

Time Scales Delta Iyengar-Type Inequalities

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Abstract

Here we give the necessary background on delta time scales approach. Then we present general related time scales delta Iyengar type inequalities for all basic norms. We finish with applications to specific time scales like \mathbb{R} , \mathbb{Z} and $q^{\mathbb{Z}}$, $q > 1$.

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1 Introduction

We are motivated by the following famous Iyengar inequality (1938), [11].

Theorem 1.1. *Let f be a differentiable function on $[a, b]$ and $|f'(x)| \leq M$. Then*

$$\left| \int_a^b f(x) dx - \frac{1}{2}(b-a)(f(a) + f(b)) \right| \leq \frac{M(b-a)^2}{4} - \frac{(f(b) - f(a))^2}{4M}. \quad (1.1)$$

We present generalized analogs of (1.1) to time scales. Motivation comes also from [2–4, 6, 7].

2 Background

Here basics on time scales come from [3, 4, 6–8]. We need the following definition.

Definition 2.1. A time scale is an arbitrary nonempty closed subset of the real numbers, e.g. \mathbb{R} , \mathbb{Z} , $q^{\mathbb{N}_0} = \{q^k | k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, q > 1\}$.

Definition 2.2. If \mathbb{T} is a time scale, then we define the forward jump operator $\sigma : \mathbb{T} \mapsto \mathbb{T}$ by $\sigma(t) = \inf\{s \in \mathbb{T} | s > t\}$, $\forall t \in \mathbb{T}$; the backward jump operator $\rho : \mathbb{T} \mapsto \mathbb{T}$ by $\rho(t) = \sup\{s \in \mathbb{T} | s < t\}$, $\forall t \in \mathbb{T}$; and the graininess function $\mu : \mathbb{T} \rightarrow \mathbb{R}_+ = [0, \infty)$, by $\mu(t) = \sigma(t) - t$, $\forall t \in \mathbb{T}$. Furthermore for a function $f : \mathbb{T} \rightarrow \mathbb{R}$, we define $f^\sigma(t) = f(\sigma(t))$, $\forall t \in \mathbb{T}$; and $f^\rho(t) = f(\rho(t))$, $\forall t \in \mathbb{T}$.

In this definition we use $\inf \emptyset = \sup \mathbb{T}$ (i.e., $\sigma(t) = t$ if t is the maximum of \mathbb{T}) and $\sup \emptyset = \inf \mathbb{T}$ (i.e., $\rho(t) = t$ if t is the minimum of \mathbb{T}).

We call $t \in \mathbb{T}$ right-scattered if $t < \sigma(t)$, $t \in \mathbb{T}$ right-dense if $t = \sigma(t)$, $t \in \mathbb{T}$ left-scattered if $\rho(t) < t$, $t \in \mathbb{T}$ left-dense if $\rho(t) = t$, $t \in \mathbb{T}$ isolated if $\rho(t) < t < \sigma(t)$, $t \in \mathbb{T}$ dense if $\rho(t) = t = \sigma(t)$.

We notice that ρ is an increasing function, so is $\rho^2(t) = \rho(\rho(t))$, \dots , so that $\rho^n(t) = \rho(\rho^{n-1}(t))$ is increasing in t for $n \in \mathbb{N}$. Since \mathbb{T} is closed subset of \mathbb{R} we have that $\sigma(t), \rho(t) \in \mathbb{T}$, for $t \in \mathbb{T}$.

Definition 2.3 (See [8]). A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is called rd-continuous (denoted by C_{rd}) if it is continuous at right-dense points of \mathbb{T} and its left-sided limits are finite at left-dense points of \mathbb{T} .

If $\mathbb{T} = \mathbb{R}$, then $f : \mathbb{R} \rightarrow \mathbb{R}$ is rd-continuous iff f is continuous. Also, if $\mathbb{T} = \mathbb{Z}$, then any function defined on \mathbb{Z} is rd-continuous ([9]).

