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Unboundedness for some Classes of Rational Difference Equations

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Abstract

We study the rational difference equation

$$x_n = \frac{\alpha + x_{n-1}}{Cx_{n-2} + x_{n-3}}, \quad n \in \mathbb{N}.$$

Particularly, we show that for nonnegative α and C, whenever $C\alpha = 0$ and $C+\alpha > 0$, unbounded solutions exist for some choice of nonnegative initial conditions. Moreover, we study the rational difference equation

$$x_n = \frac{\alpha + \beta x_{n-1} + x_{n-2}}{x_{n-3}}, n \in \mathbb{N}.$$

Particularly, we show that whenever $0 < \beta < \frac{1}{3}$ and $\alpha \in [0, 1]$, unbounded solutions exist for some choice of nonnegative initial conditions.

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1 Introduction

In [1] Camouzis and Ladas devote a chapter to the study of unbounded solutions for the kth order rational difference equation with nonnegative parameters and nonnegative initial conditions

$$x_n = \frac{\alpha + \sum_{i=1}^{k} \beta_i x_{n-i}}{A + \sum_{j=1}^{k} B_j x_{n-j}}, \quad n \in \mathbb{N}.$$

In the introduction of said chapter, the authors of [1] pose five conjectures regarding the boundedness character of five different special cases of the third order rational difference equation. Particularly we are referring to the special cases #28,#44,#56,#70, and #120. These are the only remaining cases of third order for which the boundedness character has not been established.

First we study special cases #56 and #120

$$x_n = \frac{\alpha + \beta x_{n-1} + x_{n-2}}{x_{n-3}}, \quad n \in \mathbb{N}.$$

Using a standard induction technique, we show that whenever $0 < \beta < \frac{1}{3}$ and $\alpha \in [0, 1]$, unbounded solutions exist for some choice of nonnegative initial conditions.

We then study special cases #44 and #28

$$x_n = \frac{\alpha + x_{n-1}}{C x_{n-2} + x_{n-3}}, \quad n \in \mathbb{N}.$$

We show that for nonnegative α and C, whenever $C\alpha = 0$ and $C + \alpha > 0$, unbounded solutions exist for some choice of nonnegative initial conditions. The proof is presented in two special cases. The case where $\alpha > 0$ and the case where $C > 0$.

2 Todd's Equation

Consider the third order rational difference equation

$$x_n = \frac{\alpha + \beta x_{n-1} + x_{n-2}}{x_{n-3}}, \quad n \in \mathbb{N}. \tag{2.1}$$

There have been significant results concerning the case where $\beta = 1$. In this case the equation is generally referred to by the cognomen "Todd's equation" and possesses the invariant

$$(\alpha + x_n + x_{n-1} + x_{n-2}) \left(1 + \frac{1}{x_n}\right) \left(1 + \frac{1}{x_{n-1}}\right) \left(1 + \frac{1}{x_{n-2}}\right) = \text{constant}.$$

For more information regarding Todd's equation see [5–7]. In the following theorem we show that whenever $0 < \beta < \frac{1}{3}$ and $\alpha \in [0, 1]$, (2.1) has unbounded solutions for some choice of nonnegative initial conditions.

Theorem 2.1. *Suppose* $0 < \beta < \dfrac{1}{3}$ *and* $\alpha \in [0, 1]$. *Then Equation* (2.1) *has unbounded solutions for some initial conditions.*

Proof. Choose initial conditions so that

$$\min\{x_0, x_{-2}\} > \max\left\{\frac{1}{\beta}, \frac{x_{-1}}{\beta}\right\}.$$

We shall first prove by induction that for all $j \in \mathbb{N}$,

$$\min\{x_{2j}, x_{2j-2}\} > \max\left\{\frac{1}{\beta}, \frac{x_{2j-1}}{\beta}\right\}. \tag{2.2}$$

The initial conditions provide the base case. Assume the following holds for some $j \in \mathbb{N}$:

$$\min\{x_{2j-2}, x_{2j-4}\} > \max\left\{\frac{1}{\beta}, \frac{x_{2j-3}}{\beta}\right\}.$$

Since $\beta x_{2j-2} > x_{2j-3}$,$x_{2j-2} > \dfrac{1}{\beta}$, $\beta x_{2j-2} > 1 \geq \alpha$, and $x_{2j-4} > \dfrac{1}{\beta} > 3$, we see that

$$x_{2j-1} = \frac{\alpha + \beta x_{2j-2} + x_{2j-3}}{x_{2j-4}} < \frac{3\beta x_{2j-2}}{x_{2j-4}} < \beta x_{2j-2}.$$

Thus we have shown

$$x_{2j-2} > \max\left\{\frac{1}{\beta}, \frac{x_{2j-1}}{\beta}\right\}.$$

Since $\beta x_{2j-4} > x_{2j-3}$ and $0 < \beta < \dfrac{1}{3}$, we have

$$x_{2j} = \frac{\alpha + \beta x_{2j-1} + x_{2j-2}}{x_{2j-3}} > \frac{x_{2j-2}}{x_{2j-3}} > \frac{x_{2j-2}}{\beta x_{2j-4}} > \frac{3x_{2j-2}}{x_{2j-4}} = \frac{3\beta x_{2j-2}}{\beta x_{2j-4}} > \frac{x_{2j-1}}{\beta}.$$

Also

$$x_{2j} > \frac{x_{2j-2}}{x_{2j-3}} > \frac{x_{2j-2}}{\beta x_{2j-2}} = \frac{1}{\beta}.$$

Thus

$$\min\{x_{2j}, x_{2j-2}\} > \max\left\{\frac{1}{\beta}, \frac{x_{2j-1}}{\beta}\right\}.$$

This completes the induction proof for (2.2).

