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On the Asymptotic Behavior of a Nonautonomous Difference Equation

A. Aghajani and **M. Ejtehadi**
Iran University of Science & Technology
School of Mathematics
Tehran, Iran
aghajani@iust.ac.ir
ma_ejtehadi@mathdep.iust.ac.ir

Abstract

We consider the difference equation $x_{n+1} = \dfrac{\beta_n x_n}{x_{n-1}}$, where $\{\beta_n\}_{n=0}^{\infty}$ is a positive periodic sequence with prime period $k \geq 2$. We completely determine the asymptotic behavior of every solution of this equation. In particular we show that every solution is periodic when $6 \nmid k$. Our results give necessary and sufficient conditions for the boundedness of all solutions of this equation. This solves one of the open problems proposed by E. Camouzis and G. Ladas in [3].

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1 Introduction

Consider the nonautonomous difference equation

$$x_{n+1} = \frac{\beta_n x_n}{x_{n-1}}, \quad n = 0, 1, \ldots \tag{1.1}$$

with nonzero initial conditions x_{-1}, x_0 where $\{\beta_n\}_{n=0}^{\infty}$ is a sequence of nonzero real numbers. In the case that $\{\beta_n\}_{n=0}^{\infty}$ is a constant sequence, it is easy to see that every solution of Eq. (1.1) is periodic with period six, and in particular every solutions are bounded, see [4].

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When $\{\beta_n\}_{n=0}^{\infty}$ is a periodic sequence, its periodicity may destroy the boundedness of solutions of Eq. (1.1). For example when

$$\beta_n = \begin{cases} 1, & \text{if } n = 6k+i \text{ with } i = \{0,1,2,3,4\} \\ \beta, & \text{if } n = 6k+5 \end{cases}, k = 0, 1, \dots$$

with $\beta > 0$, the solution of Eq. (1.1) with initial condition $x_{-1} = x_0 = 1$ is unbounded if and only if $\beta \neq 1$, see [2].

E. Camouzis and G. Ladas proposed the following open problem in [3].

[3, Open Problem 5.7.2]. Let $\{\beta_n\}_{n=0}^{\infty}$ be a positive periodic sequence with prime period $k \geq 2$. Obtain necessary and sufficient conditions on k and

$$\beta_0, \dots, \beta_{k-1}$$

such that every solution of Eq. (1.1) is bounded.

Our results in this paper solve the above open problem. For some results on the asymptotic behavior of Eq. (1.1) see [1].

2 Explicit General Solutions

Lemma 2.1. *If* $\{x_n\}_{n=-1}^{\infty}$ *is a solution of Eq.* (1.1)*, then*

$$x_{n+6} = \alpha_n x_n, \quad n = -1, 0, \dots, \tag{2.1}$$

where

$$\alpha_n = \frac{\beta_{n+5}\beta_{n+4}}{\beta_{n+2}\beta_{n+1}}, \quad n = -1, 0, \dots. \tag{2.2}$$

Proof. From the Eq. (1.1) we have

$$x_{n+6} = \frac{\beta_{n+5}x_{n+5}}{x_{n+4}} = \frac{\beta_{n+5}\beta_{n+4}}{x_{n+3}}, \quad n \geq -4,$$

and by replacing n with $n-3$ in (2.1) and using it again we obtain

$$x_{n+6} = \frac{\beta_{n+5}\beta_{n+4}}{\beta_{n+2}\beta_{n+1}} x_n = \alpha_n x_n, \quad n = -1, 0, \dots.$$

The proof is complete. □

Now by using Lemma 2.1 we can derive an explicit formula for the solutions of Eq. (1.1).

Theorem 2.2. *Let* $\{x_n\}_{n=-1}^{\infty}$ *be a solution of Eq.* (1.1)*. Then*

$$x_{6m+i} = \left(\prod_{s=0}^{m-1} \alpha_{6s+i}\right) x_i, \quad i = 0, 1, \dots, 5 \quad and \quad m \in \mathbb{N}, \tag{2.3}$$

where α_n, $n = -1, 0, \dots$ *is given by* (2.2).

Proof. The proof is easily completed by using Lemma 2.1 and induction. □

3 Asymptotic Behavior

Suppose that $\{\beta_n\}_{n=0}^{\infty}$ is a periodic sequence with prime period $k \geq 2$. Then $\{\alpha_n\}_{n=-1}^{\infty}$ given by (2.2) is also k-periodic. Now if

$$6s + i \equiv r_s^i \ (\text{mod } k)$$

$$0 \leq r_s^i \leq k-1, \quad i = 0, 1, \ldots, 5, \quad s = 0, 1, \ldots$$

and

$$B_i = \{r_s^i, s = 0, 1, \ldots\},$$

then

$$k' := \text{card}(B_i) = \frac{k}{\gcd\{6, k\}}.$$

Let

$$B_i = \{r_1^i, \ldots, r_{k'}^i\}, \quad m = m_1 k' + R_1, \quad 0 \leq R_1 \leq k' - 1$$

and define

$$\lambda_i = \alpha_{r_1^i} \ldots \alpha_{r_{k'}^i}, \quad i = 0, 1, \ldots, 5. \tag{3.1}$$