Definition 2.4 (See [8]). If $\sup \mathbb{T} < \infty$ and $\sup \mathbb{T}$ is left-scattered, we let $\mathbb{T}^k := \mathbb{T} - \{\sup \mathbb{T}\}$, otherwise we let $\mathbb{T}^k := \mathbb{T}$ the time scale.

In summary, $\mathbb{T}^k = \begin{cases} \mathbb{T} - (\rho(\sup \mathbb{T}), \sup \mathbb{T}], & \text{if } \sup \mathbb{T} < \infty, \\ \mathbb{T}, & \text{if } \sup \mathbb{T} = \infty. \end{cases}$

Definition 2.5 (See [8]). Assume $f : \mathbb{T} \rightarrow \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. Then we define $f^\Delta(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t such that

$$|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]| \leq \varepsilon |\sigma(t) - s|, \quad \forall s \in U.$$

We call $f^\Delta(t)$ the delta (or Hilger [10]) derivative of f at t . If $\mathbb{T} = \mathbb{R}$, then $f^\Delta = f'$, whereas if $\mathbb{T} = \mathbb{Z}$, then $f^\Delta(t) = \Delta f(t) = f(t+1) - f(t)$, the usual forward difference operator.

Theorem 2.6 (Existence of Antiderivatives, see [8]). *Let f be rd-continuous. Then f has an antiderivative F satisfying $F^\Delta = f$.*

Definition 2.7 (See [8]). If f is rd-continuous and $t_0 \in \mathbb{T}$, then we define the integral

$$F(t) = \int_{t_0}^t f(\tau) \Delta\tau \quad \text{for } t \in \mathbb{T}.$$

Therefore for $f \in C_{rd}(\mathbb{T})$ we have by definition

$$\int_a^b f(\tau) \Delta\tau = F(b) - F(a),$$

where $F^\Delta = f$.

If $\mathbb{T} = \mathbb{R}$, then

$$\int_a^b f(t) \Delta t = \int_a^b f(t) dt,$$

where the integral on the right hand side is the Riemann integral ([9]).

If every point in \mathbb{T} is isolated and $a < b$ are in \mathbb{T} , then ([9])

$$\int_a^b f(t) \Delta t = \sum_{t=a}^{\rho(b)} f(t) \mu(t).$$

Theorem 2.8 (See [8]). *Let f, g be rd-continuous on \mathbb{T} , $a, b, c \in \mathbb{T}$ and $\alpha, \beta \in \mathbb{R}$. Then*

- (1) $\int_a^b (\alpha f(t) + \beta g(t)) \Delta t = \alpha \int_a^b f(t) \Delta t + \beta \int_a^b g(t) \Delta t,$
- (2) $\int_a^b f(t) \Delta t = - \int_b^a f(t) \Delta t,$
- (3) $\int_a^b f(t) \Delta t = \int_a^c f(t) \Delta t + \int_c^b f(t) \Delta t,$
- (4) $\int_a^b f(t) g^\Delta(t) \Delta t = (fg)(b) - (fg)(a) - \int_a^b f^\Delta(t) g(\sigma(t)) \Delta t,$
- (5) $\int_a^a f(t) \Delta t = 0,$
- (6) $\int_a^b 1 \Delta t = b - a.$

Theorem 2.9 (Hölder's Inequality, see [2]). *Let $a, b \in \mathbb{T}$, $a \leq b$, and $f, g : \mathbb{T} \rightarrow \mathbb{R}$ be rd-continuous. Then*

$$\int_a^b |f(t)| |g(t)| \Delta t \leq \left(\int_a^b |f(t)|^p \Delta t \right)^{\frac{1}{p}} \left(\int_a^b |g(t)|^q \Delta t \right)^{\frac{1}{q}}, \quad (2.1)$$

where $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$.