Using (2.2), we now prove that $x_{8\eta} > \dfrac{x_{8\eta-8}}{9\beta^2}$ for all $\eta \in \mathbb{N}$:

$$\begin{aligned} x_{8\eta} &= \frac{\alpha + \beta x_{8\eta-1} + x_{8\eta-2}}{x_{8\eta-3}} > \frac{x_{8\eta-2}}{x_{8\eta-3}} \\ &= \left(\frac{x_{8\eta-6}}{\alpha + \beta x_{8\eta-4} + x_{8\eta-5}}\right)\left(\frac{\alpha + \beta x_{8\eta-3} + x_{8\eta-4}}{x_{8\eta-5}}\right) \\ &> \left(\frac{x_{8\eta-6}}{3\beta x_{8\eta-4}}\right)\left(\frac{x_{8\eta-4}}{x_{8\eta-5}}\right) = \frac{x_{8\eta-6}}{3\beta x_{8\eta-5}} \\ &= \frac{x_{8\eta-6}x_{8\eta-8}}{3\beta(\alpha + \beta x_{8\eta-6} + x_{8\eta-7})} > \frac{x_{8\eta-6}x_{8\eta-8}}{9\beta^2 x_{8\eta-6}} = \frac{x_{8\eta-8}}{9\beta^2}. \end{aligned}$$

Since $0 < \beta < \dfrac{1}{3}$, $9\beta^2 < 1$. Thus we have a subsequence of our solution which diverges to ∞. Hence the solution is unbounded. □

3 Special Case #44

We now study special case #44

$$x_n = \frac{\alpha + x_{n-1}}{x_{n-3}}, \quad n \in \mathbb{N}. \tag{3.1}$$

Particularly, we show that, whenever $\alpha > 0$, Equation (3.1) has unbounded solutions for some initial conditions.

The following lemma provides a useful technique for constructing divergent subsequences of solutions for rational difference equations.

Lemma 3.1. *Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $[0, \infty)$. Suppose that there exists $D > 1$ and hypotheses $H_1, \ldots, H_k$ so that for all $n \in \mathbb{N}$ there exists $p_n \in \mathbb{N}$ so that the following holds: Whenever x_{n-i} satisfies H_i for all $i \in \{1, \ldots, k\}$, then x_{n+p_n-i} satisfies H_i for all $i \in \{1, \ldots, k\}$ and $x_{n+p_n-1} \geq Dx_{n-1}$. Further assume that for some $N \in \mathbb{N}$, x_{N-i} satisfies H_i for all $i \in \{1, \ldots, k\}$ and $x_{N-1} > 0$. Then $\{x_n\}_{n=1}^{\infty}$ is unbounded. Particularly $\{x_{z_m-1}\}_{m=1}^{\infty}$ is a subsequence of $\{x_n\}_{n=1}^{\infty}$ which diverges to ∞, where $z_m = z_{m-1} + p_{z_{m-1}}$ and $z_0 = N$.*

Proof. Let $z_m = z_{m-1} + p_{z_{m-1}}$ and $z_0 = N$. Using induction, we prove that given $m \in \mathbb{N}$ the following holds: $x_{z_m-1} \geq D^m x_{N-1}$ and x_{z_m-i} satisfies H_i for all $i \in \{1, \ldots, k\}$. By assumption, x_{N-i} satisfies H_i for all $i \in \{1, \ldots, k\}$ and $x_{N-1} \geq D^0 x_{N-1}$. This provides the base case. Assume $x_{z_{m-1}-i}$ satisfies H_i for all $i \in \{1, \ldots, k\}$ and $x_{z_{m-1}-1} \geq D^{m-1}x_{N-1}$. Using our earlier assumption, this implies that there exists $p_{z_{m-1}}$ so that $x_{z_{m-1}+p_{z_{m-1}}-i}$ satisfies H_i for all $i \in \{1, \ldots, k\}$ and $x_{z_{m-1}+p_{z_{m-1}}-1} \geq Dx_{z_{m-1}-1} \geq (D)D^{m-1}x_{N-1} = D^m x_{N-1}$. So we have shown that $x_{z_m-1} \geq D^m x_{N-1}$ for all $m \in \mathbb{N}$. Hence the subsequence $\{x_{z_m-1}\}_{m=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ clearly diverges to ∞ since $D > 1$. □

The above argument merely simplifies the following arguments by removing a somewhat onerous construction.

Theorem 3.2. *If* $\alpha > 0$, *then Equation* (3.1) *has unbounded solutions for some initial conditions.*

Proof. We choose initial conditions so that

$$x_0 > \max\left\{\frac{2^{15}}{\alpha^3}, \frac{(\alpha+1)^4 2^{15}}{\alpha}\right\},$$

$$x_{-1} > \max\left\{\frac{2^{11}}{\alpha^2}, (\alpha+1)^2 2^{11}\right\},$$

$$x_{-2} > \frac{\alpha}{2}.$$

We show that there exists $D = \frac{4}{3}$ so that for all $n \in \mathbb{N}$ there exists $p_n \in \{7, 8\}$ so that the following holds: Whenever

$$x_{n-1} > \max\left\{\frac{2^{15}}{\alpha^3}, \frac{(\alpha+1)^4 2^{15}}{\alpha}\right\},$$

$$x_{n-2} > \max\left\{\frac{2^{11}}{\alpha^2}, (\alpha+1)^2 2^{11}\right\},$$

$$x_{n-3} > \frac{\alpha}{2},$$

then we have

$$x_{n+p_n-1} > \max\left\{\frac{2^{15}}{\alpha^3}, \frac{(\alpha+1)^4 2^{15}}{\alpha}\right\},$$

$$x_{n+p_n-2} > \max\left\{\frac{2^{11}}{\alpha^2}, (\alpha+1)^2 2^{11}\right\},$$

$$x_{n+p_n-3} > \frac{\alpha}{2},$$

$$x_{n+p_n-1} \geq \left(\frac{4}{3}\right) x_{n-1}.$$

First assume

$$x_{n-1} > \max\left\{\frac{2^{15}}{\alpha^3}, \frac{(\alpha+1)^4 2^{15}}{\alpha}\right\},$$

$$x_{n-2} > \max\left\{\frac{2^{11}}{\alpha^2}, (\alpha+1)^2 2^{11}\right\},$$

$$x_{n-3} > \frac{\alpha}{2}.$$

Since $x_{n-1}, x_{n-2}, x_{n-3} > 0$, we may write $\eta = \log_2(x_{n-1})$, $\ell = \log_2(x_{n-2})$, and $\rho = \log_2(x_{n-3})$. Hence $2^\eta = x_{n-1}$, $2^\ell = x_{n-2}$, and $2^\rho = x_{n-3}$. We use such representations for ease of computations. First we see that

$$x_n = \frac{\alpha + x_{n-1}}{x_{n-3}} = \frac{\alpha}{x_{n-3}} + \frac{x_{n-1}}{x_{n-3}} = \frac{\alpha}{2^\rho} + 2^{\eta-\rho}, \tag{3.2}$$

$$x_{n+1} = \frac{\alpha}{x_{n-2}} + \frac{x_n}{x_{n-2}} = \frac{\alpha}{2^\ell} + \left(\frac{\alpha}{2^\rho} + 2^{\eta-\rho}\right)\frac{1}{2^\ell} = \frac{\alpha}{2^\ell} + \frac{\alpha}{2^{\ell+\rho}} + \frac{1}{2^{\ell+\rho-\eta}}, \tag{3.3}$$