So, if $R_1 = 0$, then

$$\prod_{s=0}^{m-1} \alpha_{6s+i} = \lambda_i^{m_1}$$

and if $R_1 \neq 0$, we will have

$$\prod_{s=0}^{m-1} \alpha_{6s+i} = \lambda_i^{m_1} \alpha_i \ldots \alpha_{6(R_1-1)+i}.$$

Thus by (2.2) we have:

$$x_{6m+i} = \begin{cases} \lambda_i^{m_1} x_i & \text{if } R_1 = 0, \\ \lambda_i^{m_1} \alpha_i \ldots \alpha_{6(R_1-1)+i} & \text{otherwise}. \end{cases}$$

Hence, if $\lambda_i > 1$, $\{x_{6m+i}\}_{m=0}^{\infty}$ is unbounded and if $\lambda_i < 1$, $\{x_{6m+i}\}_{m=0}^{\infty}$ tends to zero. If $\lambda_i = 1$, $\{x_{6m+i}\}_{m=0}^{\infty}$ is periodic with period k'. We summarize the above results in the following theorem.

Theorem 3.1. *Suppose that $\{x_n\}_{n=-1}^{\infty}$ is a solution of Eq.* (1.1) *and $\{\beta_n\}_{n=0}^{\infty}$ is a periodic sequence with prime period $k \geq 2$. Let $k' = \dfrac{k}{\gcd\{6, k\}}$ and λ_i as given by* (3.1). *Then*

- $\{x_{6m+i}\}_{m=0}^{\infty}$ *is an unbounded sequence when $\lambda_i > 1$;*

- $\{x_{6m+i}\}_{m=0}^{\infty}$ *vanishes at infinity when* $\lambda_i < 1$*;*
- $\{x_{6m+i}\}_{m=0}^{\infty}$ *is periodic with (not necessarily prime) period* k' *when* $\lambda_i = 1$.

Lemma 3.2. *Suppose that* $\{\beta_n\}_{n=0}^{\infty}$ *is a periodic sequence with prime period* k *and* $\{\alpha_n\}_{n=-1}^{\infty}$ *is given by* (2.2)*. Then for* $i \geq 0$ *we have*

$$\alpha_i\alpha_{1+i}\dots\alpha_{k-1+i} = 1.$$

If k *is even, then*

$$\alpha_i\alpha_{2+i}\dots\alpha_{k-2+i} = 1,$$

and when $3 \mid k$*, then*

$$\alpha_i\alpha_{3+i}\dots\alpha_{k-3+i} = 1.$$

Proof. Since the product of k consecutive terms of the k-periodic sequence $\{\beta_n\}_{n=0}^{\infty}$ is constant we have

$$\begin{aligned}\alpha_i\alpha_{1+i}\dots\alpha_{k-1+i} &= \frac{\beta_{4+i}\beta_{5+i}}{\beta_{1+i}\beta_{2+i}}\cdot\frac{\beta_{5+i}\beta_{6+i}}{\beta_{2+i}\beta_{3+i}}\cdots\frac{\beta_{k+3+i}\beta_{k+4+i}}{\beta_{k+i}\beta_{k+1+i}}\\ &= \frac{(\beta_{4+i}\beta_{5+i}\dots\beta_{k+3+i})(\beta_{5+i}\beta_{6+i}\dots\beta_{k+4+i})}{(\beta_{1+i}\beta_{2+i}\dots\beta_{k+i})(\beta_{2+i}\beta_{3+i}\dots\beta_{k+1+i})} = 1\end{aligned}$$

and

$$\alpha_i\alpha_{2+i}\dots\alpha_{k-2+i} = \frac{\beta_{4+i}\beta_{5+i}\beta_{6+i}\dots\beta_{k+2+i}\beta_{k+3+i}}{\beta_{1+i}\beta_{2+i}\beta_{3+i}\dots\beta_{k-1+i}\beta_{k+i}} = 1.$$

When $3 \mid k$, it is easy to see that

$$\begin{aligned}\alpha_i\alpha_{3+i}\alpha_{6+i}\dots\alpha_{k-3+i} &= \frac{(\beta_{4+i}\beta_{5+i})(\beta_{7+i}\beta_{8+i})\dots(\beta_{k+1+i}\beta_{k+2+i})}{(\beta_{1+i}\beta_{2+i})(\beta_{4+i}\beta_{5+i})\dots(\beta_{k-2+i}\beta_{k-1+i})}\\ &= \frac{\beta_{k+1+i}\beta_{k+2+i}}{\beta_{1+i}\beta_{2+i}} = 1,\end{aligned}$$

and the proof is complete. □

Theorem 3.3. *Suppose that* $\{\beta_n\}_{n=0}^{\infty}$ *is a positive periodic sequence with prime period* k *such that* $6 \nmid k$*. Then every solution of Eq.* (1.1) *is periodic with (not necessarily prime) period* $6k'$ *where* $k' = \dfrac{k}{\gcd\{6,k\}}$.