Theorem 2.10 (See [8]). *Let $f, g \in C_{rd}(\mathbb{T})$, $a, b \in \mathbb{T}$, $a \leq b$. Then*

- 1) if $|f(t)| \leq g(t)$ on $[a, b] \cap \mathbb{T}$, then $\left| \int_a^b f(t) \Delta t \right| \leq \int_a^b g(t) \Delta t,$
- 2) if $f(t) \geq 0$, for all $a \leq t < b$ and $t \in \mathbb{T}$, then $\int_a^b f(t) \Delta t \geq 0.$

Corollary 2.11 (See [3]). *Let $f \in C_{rd}(\mathbb{T})$; $a, b, c \in \mathbb{T}$, with $c \in [a, b]$; $f(t) \geq 0$, $\forall t \in [a, b]$. Then*

$$\int_a^c f(t) \Delta t \leq \int_a^b f(t) \Delta t.$$

Definition 2.12 (See [3]). For a function $f : \mathbb{T} \rightarrow \mathbb{R}$ we consider the second derivative $f^{\Delta\Delta}$ provided f^Δ is differentiable on $\mathbb{T}^{k^2} = (\mathbb{T}^k)^k$ with derivative $f^{\Delta\Delta} = (f^\Delta)^\Delta : \mathbb{T}^{k^2} \rightarrow \mathbb{R}$. Similarly we define higher order derivatives $f^{\Delta^n} : \mathbb{T}^{k^n} \rightarrow \mathbb{R}$.

Similarly we define $\sigma^2(t) = \sigma(\sigma(t)), \dots, \sigma^n(t) = \sigma(\sigma^{n-1}(t))$, $n \in \mathbb{N}$. For convenience we put $\rho^0(t) = \sigma^0(t) = t$, $f^{\Delta^0} = f$, $\mathbb{T}^{k^0} = \mathbb{T}$.

Notice $\mathbb{T}^{k^n} \subset \mathbb{T}^{k^l}$, $l \in \{0, 1, \dots, n\}$.

Denote by $C_{rd}^n(\mathbb{T})$ the space of all functions $f \in C_{rd}(\mathbb{T})$ such that $f^{\Delta^i} \in C_{rd}(\mathbb{T})$ for $i = 1, \dots, n \in \mathbb{N}$. In this last case $\mathbb{T}^k = \mathbb{T}$ is needed.

We need the following Taylor formula.

Theorem 2.13 (Taylor's Formula, see [5, 9]). *Assume $\mathbb{T}^k = \mathbb{T}$ and $f \in C_{rd}^n(\mathbb{T})$, $n \in \mathbb{N}$ and $s, t \in \mathbb{T}$. Here $h_0(t, s) = 1$, $\forall s, t \in \mathbb{T}$; $k \in \mathbb{N}_0$, and*

$$h_{k+1}(t, s) = \int_s^t h_k(\tau, s) \Delta\tau, \quad \forall s, t \in \mathbb{T}.$$

(then $h_k^\Delta(t, s) = h_{k-1}(t, s)$, for $k \in \mathbb{N}$, $\forall t \in \mathbb{T}$, for each $s \in \mathbb{T}$ fixed). Then

$$f(t) = \sum_{k=0}^{n-1} f^{\Delta^k}(s) h_k(t, s) + \int_s^t h_{n-1}(t, \sigma(\tau)) f^{\Delta^n}(\tau) \Delta\tau. \quad (2.2)$$

Remark 2.14 (to Theorem 2.13). By [9], we have $h_1(t, s) = t - s$, $\forall s, t \in \mathbb{T}$.

So if $t \geq s$ then $h_1(t, s) \geq 0$, $h_2(t, s) \geq 0$, \dots , $h_{n-1}(t, s) \geq 0$. However for n odd number $h_{n-1}(t, \sigma(\tau)) \geq 0$ for all $s \leq \tau \leq t$ (see [4], p. 635).