$$\begin{aligned} x_{n+2} &= \frac{\alpha}{x_{n-1}} + \frac{x_{n+1}}{x_{n-1}} = \frac{\alpha}{2^\eta} + \left(\frac{\alpha}{2^\ell} + \frac{\alpha}{2^{\ell+\rho}} + \frac{1}{2^{\ell+\rho-\eta}}\right)\left(\frac{1}{2^\eta}\right) \\ &= \frac{\alpha}{2^\eta} + \frac{\alpha}{2^{\eta+\ell}} + \frac{\alpha}{2^{\eta+\ell+\rho}} + \frac{1}{2^{\ell+\rho}}, \end{aligned} \tag{3.4}$$

$$\begin{aligned} x_{n+3} &= \frac{\alpha}{x_n} + \frac{x_{n+2}}{x_n} \\ &= \frac{\alpha 2^\rho}{\alpha + 2^\eta} + \left(\frac{\alpha}{2^\eta} + \frac{\alpha}{2^{\eta+\ell}} + \frac{\alpha}{2^{\eta+\ell+\rho}} + \frac{1}{2^{\ell+\rho}}\right)\left(\frac{2^\rho}{\alpha + 2^\eta}\right). \end{aligned} \tag{3.5}$$

We will make use of these identities later. We prove the result in two cases. Let us first assume $\ell + \rho \geq \eta$. We show that if this inequality is satisfied for some $n \in \mathbb{N}$, then $p_n = 7$. First we prove that $x_{n+p_n-3} = x_{n+4} > \frac{\alpha}{2}$. Notice that

$$x_{n+4} = \frac{\alpha + x_{n+3}}{x_{n+1}} > \frac{\alpha}{x_{n+1}}.$$

From (3.3) we see that

$$\frac{\alpha}{x_{n+1}} = \frac{\alpha}{\frac{\alpha}{2^\ell} + \frac{\alpha}{2^{\ell+\rho}} + \frac{1}{2^{\ell+\rho-\eta}}} = \frac{\alpha}{\alpha 2^{-\ell} + \alpha 2^{-\ell-\rho} + 2^{\eta-\ell-\rho}}.$$

We now use the assumption $\ell + \rho \geq \eta$. This assumption implies that $2^{\eta-\ell-\rho} \leq 2^0 = 1$. Earlier we assumed that $2^{-\rho} < \frac{2}{\alpha}$. Moreover, from our assumptions,

$$2^\ell > (\alpha+1)^2 2^{11} = (\alpha^2 + 2\alpha + 1)2^{11}$$

so $2^{-\ell} < \frac{1}{\alpha 2^{12}}$ and $2^{-\ell} < 2^{-11}$. Using these inequalities we obtain

$$\frac{\alpha}{\alpha 2^{-\ell} + \alpha 2^{-\ell-\rho} + 2^{\eta-\ell-\rho}} > \frac{\alpha}{\alpha\frac{1}{\alpha 2^{12}} + \alpha 2^{-11}\frac{2}{\alpha} + 1} = \frac{\alpha}{2^{-12} + 2^{-10} + 1} > \frac{\alpha}{2}.$$

So we have shown $x_{n+p_n-3} = x_{n+4} > \frac{\alpha}{2}$.

We now prove that $x_{n+p_n-2} = x_{n+5} > \max\left\{\frac{2^{11}}{\alpha^2}, (\alpha+1)^2 2^{11}\right\}$. Notice that

$$x_{n+5} = \frac{\alpha + x_{n+4}}{x_{n+2}} > \frac{x_{n+4}}{x_{n+2}} > \frac{\alpha}{2x_{n+2}}.$$

Since $\ell + \rho \geq \eta$, $2^{\ell+\rho} \geq 2^\eta$. Moreover as we have recently shown $2^\ell > 2^{11}$, similarly $2^\eta > 2^{15}$. So $\ell > 11 > 0$ and $\eta > 15 > 0$. So from (3.4),

$$x_{n+2} = \frac{\alpha}{2^\eta} + \frac{\alpha}{2^{\eta+\ell}} + \frac{\alpha}{2^{\eta+\ell+\rho}} + \frac{1}{2^{\ell+\rho}} < \frac{\alpha}{2^\eta} + \frac{\alpha}{2^\eta} + \frac{\alpha}{2^\eta} + \frac{1}{2^\eta} = \frac{3\alpha+1}{2^\eta}. \tag{3.6}$$

Hence,

$$x_{n+5} > \frac{\alpha}{2x_{n+2}} > \frac{\alpha}{2\frac{3\alpha+1}{2^\eta}} = \frac{\alpha}{3\alpha+1}2^{\eta-1} > \left(\frac{1}{3\alpha+1}\right)\max\left\{\frac{2^{14}}{\alpha^2}, (\alpha+1)^4 2^{14}\right\}.$$

So,

$$\begin{aligned} x_{n+5} &> \left(\frac{1}{3\alpha+1}\right)\max\left\{\frac{2^{14}}{\alpha^2}, (\alpha+1)^4 2^{14}\right\} \geq \frac{(\alpha+1)^4 2^{14}}{3\alpha+1} > \frac{(\alpha+1)^4 2^{14}}{3\alpha+3} \\ &= \frac{(\alpha+1)^3 2^{14}}{3} > (\alpha+1)^2 2^{11}. \end{aligned}$$

When $\alpha \geq 1$, $\dfrac{1}{\alpha^2} \leq 1 < (\alpha+1)^2$ so

$$(\alpha+1)^2 2^{11} = \max\left\{\frac{2^{11}}{\alpha^2}, (\alpha+1)^2 2^{11}\right\}.$$

Thus the only remaining case is when $\alpha < 1$. In this case we have the following:

$$x_{n+5} > \left(\frac{1}{3\alpha+1}\right)\max\left\{\frac{2^{14}}{\alpha^2}, (\alpha+1)^4 2^{14}\right\} \geq \frac{2^{14}}{(3\alpha+1)\alpha^2} > \frac{2^{14}}{4\alpha^2} > \frac{2^{11}}{\alpha^2}.$$

So we have shown $x_{n+p_n-2} = x_{n+5} > \max\left\{\dfrac{2^{11}}{\alpha^2}, (\alpha+1)^2 2^{11}\right\}$.

