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Symmetric Functions and Difference Equations with Asymptotically Period-two Solutions

Kenneth S. Berenhaut and **Richard T. Guy**
Wake Forest University
Department of Mathematics
Winston-Salem, NC 27109
berenhks@wfu.edu and guyrt7@wfu.edu

Abstract

This paper introduces easily verified conditions which guarantee that all solutions to the equation $y_n = f(y_{n-k}, y_{n-m})$, with $k, m \geq 1$ and $\gcd(k, m) = 1$ are asymptotically periodic with period two. A recent result of Sun and Xi is employed. Several examples are included.

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1 Introduction

In this paper, we consider recursive equations of the form

$$y_n = f(y_{n-k}, y_{n-m}), \tag{1.1}$$

for $n \geq 0$, where $k, m \geq 1$, $\gcd(k, m) = 1$, $s = \max\{k, m\}$ and $y_{-s}, y_{-s+1}, \ldots, y_{-1} \in (0, \infty)$.

Recently, Sun and Xi [6] and Stević [4] proved the following interesting result regarding criteria for asymptotically two-periodic behavior of solutions to (1.1).

Theorem 1.1. *Suppose that* $\{y_i\}$ *satisfies* (1.1)*, and in addition*

(*i*) $f \in C((0, \infty)^2, (a, \infty))$ *with*

$$a = \inf_{(u,v) \in (0,\infty)^2} f(u, v) \geq 0, \tag{1.2}$$

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(*ii*) $f(u, v)$ *is increasing in* u *and decreasing in* v*,*

(*iii*) *there exists a decreasing function* $g \in C((a, \infty), (a, \infty))$ *such that*

$$g(g(x)) = x, \text{ for } x > a, \tag{1.3}$$

$$x = f(x, g(x)), \text{ for } x > a, \tag{1.4}$$

and

$$\lim_{x \to a^+} g(x) = \infty \text{ and } \lim_{x \to \infty} g(x) = a. \tag{1.5}$$

Then, if k *is even and* m *is odd, every positive solution to* (1.1) *converges to a (not necessarily prime) two-periodic solution. Otherwise, every solution converges to the unique equilibrium of* (1.1).

For earlier related results and motivation, see [1, 2, 5].

Remark 1.2. Note that the self-inverse property in (1.3) gives that the range of g must comprise all of (a, ∞). Hence under the assumption that g is decreasing, (1.5) is immediately satisfied and may be removed from the statement of Theorem 1.1.

2 Main Result and Examples

In this short note, we will give easily verified conditions for existence of a function g satisfying the requirements of Theorem 1.1. In particular, we will prove the following.

Theorem 2.1. *Suppose that* $\{y_i\}$ *satisfies* (1.1) *with* $f(u, v) = h(u, v)/v$ *for some function* h *where*

(*i*) $h \in C((0, \infty)^2, (0, \infty))$ *is such that* $h(u, v)$ *is symmetric in* u *and* v *and increasing in* u*,*

(*ii*) *the function* f *is decreasing in* v*,*

(*iii*) *with* a *as in* (1.2)*, for all* $v > a$*, there exist* C_v *and* D_v *(possibly infinite) such that*

$$\lim_{u \to a^+} f(u, v)/u = C_v > 1 \text{ and } \lim_{u \to \infty} f(u, v)/u = D_v < 1. \tag{2.1}$$

Then, there exists a continuous function g *satisfying* (1.3), (1.4) *and* (1.5)*. Hence by Theorem 1.1, if* k *is even and* m *is odd, every positive solution to* (1.1) *converges to a (not necessarily prime) two-periodic solution, and otherwise, every positive solution converges to the unique equilibrium of* (1.1).

Now, note that under assumptions (1) and (2) in Theorem 2.1, we have that, for fixed $v > a$,

$$\frac{f(u,v)}{u} = \frac{h(u,v)}{uv} = \frac{h(v,u)}{u}\frac{1}{v} = \frac{f(v,u)}{v} \tag{2.2}$$

is decreasing in u and hence the (possibly infinite) limits in (2.1) exist. A key point here is that there is no need to have a closed form for g to verify the hypotheses of Theorem 1.1. In fact, finding such a closed form may not be very practical in practice (see for instance Example 2.6 below).