Also it holds ([1])

$$h_k(t, s) \leq \frac{(t-s)^k}{k!}, \quad \forall t \geq s, k \in \mathbb{N}_0.$$

Corollary 2.15 (to Theorem 2.13, see [3]). *Assume $f \in C_{rd}^n(\mathbb{T})$ and $s, t \in \mathbb{T}$. Let $m \in \mathbb{N}$ with $m < n$ Then*

$$f^{\Delta^m}(t) = \sum_{k=0}^{n-m-1} f^{\Delta^{k+m}}(s) h_k(t, s) + \int_s^t h_{n-m-1}(t, \sigma(\tau)) f^{\Delta^n}(\tau) \Delta\tau. \quad (2.3)$$

Proof. Use Theorem 2.13 with n and f substituted by $n-m$ and f^{Δ^m} , respectively. \square

Corollary 2.16 (See [3]). *Let $f \in C_{rd}(\mathbb{T})$; $a, b \in \mathbb{T}$, such that $f(t) > 0, \forall t \in [a, b] \cap \mathbb{T}$, then $\int_a^b f(t) \Delta t > 0$.*

We mention also the following.

Lemma 2.17 (See [4, p. 631]). *Let the time scale \mathbb{T} be such that $\mathbb{T}^k = \mathbb{T}$. Let $h_k : \mathbb{T}^2 \rightarrow \mathbb{R}, k \in \mathbb{N}_0$, such that $h_0(t, s) \equiv 1, \forall s, t \in \mathbb{T}$, and $h_{k+1}(t, s) = \int_s^t h_k(\tau, s) \Delta \tau, \forall s, t \in \mathbb{T}$, for all $k \in \mathbb{N}_0$.*

Then $h_k(t, s)$ is continuous in $s \in \mathbb{T}$, for each fixed $t \in \mathbb{T}$; and continuous in $t \in \mathbb{T}$, for each fixed $s \in \mathbb{T}$. Also it holds that $h_k(t, \sigma(s))$ is rd-continuous in $s \in \mathbb{T}$, for each fixed $t \in \mathbb{T}$; for all $k \in \mathbb{N}_0$.

3 Main Results

In this article, we assume that $\mathbb{T}^k = \mathbb{T}$. Next, we present Iyengar type inequality on time scales for all norms $\|\cdot\|_p, 1 \leq p \leq \infty$.

Theorem 3.1. *Let $f \in C_{rd}^n(\mathbb{T})$, n is an odd number, $a, b \in \mathbb{T}$; $a \leq b$. Here σ is continuous and $h_{n-1}(t, s)$ jointly continuous. Then*

1)

$$\left| \int_a^b f(t) \Delta t - \sum_{k=0}^{n-1} \left(f^{\Delta^k}(a) h_{k+1}(x, a) - f^{\Delta^k}(b) h_{k+1}(x, b) \right) \right| \leq \|f^{\Delta^n}\|_{\infty, [a, b] \cap \mathbb{T}} \left[\left(\int_a^x \left(\int_a^t h_{n-1}(t, \sigma(\tau)) \Delta \tau \right) \Delta t \right) + \left(\int_x^b \left(\int_t^b h_{n-1}(t, \sigma(\tau)) \Delta \tau \right) \Delta t \right) \right], \quad (3.1)$$

$\forall x \in [a, b] \cap \mathbb{T}$,

2) assuming $f^{\Delta^k}(a) = f^{\Delta^k}(b) = 0, k = 0, 1, \dots, n-1$, we get from (3.1) that

$$\left| \int_a^b f(t) \Delta t \right| \leq \|f^{\Delta^n}\|_{\infty, [a, b] \cap \mathbb{T}}$$

$$\left[\left(\int_a^x \left(\int_a^t h_{n-1}(t, \sigma(\tau)) \Delta \tau \right) \Delta t \right) + \left(\int_x^b \left(\int_t^b h_{n-1}(t, \sigma(\tau)) \Delta \tau \right) \Delta t \right) \right], \quad (3.2)$$

$\forall x \in [a, b] \cap \mathbb{T}$,

2₁) when $x = a$ we get from (3.2) that

$$\left| \int_a^b f(t) \Delta t \right| \leq \|f^{\Delta^n}\|_{\infty, [a, b] \cap \mathbb{T}} \left(\int_a^b \left(\int_t^b h_{n-1}(t, \sigma(\tau)) \Delta \tau \right) \Delta t \right), \quad (3.3)$$