We now prove that $x_{n+p_n-1} = x_{n+6} \geq (\frac{4}{3})x_{n-1} > \max\left\{\dfrac{2^{15}}{\alpha^3}, \dfrac{(\alpha+1)^4 2^{15}}{\alpha}\right\}$. First assume

$$\max\left\{(\alpha+1)2^5, \frac{(\alpha+1)2^5}{\alpha}\right\} \geq 2^{\eta-\rho}. \tag{3.7}$$

Notice that

$$x_{n+6} = \frac{\alpha + x_{n+5}}{x_{n+3}} > \frac{x_{n+5}}{x_{n+3}} = \frac{\alpha + x_{n+4}}{x_{n+2}x_{n+3}} > \frac{x_{n+4}}{x_{n+2}x_{n+3}} = \frac{\alpha + x_{n+3}}{x_{n+2}x_{n+3}x_{n+1}} > \frac{1}{x_{n+2}x_{n+1}}.$$

We use (3.3), our induction assumption, our assumption (3.7), and the fact that $2^{-\rho} < \dfrac{2}{\alpha}$ to obtain

$$x_{n+1} = \frac{\alpha}{2^\ell} + \frac{\alpha}{2^{\ell+\rho}} + \frac{2^{\eta-\rho}}{2^\ell} < \frac{\alpha}{2^\ell} + \frac{1}{2^{\ell-1}} + \frac{\max\left\{(\alpha+1)2^5, \frac{(\alpha+1)2^5}{\alpha}\right\}}{\max\left\{\frac{2^{11}}{\alpha^2}, (\alpha+1)^2 2^{11}\right\}}.$$

Notice that if $\alpha \geq 1$,

$$\frac{\max\left\{(\alpha+1)2^5, \frac{(\alpha+1)2^5}{\alpha}\right\}}{\max\left\{\frac{2^{11}}{\alpha^2}, (\alpha+1)^2 2^{11}\right\}} \leq \frac{(\alpha+1)2^5}{(\alpha+1)^2 2^{11}} = \frac{1}{(\alpha+1)2^6} < \frac{1}{(\alpha+1)2^3}.$$

Also if $\alpha < 1$,

$$\frac{\max\left\{(\alpha+1)2^5, \frac{(\alpha+1)2^5}{\alpha}\right\}}{\max\left\{\frac{2^{11}}{\alpha^2}, (\alpha+1)^2 2^{11}\right\}} \leq \frac{(\alpha+1)2^5}{\alpha\left(\frac{2^{11}}{\alpha^2}\right)} = \frac{(\alpha+1)\alpha}{2^6} < \frac{(\alpha+1)^2}{2^6} < \frac{1}{(\alpha+1)2^3}.$$

So

$$\begin{aligned} x_{n+1} &< \frac{\alpha}{2^\ell} + \frac{1}{2^{\ell-1}} + \frac{1}{(\alpha+1)2^3} < \frac{\alpha}{(\alpha+1)^2 2^{11}} + \frac{1}{(\alpha+1)^2 2^{10}} + \frac{1}{(\alpha+1)2^3} \\ &< \frac{1}{(\alpha+1)2^{11}} + \frac{1}{(\alpha+1)2^{10}} + \frac{1}{(\alpha+1)2^3} = \frac{1}{\alpha+1}(2^{-11} + 2^{-10} + 2^{-3}) \\ &< \frac{1}{4\alpha+4}. \end{aligned}$$

Now using the inequality we have just shown and (3.6), we have

$$x_{n+6} > \frac{1}{x_{n+1}x_{n+2}} > \frac{4\alpha+4}{3\alpha+1}(2^\eta) > \frac{4\alpha+4}{3\alpha+3}(2^\eta) = \left(\frac{4}{3}\right) x_{n-1}.$$

Thus we have shown $x_{n+p_n-1} = x_{n+6} \geq \left(\frac{4}{3}\right) x_{n-1} > \max\left\{\frac{2^{15}}{\alpha^3}, \frac{(\alpha+1)^4 2^{15}}{\alpha}\right\}$ when (3.7) holds. Now assume the opposite inequality in (3.7). Using (3.5) and (3.6), we have the following:

$$\begin{aligned} x_{n+3} &= \frac{\alpha 2^\rho}{\alpha+2^\eta} + x_{n+2}\left(\frac{2^\rho}{\alpha+2^\eta}\right) < \frac{\alpha 2^\rho}{\alpha+2^\eta} + \left(\frac{3\alpha+1}{2^\eta}\right)\left(\frac{2^\rho}{\alpha+2^\eta}\right) \\ &< 2^{\rho-\eta}\left(\alpha + \frac{3\alpha+1}{2^\eta}\right). \end{aligned}$$

So

$$\begin{aligned} x_{n+6} &> \frac{x_{n+5}}{x_{n+3}} > \frac{\alpha + x_{n+4}}{x_{n+3}x_{n+2}} > \frac{x_{n+4}}{x_{n+3}x_{n+2}} \\ &> \frac{\alpha}{x_{n+3}x_{n+2}x_{n+1}} > \frac{\alpha}{2^{\rho-\eta}\left(\alpha + \frac{3\alpha+1}{2^\eta}\right) x_{n+2}x_{n+1}}. \end{aligned}$$

Since $2^\eta > \frac{(\alpha+1)^4 2^{15}}{\alpha} > (\alpha+1)^3 2^{15} > 3\alpha+1$, we see that $\frac{3\alpha+1}{2^\eta} < 1$ holds, and using (3.3), we get

$$x_{n+6} > \frac{\alpha}{2^{\rho-\eta}(\alpha+1)x_{n+2}x_{n+1}} = \frac{\alpha}{2^{\rho-\eta}(\alpha+1)x_{n+2}\left(\frac{\alpha}{2^\ell} + \frac{\alpha}{2^{\ell+\rho}} + \frac{1}{2^{\ell+\rho-\eta}}\right)}.$$

Distributing the $2^{\rho-\eta}$, we have

$$x_{n+6} > \frac{\alpha}{(\alpha+1)x_{n+2}\left(\frac{\alpha}{2^{\ell+\eta-\rho}} + \frac{\alpha}{2^{\ell+\eta}} + \frac{1}{2^{\ell}}\right)}. \tag{3.8}$$

Now let us assume $\alpha \geq 1$. Then we have

$$x_{n+6} > \frac{\alpha}{(\alpha+1)x_{n+2}\left(\frac{\alpha}{2^{\ell+\eta-\rho}} + \frac{\alpha}{2^{\ell+\eta}} + \frac{\alpha}{2^{\ell}}\right)} = \frac{2^{\ell}}{(\alpha+1)x_{n+2}\left(\frac{1}{2^{\eta-\rho}} + \frac{1}{2^{\eta}} + 1\right)}.$$

