.

Proof. It is easy to see that if (5.18) holds, then (5.15) holds and the result now follows from Theorem 5.7. ■

In a similar fashion to Theorem 5.9 we have the following result for (1.2).

Theorem 5.10. Consider the integral equation (1.2) with $I_{\mathbb{T}} := [a, b]_{\mathbb{T}}$. Let $\mathbf{k} : [a, b]_{\mathbb{T}}^2 \times \mathbb{R}^n \to \mathbb{R}^n$ and $\mathbf{f} : [a, b]_{\mathbb{T}} \to \mathbb{R}^n$ be both continuous. If there exists a nonnegative constant N such that (5.18) holds, then (1.2) has at least one solution.

Proof. The idea of the proof follows that of Theorem 5.9 and so is omitted. ■

6. Open Problems and Remarks

Due to the very young age of the theory, there is obviously a huge potential for further investigations into integral equations on time scales, in both the linear and nonlinear formats.

It might be interesting to obtain further results based on the approach in Section 4 applied to the alternate form of the linear integral equation, namely

$$\mathbf{x}(t) = \mathbf{f}(t) + \int_a^t \mathcal{B}(t, s)\mathbf{x}^{\sigma}(s)\ \Delta s, \quad t \in I_{\mathbb{T}};$$

as such equations that feature $\mathbf{x}^{\sigma}$ arise naturally in modelling, for example see [35, Sec. 6] for an example of this type concerning economic dynamics. Furthermore, the results on boundedness of solutions to linear equations in Section 4 may have the potential to be extended to the nonlinear case.

References

[1] Ravi Agarwal, Martin Bohner, Donal O'Regan, and Allan Peterson, Dynamic equations on time scales: a survey, *J. Comput. Appl. Math.*, 141(1-2):1–26, 2002. Dynamic equations on time scales.

[2] P. Amster, P. De Nápoli, and C.C. Tisdell, Variational methods for two resonant problems on time scales, *Int. J. Difference Equ.*, 2(1):1–12, 2007.

[3] Boris Belinskiy, John R. Graef, and Sonja Petrović, A nonlinear Sturm-Picone comparison theorem for dynamic equations on time scales, *Int. J. Difference Equ.*, 2(1):25–35, 2007.

[4] A. Bielecki, Une remarque sur la méthode de Banach-Cacciopoli-Tikhonov dans la théorie des équations différentielles ordinaires, *Bull. Acad. Polon. Sci. Cl. III.*, 4:261–264, 1956.

[5] M. Bohner and S.H. Saker, Oscillation of second order half-linear dynamic equations on discrete time scales, *Int. J. Difference Equ.*, 1(2):205–218, 2006.

[6] Martin Bohner and Allan Peterson, *Dynamic equations on time scales*, Birkhäuser Boston Inc., Boston, MA, 2001. An introduction with applications.

[7] Martin Bohner and Allan Peterson, *Advances in dynamic equations on time scales*, Birkhäuser, Boston, 2003.

[8] Martin Bohner and Christopher C. Tisdell, Oscillation and nonoscillation of forced second order dynamic equations, *Pacific J. Math.*, 230(1):59–71, 2007.

[9] T.A. Burton, *Volterra integral and differential equations*, volume 202 of *Mathematics in Science and Engineering*, Elsevier B. V., Amsterdam, second edition, 2005.

[10] E.T. Copson, *Metric spaces*, Cambridge Tracts in Mathematics and Mathematical Physics, No. 57. Cambridge University Press, London, 1968.

[11] C. Corduneanu, *Integral equations and applications*, Cambridge University Press, Cambridge, 1991.

[12] Constantin Corduneanu, *Integral equations and stability of feedback systems*, Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1973. Mathematics in Science and Engineering, Vol. 104.

[13] Qiuyi Dai and Christopher C. Tisdell, Existence of solutions to first-order dynamic boundary value problems, *Int. J. Difference Equ.*, 1(1):1–17, 2006.

[14] James Dugundji and Andrzej Granas, *Fixed point theory. I*, volume 61 of *Monografie Matematyczne [Mathematical Monographs]*, Państwowe Wydawnictwo Naukowe (PWN), Warsaw, 1982.

[15] R.E. Edwards, *Functional analysis*, Dover Publications Inc., New York, 1995. Theory and applications, Corrected reprint of the 1965 original.