2₂) when $x = b$ we get from (3.2) that

$$\left| \int_a^b f(t) \Delta t \right| \leq \|f^{\Delta^n}\|_{\infty, [a, b] \cap \mathbb{T}} \left(\int_a^b \left(\int_a^t h_{n-1}(t, \sigma(\tau)) \Delta \tau \right) \Delta t \right), \quad (3.4)$$

2₃) by (3.3) and (3.4) we get

$$\left| \int_a^b f(t) \Delta t \right| \leq \|f^{\Delta^n}\|_{\infty, [a, b] \cap \mathbb{T}} \min \left\{ \left(\int_a^b \left(\int_t^b h_{n-1}(t, \sigma(\tau)) \Delta \tau \right) \Delta t \right), \left(\int_a^b \left(\int_a^t h_{n-1}(t, \sigma(\tau)) \Delta \tau \right) \Delta t \right) \right\}, \quad (3.5)$$

and

3) assuming $f^{\Delta^k}(a) = f^{\Delta^k}(b) = 0$, $k = 1, \dots, n-1$, by (3.1) we have

$$\left| \int_a^b f(t) \Delta t - [f(a)(x-a) + f(b)(b-x)] \right| \leq \|f^{\Delta^n}\|_{\infty, [a, b] \cap \mathbb{T}} \left[\left(\int_a^x \left(\int_a^t h_{n-1}(t, \sigma(\tau)) \Delta \tau \right) \Delta t \right) + \left(\int_x^b \left(\int_t^b h_{n-1}(t, \sigma(\tau)) \Delta \tau \right) \Delta t \right) \right], \quad (3.6)$$

$\forall x \in [a, b] \cap \mathbb{T}$.

Proof. By [8, p. 23], we have that $\|f^{\Delta^n}\|_{\infty, [a, b] \cap \mathbb{T}} < \infty$. By Theorem 2.13, see (2.2), we have

$$f(t) - \sum_{k=0}^{n-1} f^{\Delta^k}(a) h_k(t, a) = \int_a^t h_{n-1}(t, \sigma(\tau)) f^{\Delta^n}(\tau) \Delta \tau, \quad (3.7)$$

and

$$f(t) - \sum_{k=0}^{n-1} f^{\Delta^k}(b) h_k(t, b) = \int_b^t h_{n-1}(t, \sigma(\tau)) f^{\Delta^n}(\tau) \Delta \tau, \quad (3.8)$$

$\forall t \in [a, b] \cap \mathbb{T}$.

Then we get

$$\left| f(t) - \sum_{k=0}^{n-1} f^{\Delta^k}(a) h_k(t, a) \right| \stackrel{(3.7)}{\leq} \|f^{\Delta^n}\|_{\infty, [a, b] \cap \mathbb{T}} \int_a^t h_{n-1}(t, \sigma(\tau)) \Delta \tau, \quad (3.9)$$

and

$$\left| f(t) - \sum_{k=0}^{n-1} f^{\Delta^k}(b) h_k(t, b) \right| \stackrel{(3.8)}{=} \left| \int_t^b h_{n-1}(t, \sigma(\tau)) f^{\Delta^n}(\tau) \Delta \tau \right|$$

$$\leq \left(\int_t^b h_{n-1}(t, \sigma(\tau)) \Delta\tau \right) \|f^{\Delta^n}\|_{\infty, [a, b] \cap \mathbb{T}}. \quad (3.10)$$

Therefore it holds by (3.9), (3.10)

$$\begin{aligned} - \|f^{\Delta^n}\|_{\infty, [a, b] \cap \mathbb{T}} \int_a^t h_{n-1}(t, \sigma(\tau)) \Delta\tau &\leq f(t) - \sum_{k=0}^{n-1} f^{\Delta^k}(a) h_k(t, a) \\ &\leq \|f^{\Delta^n}\|_{\infty, [a, b] \cap \mathbb{T}} \int_a^t h_{n-1}(t, \sigma(\tau)) \Delta\tau \end{aligned}$$

and

$$\begin{aligned} - \|f^{\Delta^n}\|_{\infty, [a, b] \cap \mathbb{T}} \left(\int_t^b h_{n-1}(t, \sigma(\tau)) \Delta\tau \right) &\leq f(t) - \sum_{k=0}^{n-1} f^{\Delta^k}(b) h_k(t, b) \\ &\leq \|f^{\Delta^n}\|_{\infty, [a, b] \cap \mathbb{T}} \left(\int_t^b h_{n-1}(t, \sigma(\tau)) \Delta\tau \right), \end{aligned}$$

$\forall t \in [a, b] \cap \mathbb{T}$.