In this case $2^{\ell} > \max\left\{\dfrac{2^{11}}{\alpha^2}, (\alpha+1)^2 2^{11}\right\} \geq (\alpha+1)^2 2^{11}$, so

$$x_{n+6} > \frac{2^{11}(\alpha+1)}{x_{n+2}\left(1 + \frac{1}{2^{\eta-\rho}} + \frac{1}{2^{\eta}}\right)}.$$

Since we assumed the reversed inequality in (3.7), we have that $2^{\eta-\rho} > 2^5 > 1$. Furthermore we know from earlier that $2^{\eta} > 2^{15} > 1$. Using this information, we obtain

$$x_{n+6} > \frac{2^{11}(\alpha+1)}{x_{n+2}\left(1 + \frac{1}{2^{\eta-\rho}} + \frac{1}{2^{\eta}}\right)} > \frac{2^{11}(\alpha+1)}{3x_{n+2}}.$$

Now we use (3.6), and we obtain

$$x_{n+6} > \frac{2^{11}(\alpha+1)}{3(3\alpha+1)}x_{n-1} > \frac{2^{11}(\alpha+1)}{3(3\alpha+3)}x_{n-1} = \left(\frac{2^{11}}{9}\right)x_{n-1} > \left(\frac{4}{3}\right)x_{n-1}.$$

We now prove the case when $\alpha < 1$. Here we continue from (3.8) with the following:

$$x_{n+6} > \frac{\alpha}{(\alpha+1)x_{n+2}\left(\frac{1}{2^{\ell+\eta-\rho}} + \frac{1}{2^{\ell+\eta}} + \frac{1}{2^{\ell}}\right)} = \frac{\alpha 2^{\ell}}{(\alpha+1)x_{n+2}\left(\frac{1}{2^{\eta-\rho}} + \frac{1}{2^{\eta}} + 1\right)}.$$

In this case $2^{\ell} > \max\left\{\dfrac{2^{11}}{\alpha^2}, (\alpha+1)^2 2^{11}\right\} \geq \dfrac{2^{11}}{\alpha}$, so we have

$$x_{n+6} > \frac{2^{11}}{(\alpha+1)x_{n+2}\left(1 + \frac{1}{2^{\eta-\rho}} + \frac{1}{2^{\eta}}\right)}.$$

Since we assumed the reverse inequality in (3.7), we have that $2^{\eta-\rho} > 2^5 > 1$. Furthermore we know from earlier that $2^{\eta} > 2^{15} > 1$. Using this information, we obtain

$$x_{n+6} > \frac{2^{11}}{(\alpha+1)x_{n+2}\left(1 + \frac{1}{2^{\eta-\rho}} + \frac{1}{2^{\eta}}\right)} > \frac{2^{11}}{3(\alpha+1)x_{n+2}}.$$

Now we use (3.6) and the assumption $\alpha < 1$ to obtain

$$x_{n+6} > \frac{2^{11}}{3(3\alpha+1)(\alpha+1)}(x_{n-1}) > \frac{2^{11}}{24}x_{n-1} > \left(\frac{4}{3}\right)x_{n-1}.$$

Thus we have shown $x_{n+p_n-1} = x_{n+6} \geq \left(\frac{4}{3}\right) x_{n-1} > \max\left\{\frac{2^{15}}{\alpha^3}, \frac{(\alpha+1)^4 2^{15}}{\alpha}\right\}$ when the opposite inequality of (3.7) holds. Therefore we have finished the case where $\ell+\rho \geq \eta$.

We now consider the case $\ell + \rho < \eta$. We show that if this inequality is satisfied for some $n \in \mathbb{N}$, then $p_n = 8$. First we prove that $x_{n+p_n-3} = x_{n+5} > \frac{\alpha}{2}$. Notice that since our assumptions have changed, (3.6) and (3.8) no longer hold. We will now make a new analogue for (3.6), namely the forthcoming (3.9). Since $\ell + \rho < \eta$, we have $2^{\ell+\rho} < 2^{\eta}$. Moreover, since $2^{\ell} > 2^{11}$ and $2^{\eta} > 2^{15}$, we have $\ell > 0$ and $\eta > 0$. So from (3.4),

$$x_{n+2} = \frac{\alpha}{2^{\eta}} + \frac{\alpha}{2^{\eta+\ell}} + \frac{\alpha}{2^{\eta+\ell+\rho}} + \frac{1}{2^{\ell+\rho}} < \frac{\alpha}{2^{\ell+\rho}} + \frac{\alpha}{2^{\ell+\rho}} + \frac{\alpha}{2^{\ell+\rho}} + \frac{1}{2^{\ell+\rho}} = \frac{3\alpha+1}{2^{\ell+\rho}}. \quad (3.9)$$

So we have

$$x_{n+5} = \frac{\alpha + x_{n+4}}{x_{n+2}} > \frac{\alpha}{x_{n+2}} > \frac{\alpha 2^{\ell+\rho}}{3\alpha+1}.$$

Notice that

$$\begin{aligned} \frac{\alpha 2^{\ell}}{3\alpha+1} &> \max\left\{\frac{\alpha 2^{11}}{(3\alpha+1)\alpha^2}, \frac{\alpha(\alpha+1)^2 2^{11}}{3\alpha+1}\right\} \\ &\geq \max\left\{\frac{2^{11}}{(3\alpha+1)\alpha}, \frac{\alpha(\alpha+1)^2 2^{11}}{3\alpha+3}\right\} \\ &= \max\left\{\frac{2^{11}}{(3\alpha+1)\alpha}, \frac{\alpha(\alpha+1) 2^{11}}{3}\right\} > 1. \end{aligned}$$

So $x_{n+5} > 2^{\rho} > \frac{\alpha}{2}$.