[16] L. Erbe, A. Peterson, and C.C. Tisdell, Monotone solutions of dynamic systems on time scales, *J. Difference Equ. Appl.*, 12(3-4):343–355, 2006.

[17] Lynn Erbe, Allan Peterson, and Christopher Tisdell, Existence of solutions to second-order BVPs on time scales, *Appl. Anal.*, 84(10):1069–1078, 2005.

[18] M.I. Gil' and P.E. Kloeden, Solution estimates of nonlinear vector Volterra-Stieltjes equations, *Anal. Appl. (Singap.)*, 1(2):165–175, 2003.

[19] T.H. Hildebrandt, On systems of linear differentio-Stieltjes-integral equations, *Illinois J. Math.*, 3:352–373, 1959.

[20] Stefan Hilger, Analysis on measure chains—a unified approach to continuous and discrete calculus, *Results Math.*, 18(1-2):18–56, 1990.

[21] J. Hoffacker and C.C. Tisdell, Stability and instability for dynamic equations on time scales, *Comput. Math. Appl.*, 49(9-10):1327–1334, 2005.

[22] Chaim Samuel Hönig, *Volterra Stieltjes-integral equations*, North-Holland Publishing Co., Amsterdam, 1975. Functional analytic methods; linear constraints, Mathematics Studies, No. 16, Notas de Matemática, No. 56. [Notes on Mathematics, No. 56].

[23] A.L. Jensen, Dynamics of populations with nonoverlapping generations, continuous mortality, and discrete reproductive periods, *Ecol. Modelling*, 74:305–309, 1994.

[24] Jaroslav Kurzweil, Generalized ordinary differential equations and continuous dependence on a parameter, *Czechoslovak Math. J.*, 7 (82):418–449, 1957.

[25] V. Lakshmikantham, S. Sivasundaram, and B. Kaymakcalan, *Dynamic systems on measure chains*, volume 370 of *Mathematics and its Applications*, Kluwer Academic Publishers Group, Dordrecht, 1996.

[26] N.G. Lloyd, *Degree theory*, Cambridge University Press, Cambridge, 1978. Cambridge Tracts in Mathematics, No. 73.

[27] L.V. Nedorezova and B.N. Nedorezova, Correlations between models of population dynamics in continuous and discrete time, *Ecol. Modelling*, 82:93–97, 1995.

[28] Donal O'Regan, *Existence theory for nonlinear ordinary differential equations*, volume 398 of *Mathematics and its Applications*, Kluwer Academic Publishers Group, Dordrecht, 1997.

[29] Allan C. Peterson and Christopher C. Tisdell, Boundedness and uniqueness of solutions to dynamic equations on time scales, *J. Difference Equ. Appl.*, 10(13-15):1295–1306, 2004.

[30] Štefan Schwabik, *Generalized ordinary differential equations*, volume 5 of *Series in Real Analysis*, World Scientific Publishing Co. Inc., River Edge, NJ, 1992.

[31] Štefan Schwabik, Milan Tvrdý, and Otto Vejvoda, *Differential and integral equations*, D. Reidel Publishing Co., Dordrecht, 1979. Boundary value problems and adjoints.

[32] D.R. Smart, *Fixed point theorems*, Cambridge University Press, London, 1974. Cambridge Tracts in Mathematics, No. 66.

[33] Vanessa Spedding, Taming nature's numbers, *New Scientist*, 179(2404):28–31, 19 July 2003.

[34] Christopher C. Tisdell, Existence and uniqueness to nonlinear dynamic boundary value problems, *Cubo*, 8(3):11–24, 2006.

[35] Christopher C. Tisdell and Atiya Zaidi, Basic qualitative and quantitative results for solutions to nonlinear dynamic equations on time scales with an application to economic modelling, *Nonlinear Anal.*, 2008. In press (doi:10.1016/j.na.2007.03.043).

[36] D. Weld, Switching between discrete and continuous process models to predict genetic activity, In *Artificial Intelligence Laboratory*, volume 793 of *Technical Report*. Massachusetts Institute of Technology, 184.

[37] P.J.Y. Wong and Y.C. Soh, Constant-sign solutions for a system of integral equations on time scales, *Comput. Math. Appl.*, 49(2-3):271–280, 2005.

[38] Patricia J.Y. Wong and K.L. Boey, Nontrivial periodic solutions in the modelling of infectious disease, *Appl. Anal.*, 83(1):1–16, 2004.