Before turning to a proof of Theorem 2.1, we give several examples of difference equations satisfying the requirements of the theorem.

Example 2.2. Consider the equation

$$x_n = 1 + \frac{x_{n-k}}{x_{n-m}}. \tag{2.3}$$

Here we have $h(u,v) = u + v$, $f(u,v) = 1 + u/v$, $a = 1$, and for $v > 1$,

$$\lim_{u\to 1^+} \frac{f(u,v)}{u} = \lim_{u\to 1^+} \frac{1}{u} + \frac{1}{v} = 1 + \frac{1}{v} > 1 \tag{2.4}$$

and

$$\lim_{u\to\infty} \frac{f(u,v)}{u} = \lim_{u\to\infty} \frac{1}{u} + \frac{1}{v} = \frac{1}{v} < 1. \tag{2.5}$$

Hence Theorem 2.1 is applicable and all positive solutions are asymptotically two-periodic whenever k is even and m is odd, and otherwise all positive solutions converge to the unique equilibrium. See [1–3] and the references therein for further discussion of Eq. (2.3).

Example 2.3. Consider the equation

$$x_n = 1 + \frac{x_{n-k}}{x_{n-m}} + \sqrt{\frac{x_{n-k}}{x_{n-m}}}. \tag{2.6}$$

Here we have $h(u,v) = u+v+\sqrt{uv}$, $f(u,v) = 1+u/v+\sqrt{u/v}$, $a = 1$, and for $v > 1$,

$$\lim_{u\to 1^+} \frac{f(u,v)}{u} = \lim_{u\to 1^+} \frac{1}{u} + \frac{1}{v} + \frac{1}{\sqrt{uv}} = 1 + \frac{1}{v} + \frac{1}{\sqrt{v}} > 1 \tag{2.7}$$

and

$$\lim_{u\to\infty} \frac{f(u,v)}{u} = \lim_{u\to\infty} \frac{1}{u} + \frac{1}{v} + \frac{1}{\sqrt{uv}} = \frac{1}{v} < 1. \tag{2.8}$$

By Theorem 2.1, we have the required asymptotic two-periodic behavior.

Example 2.4. Consider the equation

$$x_n = \frac{1}{x_{n-m}} \left(\frac{x_{n-k}}{x_{n-k}+1} + \frac{x_{n-m}}{x_{n-m}+1} \right). \tag{2.9}$$

Here we have $h(u,v) = u/(u+1) + v/(v+1)$, $f(u,v) = (u/(u+1) + v/(v+1))/v$, $a = 0$, and for $v > 0$,

$$\lim_{u \to 0^+} \frac{f(u,v)}{u} = \lim_{u \to 0^+} \frac{1}{v(u+1)} + \frac{1}{u(v+1)} = \infty \tag{2.10}$$

and

$$\lim_{u \to \infty} \frac{f(u,v)}{u} = \lim_{u \to \infty} \frac{1}{v(u+1)} + \frac{1}{u(v+1)} = 0 < 1. \tag{2.11}$$

By Theorem 2.1, we have the required asymptotic two-periodic behavior.

Example 2.5. Consider the equation

$$x_n = 1 + \frac{x_{n-k}}{x_{n-m}} + \frac{\log(x_{n-k}x_{n-m})}{x_{n-m}}. \tag{2.12}$$

Here we have $h(u,v) = u + v + \log(uv)$, $f(u,v) = 1 + u/v + \log(uv)/v$, $a = 1$, and for $v > 1$,

$$\lim_{u \to 1^+} \frac{f(u,v)}{u} = \lim_{u \to 1^+} \frac{1}{u} + \frac{1}{v} + \frac{\log(uv)}{uv} = 1 + \frac{1}{v} + \frac{\log(v)}{v} > 1 \tag{2.13}$$

and

$$\lim_{u \to \infty} \frac{f(u,v)}{u} = \lim_{u \to \infty} \frac{1}{u} + \frac{1}{v} + \frac{\log(uv)}{uv} = \frac{1}{v} < 1. \tag{2.14}$$

By Theorem 2.1, we have the required asymptotic two-periodic behavior.