We now prove that $x_{n+p_n-2} = x_{n+6} > \max\left\{\frac{2^{11}}{\alpha^2}, (\alpha+1)^2 2^{11}\right\}$. Notice that from (3.5), we have

$$x_{n+3} = \frac{\alpha 2^{\rho}}{\alpha + 2^{\eta}} + x_{n+2}\left(\frac{2^{\rho}}{\alpha+2^{\eta}}\right) < 2^{\rho-\eta}(\alpha + x_{n+2}). \quad (3.10)$$

Since $x_{n+5} > 2^{\rho}$, we get

$$x_{n+6} > \frac{x_{n+5}}{x_{n+3}} > 2^{\eta-\rho}\frac{x_{n+5}}{\alpha + x_{n+2}} > \frac{2^{\eta}}{\alpha + x_{n+2}}. \quad (3.11)$$

We assume $\frac{1}{\alpha} \leq \alpha + 1$ and we use (3.9). We know that $2^{\rho} > \frac{\alpha}{2}$ and $2^{\ell} > (\alpha+1)^2 2^{11} \geq \frac{(\alpha+1)2^{11}}{\alpha}$. So,

$$x_{n+2} < \frac{3\alpha+1}{2^{\ell+\rho}} < \frac{3\alpha+1}{\frac{(\alpha+1)2^{11}}{\alpha}\left(\frac{\alpha}{2}\right)} = \frac{3\alpha+1}{(\alpha+1)2^{10}} < \frac{3\alpha+3}{(\alpha+1)2^{10}} < 2^{-8}.$$

So, since $\dfrac{1}{\alpha} \leq \alpha + 1$,

$$\begin{aligned} x_{n+6} &> \frac{2^{\eta}}{\alpha + x_{n+2}} > \frac{2^{\eta}}{\alpha + 2^{-8}} > \max\left\{\frac{2^{15}}{(\alpha + 2^{-8})\alpha^3}, \frac{(\alpha+1)^4 2^{15}}{(\alpha + 2^{-8})\alpha}\right\} \\ &\geq \frac{(\alpha+1)^4 2^{15}}{(\alpha + 2^{-8})\alpha} > \frac{(\alpha+1)^4 2^{15}}{(\alpha+1)^2} = (\alpha+1)^2 2^{15} > (\alpha+1)^2 2^{11} \geq \frac{2^{11}}{\alpha^2}. \end{aligned}$$

We now assume $\dfrac{1}{\alpha} > \alpha + 1$ and we use (3.9). We know that $2^{\rho} > \dfrac{\alpha}{2}$ and $2^{\ell} > \dfrac{2^{11}}{\alpha^2} \geq \dfrac{(\alpha+1)2^{11}}{\alpha}$. So,

$$x_{n+2} < \frac{3\alpha+1}{2^{\ell+\rho}} < \frac{3\alpha+1}{\frac{(\alpha+1)2^{11}}{\alpha}\left(\frac{\alpha}{2}\right)} = \frac{3\alpha+1}{(\alpha+1)2^{10}} < \frac{3\alpha+3}{(\alpha+1)2^{10}} < 2^{-8}.$$

We now use (3.11) and our assumption $\dfrac{1}{\alpha} > \alpha + 1$ to obtain the following:

$$\begin{aligned} x_{n+6} &> \frac{2^{\eta}}{\alpha + x_{n+2}} > \frac{2^{\eta}}{\alpha + 2^{-8}} > \max\left\{\frac{2^{15}}{(\alpha + 2^{-8})\alpha^3}, \frac{(\alpha+1)^4 2^{15}}{(\alpha + 2^{-8})\alpha}\right\} \\ &\geq \frac{2^{15}}{(\alpha + 2^{-8})\alpha^3} > \frac{(\alpha+1)2^{15}}{(\alpha+1)\alpha^2} = \frac{2^{15}}{\alpha^2} > \frac{2^{11}}{\alpha^2} > (\alpha+1)^2 2^{11}. \end{aligned}$$

Thus we have shown that $x_{n+p_n-2} = x_{n+6} > \max\left\{\dfrac{2^{11}}{\alpha^2}, (\alpha+1)^2 2^{11}\right\}$.

Now we prove $x_{n+p_n-1} = x_{n+7} \geq \left(\dfrac{4}{3}\right) x_{n-1} > \max\left\{\dfrac{2^{15}}{\alpha^3}, \dfrac{(\alpha+1)^4 2^{15}}{\alpha}\right\}$. Notice that

$$x_{n+7} = \frac{\alpha + x_{n+6}}{x_{n+4}} > \frac{x_{n+6}}{x_{n+4}} = \frac{\alpha + x_{n+5}}{x_{n+4}x_{n+3}} > \frac{x_{n+5}}{x_{n+4}x_{n+3}} = \frac{\alpha + x_{n+4}}{x_{n+2}x_{n+3}x_{n+4}} > \frac{1}{x_{n+2}x_{n+3}}.$$

Using (3.9) and (3.10), we have

$$x_{n+7} > \frac{1}{x_{n+2}x_{n+3}} > \frac{2^{\ell+\rho}}{(3\alpha+1)x_{n+3}} > \frac{2^{\ell+\eta}}{(3\alpha+1)(\alpha + x_{n+2})}.$$

Earlier we demonstrated that $x_{n+2} < 2^{-8}$. Furthermore we have assumed that $2^{\ell} > (\alpha+1)^2 2^{11}$. Thus,

$$\begin{aligned} x_{n+7} &> \frac{2^{\ell+\eta}}{(3\alpha+1)(\alpha + x_{n+2})} > \frac{2^{\ell+\eta}}{(3\alpha+3)(\alpha+1)} > \frac{(\alpha+1)^2 2^{11} 2^{\eta}}{(3\alpha+3)(\alpha+1)} \\ &= \left(\frac{2^{11}}{3}\right) x_{n-1} > \left(\frac{4}{3}\right) x_{n-1}. \end{aligned}$$

Hence $x_{n+p_n-1} = x_{n+7} \geq \left(\dfrac{4}{3}\right) x_{n-1} > \max\left\{\dfrac{2^{15}}{\alpha^3}, \dfrac{(\alpha+1)^4 2^{15}}{\alpha}\right\}$. Now we apply Lemma 3.1, and then the proof is done. □

4 Special Case #28

We now study special case #28

$$x_n = \frac{x_{n-1}}{Cx_{n-2} + x_{n-3}}, \quad n \in \mathbb{N}. \tag{4.1}$$

Particularly, we show that whenever $C > 0$, Equation (4.1) has unbounded solutions for some initial conditions.