Consequently, we have

$$\begin{aligned} \sum_{k=0}^{n-1} f^{\Delta^k}(a) h_k(t, a) - \|f^{\Delta^n}\|_{\infty, [a, b] \cap \mathbb{T}} \int_a^t h_{n-1}(t, \sigma(\tau)) \Delta\tau &\leq f(t) \quad (3.11) \\ &\leq \sum_{k=0}^{n-1} f^{\Delta^k}(a) h_k(t, a) + \|f^{\Delta^n}\|_{\infty, [a, b] \cap \mathbb{T}} \int_a^t h_{n-1}(t, \sigma(\tau)) \Delta\tau \end{aligned}$$

and

$$\begin{aligned} \sum_{k=0}^{n-1} f^{\Delta^k}(b) h_k(t, b) - \|f^{\Delta^n}\|_{\infty, [a, b] \cap \mathbb{T}} \left(\int_t^b h_{n-1}(t, \sigma(\tau)) \Delta\tau \right) &\leq f(t) \quad (3.12) \\ &\leq \sum_{k=0}^{n-1} f^{\Delta^k}(b) h_k(t, b) + \|f^{\Delta^n}\|_{\infty, [a, b] \cap \mathbb{T}} \left(\int_t^b h_{n-1}(t, \sigma(\tau)) \Delta\tau \right), \end{aligned}$$

$\forall t \in [a, b] \cap \mathbb{T}$.

Let any $x \in [a, b] \cap \mathbb{T}$, then integrating (3.11), (3.12) we obtain:

$$\begin{aligned} \sum_{k=0}^{n-1} f^{\Delta^k}(a) h_{k+1}(x, a) - \|f^{\Delta^n}\|_{\infty, [a, b] \cap \mathbb{T}} \left(\int_a^x \left(\int_a^t h_{n-1}(t, \sigma(\tau)) \Delta\tau \right) \Delta t \right) \\ \leq \int_a^x f(t) \Delta t \leq \end{aligned} \quad (3.13)$$

$$\sum_{k=0}^{n-1} f^{\Delta^k}(a) h_{k+1}(x, a) + \|f^{\Delta^n}\|_{\infty, [a, b] \cap \mathbb{T}} \left(\int_a^x \left(\int_a^t h_{n-1}(t, \sigma(\tau)) \Delta\tau \right) \Delta t \right),$$

and

$$\begin{aligned} & - \sum_{k=0}^{n-1} f^{\Delta^k}(b) h_{k+1}(x, b) - \|f^{\Delta^n}\|_{\infty, [a, b] \cap \mathbb{T}} \left(\int_x^b \left(\int_t^b h_{n-1}(t, \sigma(\tau)) \Delta\tau \right) \Delta t \right) \\ & \leq \int_x^b f(t) \Delta t \leq \end{aligned} \quad (3.14)$$

$$- \sum_{k=0}^{n-1} f^{\Delta^k}(b) h_{k+1}(x, b) + \|f^{\Delta^n}\|_{\infty, [a, b] \cap \mathbb{T}} \left(\int_x^b \left(\int_t^b h_{n-1}(t, \sigma(\tau)) \Delta\tau \right) \Delta t \right).$$