Proof. Suppose that $\{x_n\}$ is a solution of Eq. (1.1). If $\gcd\{6,k\} = 1$, then $k' = k$. Thus by Lemma 3.2 and (3.1), we have

$$\lambda_i = \alpha_i\dots\alpha_{k-1+i} = 1, \quad i = 0,1,\dots,5.$$

Now Theorem 3.1 gives the desired result because every subsequence $\{x_{6m+i}\}_{m=0}^{\infty}$ of sequence $\{x_n\}$ is k'-periodic so $\{x_n\}$ is periodic of period $6k'$.

In the case that $\gcd\{6,k\}=2$, we have $k'=\dfrac{k}{2}$. If B_i, $i=0,1,\dots,5$, are as in the proof of Theorem 3.1 and $r_1,r_2\in B_i$, then there exist $m_1,m_2\in\mathbb{N}$ such that

$$r_1\equiv 6m_1+i\ (\text{mod } k)\quad\text{and}\quad r_2\equiv 6m_2+i\ (\text{mod } k)$$

which implies

$$r_2-r_1\equiv 6(m_2-m_1)\ (\text{mod } k). \tag{3.2}$$

Since k is even, by (3.2), r_2-r_1 is even. Thus

$$B_i=\{0,2,\dots,k-2\}\quad\text{or}\quad B_i=\{1,3,\dots,k-1\},\quad\text{for}\quad i=0,1,\dots,5.$$

Then

$$\lambda_i=\alpha_0\alpha_2\dots\alpha_{k-2}\quad\text{or}\quad\lambda_i=\alpha_1\alpha_3\dots\alpha_{k-1},\quad\text{for}\quad i=0,1,\dots,5,$$

and by Lemma 3.2 we have $\lambda_i=1$, $i=0,1,\dots,5$. Thus Theorem 3.1 implies that $\{x_{6m+i}\}_{m=0}^{\infty}$, $i=0,1,\dots,5$ is k'-periodic. So $\{x_n\}$ is periodic with period $6k'$.

Now suppose that $\gcd\{6,k\}=3$. Then $k'=\dfrac{k}{3}$ and

$$B_i=\{i,3+i,6+i,\dots,k-3+i\},\quad i=0,1,\dots,5.$$

Thus Lemma 3.2 implies that

$$\lambda_i=\alpha_i\alpha_{3+i}\dots\alpha_{k-3+i}=1,\quad i=0,1,\dots,5.$$

So like in the previous cases by using Theorem 3.1, the proof is complete. □

Remark 3.4. Theorem 3.1 implies that if $\{\beta_n\}_{n=0}^{\infty}$ is a positive periodic sequence of prime period k such that $6\mid k$, then every solution of Eq. (1.1) is periodic with (not necessarily prime) period k if and only if $\lambda_i=1$ for $i=0,1,\dots,5$, and it is bounded if and only if $\lambda_i\le 1$ for $i=0,1,\dots,5$. Also note that in this case we have $B_i=\{i,6+i,\dots,k-6+i\}$ for $i=0,1,\dots,5$, so

$$\lambda_i=\prod_{s=0}^{\frac{k}{6}-1}\alpha_{6s+i},\quad i=0,1,\dots,5.$$

For example when $k=6$, we have $\lambda_i=\alpha_i$ for $i=0,1,\dots,5$, so every solution of Eq. (1.1) is periodic with period 6 if and only if $\alpha_i=1$ for $i=0,1,\dots,5$, and it is bounded if and only if $\alpha_i\le 1$ for $i=0,1,\dots,5$, where

$$\alpha_i=\frac{\beta_{i+4}\beta_{i+5}}{\beta_{i+1}\beta_{i+2}}.$$

The following corollary solves [3, Open Problem 5.7.2].

Corollary 3.5. *Let* $\{\beta_n\}_{n=0}^{\infty}$ *be a positive periodic sequence with prime period* k. *If* $6 \nmid k$, *then every solution of Eq.* (1.1) *is bounded. If* $6 \mid k$, *then every solution of Eq.* (1.1) *is bounded if and only if*

$$\prod_{s=0}^{\frac{k}{6}-1} \alpha_{6s+i} \leq 1, \quad i = 0, 1, \ldots, 5,$$

where

$$\alpha_n = \frac{\beta_{n+4}\beta_{n+5}}{\beta_{n+1}\beta_{n+2}} \quad \textit{for} \quad n \geq -1.$$

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