Theorem 4.1. *If $C > 0$, then Equation* (4.1) *has unbounded solutions for some initial conditions.*

Proof. We choose initial conditions so that

$$x_0 > \max\left\{1000(C+1)^3, \frac{1000(C+1)^3}{C}, \frac{100(C+1)^3}{C^3}, 100(C^2+C)\right\},$$

$$x_{-1} > \max\left\{10, \frac{10}{C}, \frac{1}{C^3}\right\}.$$

We show that there exists $D = 2$ so that for all $n \in \mathbb{N}$ there exists $p_n \in \{7, 8\}$ so that the following holds: Whenever

$$x_{n-1} > \max\left\{1000(C+1)^3, \frac{1000(C+1)^3}{C}, \frac{100(C+1)^3}{C^3}, 100(C^2+C)\right\},$$

$$x_{n-2} > \max\left\{10, \frac{10}{C}, \frac{1}{C^3}\right\},$$

then we have

$$x_{n+p_n-1} > \max\left\{1000(C+1)^3, \frac{1000(C+1)^3}{C}, \frac{100(C+1)^3}{C^3}, 100(C^2+C)\right\},$$

$$x_{n+p_n-2} > \max\left\{10, \frac{10}{C}, \frac{1}{C^3}\right\},$$

$$x_{n+p_n-1} \geq 2x_{n-1}.$$

First assume

$$x_{n-1} > \max\left\{1000(C+1)^3, \frac{1000(C+1)^3}{C}, \frac{100(C+1)^3}{C^3}, 100(C^2+C)\right\},$$

$$x_{n-2} > \max\left\{10, \frac{10}{C}, \frac{1}{C^3}\right\}.$$

Using algebra, we immediately obtain the following:

$$x_n = \frac{x_{n-1}}{Cx_{n-2} + x_{n-3}} < \frac{x_{n-1}}{Cx_{n-2}} < \frac{x_{n-1}}{10},$$

$$x_{n+1} = \frac{x_n}{Cx_{n-1} + x_{n-2}} < \frac{x_n}{x_{n-2}} < \frac{x_n}{10},$$

$$x_{n+1} = \frac{x_n}{Cx_{n-1} + x_{n-2}} < \frac{x_n}{Cx_{n-1}} < \frac{x_n}{1000C},$$

$$x_n = \frac{x_{n-1}}{Cx_{n-2} + x_{n-3}} < \frac{x_{n-1}}{Cx_{n-2}} < \frac{x_{n-1}}{10C},$$

$$x_{n+2} = \frac{x_{n+1}}{Cx_n + x_{n-1}} < \frac{x_{n+1}}{x_{n-1}} < \frac{x_{n+1}}{100C},$$

$$x_{n+2} = \frac{x_{n+1}}{Cx_n + x_{n-1}} < \frac{x_{n+1}}{x_{n-1}} < \frac{x_{n+1}}{1000}.$$

So we get the following inequalities:

$$100000x_{n+2} < 100x_{n+1} < 10x_n < x_{n-1}, \tag{4.2}$$

$$1000000C^3x_{n+2} < 10000C^2x_{n+1} < 10Cx_n < x_{n-1}. \tag{4.3}$$

Using (4.3) we get

$$\frac{x_{n+2}}{x_{n+3}} = Cx_{n+1} + x_n < 2x_n. \tag{4.4}$$

We use (4.2) and (4.3) to get

$$\begin{aligned}
Cx_{n+3} + x_{n+2} &= x_{n+2}\left(1 + \frac{C}{Cx_{n+1} + x_n}\right) < x_{n+2}\left(1 + \frac{1}{x_{n+1}}\right) \\
&< x_{n+2}\left(1 + \frac{1}{1000x_{n+2}}\right) \\
&= x_{n+2} + \frac{1}{1000} = \frac{x_n}{(Cx_n + x_{n-1})(Cx_{n-1} + x_{n-2})} + \frac{1}{1000} \\
&< \frac{1}{Cx_{n-2}} + \frac{1}{1000} < \frac{1}{10} + \frac{1}{1000} < 1.
\end{aligned}$$

In short,

$$Cx_{n+3} + x_{n+2} < 1.$$

Using (4.3) and (4.4), we have

$$x_{n+5} = \frac{x_{n+4}}{Cx_{n+3} + x_{n+2}} = \frac{1}{Cx_{n+2} + x_{n+1}}\left(\frac{x_{n+3}}{Cx_{n+3} + x_{n+2}}\right),$$

$$x_{n+5} = \frac{1}{Cx_{n+2} + x_{n+1}}\left(\frac{1}{C + \frac{x_{n+2}}{x_{n+3}}}\right) > \frac{1}{2x_{n+1}}\left(\frac{1}{C + 2x_n}\right). \tag{4.5}$$

Furthermore we have

$$x_{n+4} = \frac{x_{n+3}}{Cx_{n+2} + x_{n+1}} < \frac{x_{n+3}}{x_{n+1}} = \frac{1}{(Cx_{n+1} + x_n)(Cx_n + x_{n-1})} < \frac{1}{x_{n-1}x_n}. \quad (4.6)$$

Using (4.5) and (4.6), we obtain

$$\begin{aligned} x_{n+6} &= \frac{x_{n+5}}{Cx_{n+4} + x_{n+3}} > \left(\frac{1}{2x_{n+1}}\right)\left(\frac{1}{C + 2x_n}\right)\left(\frac{1}{\frac{C}{x_{n-1}x_n} + x_{n+3}}\right) \\ &= \left(\frac{x_{n-1}}{2x_{n+1}}\right)\left(\frac{1}{C + 2x_n}\right)\left(\frac{1}{\frac{C}{x_n} + x_{n-1}x_{n+3}}\right) \\ &= \left(\frac{Cx_{n-1} + x_{n-2}}{2C + 4x_n}\right)\left(\frac{Cx_{n-2} + x_{n-3}}{\frac{C}{x_n} + x_{n-1}x_{n+3}}\right) \\ &= \left(\frac{(Cx_{n-1} + x_{n-2})^2}{2C + 4x_n}\right)\left(\frac{Cx_{n-2} + x_{n-3}}{\frac{C}{x_{n+1}} + x_{n-1}x_{n+3}(Cx_{n-1} + x_{n-2})}\right). \end{aligned}$$

Using (4.2) and (4.3), we see

$$\begin{aligned} x_{n-1}x_{n+3}(Cx_{n-1} + x_{n-2}) &= \frac{x_{n-1}x_{n+2}(Cx_{n-1} + x_{n-2})}{Cx_{n+1} + x_n} \\ &= \frac{x_{n-1}x_{n+1}(Cx_{n-1} + x_{n-2})}{(Cx_{n+1} + x_n)(Cx_n + x_{n-1})} \\ &= \frac{x_{n-1}x_n}{(Cx_{n+1} + x_n)(Cx_n + x_{n-1})} \\ &< 1 < C^3 x_{n-2} < \frac{C(Cx_{n-1} + x_{n-2})(Cx_{n-2} + x_{n-3})}{x_{n-1}} \\ &= \frac{C}{x_{n+1}}. \end{aligned}$$