Example 2.6. Consider the equation

$$x_n = \frac{x_{n-k}^{\alpha} + x_{n-k}^{\beta} + x_{n-m}^{\alpha} + x_{n-m}^{\beta}}{x_{n-m}}. \tag{2.15}$$

Here, for $0 < \alpha, \beta < 1$, we have $h(u,v) = u^\alpha + u^\beta + v^\alpha + v^\beta$, $f(u,v) = u^\alpha/v + u^\beta/v + v^{-(1-\alpha)} + v^{-(1-\beta)}$, $a = 0$, and for $v > 0$,

$$\lim_{u \to 0^+} \frac{f(u,v)}{u} = \lim_{u \to 0^+} \frac{1}{vu^{1-\alpha}} + \frac{1}{vu^{1-\beta}} + \frac{1}{uv^{1-\alpha}} + \frac{1}{uv^{1-\beta}} = \infty \tag{2.16}$$

and

$$\lim_{u \to \infty} \frac{f(u,v)}{u} = \lim_{u \to \infty} \frac{1}{vu^{1-\alpha}} + \frac{1}{vu^{1-\beta}} + \frac{1}{uv^{1-\alpha}} + \frac{1}{uv^{1-\beta}} = 0 < 1. \tag{2.17}$$

By Theorem 2.1, we have the required asymptotic two-periodic behavior.

Remark 2.7. To see that the requirement in (3) cannot be removed, consider the equation

$$x_n = 1 + \frac{x_{n-k}}{x_{n-m}} + x_{n-k}. \tag{2.18}$$

Here, we have $h(u,v) = u + v + uv$, $f(u,v) = 1 + u/v + u$ and $a = 1$, yet the equation possesses no equilibrium. Indeed, for $v > 1$,

$$\lim_{u \to 1^+} \frac{f(u,v)}{u} = \lim_{u \to 1^+} 1 + \frac{1}{u} + \frac{1}{v} = 2 + \frac{1}{v} > 1, \tag{2.19}$$

but

$$\lim_{u \to \infty} \frac{f(u,v)}{u} = \lim_{u \to \infty} 1 + \frac{1}{u} + \frac{1}{v} = 1 + \frac{1}{v} > 1. \tag{2.20}$$

Condition (3) is not satisfied and Theorem 2.1 is not applicable.

3 Proof of the Main Result

We now turn to a proof of Theorem 2.1.

Proof of Theorem 2.1. By Theorem 1.1 and Remark 1.2, we need only show that there exists a decreasing continuous function g which satisfies (1.3) and (1.4). To that end, note that by (2.2), $f(u,v)/u$ is decreasing in u and hence for fixed $v > a$, by (3), there exists a unique $u = g(v)$, say, which satisfies

$$f(g(v), v)/g(v) = 1. \tag{3.1}$$

To see that the function g is decreasing, note that for $x > a$ and $\epsilon > 0$, by (2) and the definition of g,

$$\frac{f(g(x), x + \epsilon)}{g(x)} < \frac{f(g(x), x)}{g(x)} = 1 = \frac{f(g(x + \epsilon), x + \epsilon)}{g(x + \epsilon)}, \tag{3.2}$$

and hence since $f(u,v)/u$ is decreasing in u, $g(x + \epsilon) < g(x)$, as required.

Now, note that for $x > a$, by the definition of g and the assumptions on f and h, we have

$$f(x, g(x)) = \frac{h(x, g(x))}{g(x)} = \frac{h(g(x), x)}{g(x)} = \frac{f(g(x), x)x}{g(x)} = x, \tag{3.3}$$

and (1.4) is satisfied. Since $u = g(g(x))$ is the unique value satisfying $f(u, g(x)) = u$, (3.3) gives that (1.3) also holds.

The continuity of g follows from its monotonicity and the fact that the range of g is (a, ∞) (see Remark 1.2), and the theorem is proven. □

Remark 3.1. Expanding on ideas in [1], Stević [4,5] showed that similar asymptotically two-periodic behavior can occur for multivariable functions f in (1.1), with varying delays. The interested reader may verify that the ideas introduced here are applicable in that case as well.

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