Using this in the prior inequality, we get

$$x_{n+6} > \frac{x_{n-1}(Cx_{n-1} + x_{n-2})}{4C^2 + 8Cx_n}. \quad (4.7)$$

Using this fact, we have

$$\begin{aligned} x_{n+7} &= \frac{x_{n+6}}{Cx_{n+5} + x_{n+4}} = \frac{x_{n+6}}{\left(\frac{Cx_{n+4}}{Cx_{n+3} + x_{n+2}}\right) + x_{n+4}} = \left(\frac{x_{n+6}}{x_{n+4}}\right)\left(\frac{1}{1 + \frac{C}{Cx_{n+3} + x_{n+2}}}\right) \\ &= \left(\frac{x_{n+4}}{Cx_{n+3} + x_{n+2}}\right)\left(\frac{1}{Cx_{n+4} + x_{n+3}}\right)\left(\frac{1}{x_{n+4}}\right)\left(\frac{1}{1 + \frac{C}{Cx_{n+3} + x_{n+2}}}\right) \end{aligned}$$

$$\begin{aligned} &= \left(\frac{1}{C + Cx_{n+3} + x_{n+2}}\right)\left(\frac{1}{Cx_{n+4} + x_{n+3}}\right) \\ &> \left(\frac{1}{Cx_{n+4} + x_{n+3}}\right)\left(\frac{1}{C+1}\right). \end{aligned}$$

We now use (4.2) and (4.3) to show

$$\begin{aligned} x_{n+3} &= \frac{x_{n+2}}{Cx_{n+1} + x_n} = \frac{x_{n+1}}{(Cx_{n+1} + x_n)(Cx_n + x_{n-1})} \\ &< \frac{x_{n+1}}{(Cx_{n+1} + 500Cx_{n+1} + 5x_{n+1})(Cx_n + x_{n-1})} \\ &< \frac{1}{(501C + 5)x_{n-1}} < \frac{1}{(4C + 4)x_{n-1}}. \end{aligned}$$

We now use this fact and (4.6) to obtain

$$x_{n+7} > \left(\frac{1}{\frac{C}{x_n x_{n-1}} + \frac{1}{x_{n-1}(4C+4)}}\right)\left(\frac{1}{C+1}\right) = \left(\frac{x_{n-1}}{\frac{C}{x_n} + \frac{1}{4(C+1)}}\right)\left(\frac{1}{C+1}\right). \quad (4.8)$$

Suppose $x_n \leq 4C(C+1)$. We will show that in this case $p_n = 7$. Using (4.5), we have

$$\begin{aligned} x_{n+p_n-2} &= x_{n+5} > \frac{1}{2x_{n+1}}\left(\frac{1}{C + 2x_n}\right) \\ &= \frac{Cx_{n-1} + x_{n-2}}{2x_n}\left(\frac{1}{C + 2x_n}\right) > \frac{Cx_{n-1}}{2x_n(C + 2x_n)} \\ &\geq \frac{Cx_{n-1}}{8C(C+1)(C + 8C(C+1))} \\ &= \frac{x_{n-1}}{8(C+1)(C + 8C(C+1))} > \frac{x_{n-1}}{100(C+1)^3} > \max\left\{10, \frac{10}{C}, \frac{1}{C^3}\right\}. \end{aligned}$$

Also using (4.7), we have

$$\begin{aligned} x_{n+p_n-1} &= x_{n+6} > \frac{x_{n-1}(Cx_{n-1} + x_{n-2})}{4C^2 + 8Cx_n} > \frac{Cx_{n-1}^2}{4C^2 + 8C(4C(C+1))} \\ &= \frac{x_{n-1}^2}{4C + 8(4C(C+1))} > \frac{x_{n-1}^2}{50(C^2 + C)} > 2x_{n-1}. \end{aligned}$$

Now suppose $x_n > 4C(C+1)$. We will show that in this case $p_n = 8$. Using (4.7), we have

$$\begin{aligned} x_{n+p_n-2} &= x_{n+6} > \frac{x_{n-1}(Cx_{n-1} + x_{n-2})}{4C^2 + 8Cx_n} > \frac{x_{n-1}(Cx_{n-1} + x_{n-2})}{9Cx_n} \\ &= \frac{(Cx_{n-1} + x_{n-2})(Cx_{n-2} + x_{n-3})}{9C} > x_{n-2} > \max\left\{10, \frac{10}{C}, \frac{1}{C^3}\right\}. \end{aligned}$$

Also using (4.8), we have

$$\begin{aligned} x_{n+p_n-1} &= x_{n+7} > \left(\frac{x_{n-1}}{\frac{C}{x_n} + \frac{1}{4(C+1)}}\right)\left(\frac{1}{C+1}\right) \\ &> \left(\frac{x_{n-1}}{\frac{C}{4C(C+1)} + \frac{1}{4(C+1)}}\right)\left(\frac{1}{C+1}\right) = 2x_{n-1}. \end{aligned}$$

Hence after application of Lemma 3.1, the proof is complete. □

5 Conclusion

Theorem 2.1 establishes the boundedness character of the special cases #56 and #120 in a range of their parameters. Theorems 3.2 and 4.1 establish the boundedness character of the special cases #44 and #28 respectively. Further work should focus on expanding the range for which the boundedness character of special cases #56 and #120 is known and resolving [1, Conjecture 3.0.1]. There remains only one special case for which Conjecture 3.0.1 has not yet been established. This is special case #70. It is worthwhile to note that special case #70 is part of the period-six trichotomy conjecture. The resolution of the period-six trichotomy conjecture will immediately resolve [1, Conjecture 3.0.1]. See [2] for more details regarding the period-six trichotomy conjecture. We restate, for the convenience of the reader, the period-six trichotomy conjecture.

Conjecture 5.1. Assume that $\alpha, C \in [0,\infty)$. Then the following period-six trichotomy result is true for the rational equation

$$x_n = \frac{\alpha + x_{n-1}}{Cx_{n-2} + x_{n-3}}, \quad n \in \mathbb{N}. \tag{5.1}$$

(i) Every solution of Equation (5.1) converges to its positive equilibrium if and only if $\alpha C^2 > 1$.

(ii) Every solution of Equation (5.1) converges to a not necessarily prime period-six solution of Equation (5.1) if and only if $\alpha C^2 = 1$.

(iii) Equation (5.1) has unbounded solutions if and only if $\alpha C^2 < 1$.

For more on boundedness character see [1–4].

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