Adding (3.13) and (3.14) we derive

$$\begin{aligned} & \sum_{k=0}^{n-1} \left(f^{\Delta^k}(a) h_{k+1}(x, a) - f^{\Delta^k}(b) h_{k+1}(x, b) \right) - \|f^{\Delta^n}\|_{\infty, [a, b] \cap \mathbb{T}} \\ & \left[\left(\int_a^x \left(\int_a^t h_{n-1}(t, \sigma(\tau)) \Delta\tau \right) \Delta t \right) + \left(\int_x^b \left(\int_t^b h_{n-1}(t, \sigma(\tau)) \Delta\tau \right) \Delta t \right) \right] \\ & \leq \int_a^b f(t) \Delta t \leq \end{aligned} \quad (3.15)$$

$$\begin{aligned} & \sum_{k=0}^{n-1} \left(f^{\Delta^k}(a) h_{k+1}(x, a) - f^{\Delta^k}(b) h_{k+1}(x, b) \right) + \|f^{\Delta^n}\|_{\infty, [a, b] \cap \mathbb{T}} \\ & \left[\left(\int_a^x \left(\int_a^t h_{n-1}(t, \sigma(\tau)) \Delta\tau \right) \Delta t \right) + \left(\int_x^b \left(\int_t^b h_{n-1}(t, \sigma(\tau)) \Delta\tau \right) \Delta t \right) \right], \end{aligned}$$

$\forall x \in [a, b] \cap \mathbb{T}$.

The proof is now complete. \square

We continue with the next result.

Theorem 3.2. Let $f \in C_{rd}^n(\mathbb{T})$, $n \in \mathbb{N}$ is odd, $a, b \in \mathbb{T}$; $a \leq b$. Then

1)

$$\begin{aligned} & \left| \int_a^b f(t) \Delta t - \sum_{k=0}^{n-1} \left(f^{\Delta^k}(a) h_{k+1}(x, a) - f^{\Delta^k}(b) h_{k+1}(x, b) \right) \right| \leq \\ & \|f^{\Delta^n}\|_{L_1([a, b] \cap \mathbb{T})} \left\{ \int_a^x (t - \sigma(a))^{n-1} \Delta t + \int_x^b (\sigma(b) - t)^{n-1} \Delta t \right\}, \end{aligned} \quad (3.16)$$

$\forall x \in [a, b] \cap \mathbb{T}$,

2) assuming $f^{\Delta^k}(a) = f^{\Delta^k}(b) = 0$, $k = 0, 1, \dots, n-1$, from (3.16) we obtain

$$\left| \int_a^b f(t) \Delta t \right| \leq \|f^{\Delta^n}\|_{L_1([a,b] \cap \mathbb{T})} \cdot \left\{ \int_a^x (t - \sigma(a))^{n-1} \Delta t + \int_x^b (\sigma(b) - t)^{n-1} \Delta t \right\}, \quad (3.17)$$

$\forall x \in [a, b] \cap \mathbb{T}$,

2₁) when $x = a$ by (3.17) we get

$$\left| \int_a^b f(t) \Delta t \right| \leq \|f^{\Delta^n}\|_{L_1([a,b] \cap \mathbb{T})} \left(\int_a^b (\sigma(b) - t)^{n-1} \Delta t \right), \quad (3.18)$$

2₂) when $x = b$ by (3.17) we get

$$\left| \int_a^b f(t) \Delta t \right| \leq \|f^{\Delta^n}\|_{L_1([a,b] \cap \mathbb{T})} \left(\int_a^x (t - \sigma(a))^{n-1} \Delta t \right), \quad (3.19)$$

2₃) by (3.18), (3.19) we have

$$\left| \int_a^b f(t) \Delta t \right| \leq \|f^{\Delta^n}\|_{L_1([a,b] \cap \mathbb{T})} \cdot \min \left\{ \left(\int_a^b (\sigma(b) - t)^{n-1} \Delta t \right), \left(\int_a^b (t - \sigma(a))^{n-1} \Delta t \right) \right\}, \quad (3.20)$$

3) assuming $f^{\Delta^k}(a) = f^{\Delta^k}(b) = 0$, $k = 1, \dots, n-1$, by (3.16) we derive

$$\left| \int_a^b f(t) \Delta t - [f(a)(x-a) + f(b)(b-x)] \right| \leq \|f^{\Delta^n}\|_{L_1([a,b] \cap \mathbb{T})} \left\{ \int_a^x (t - \sigma(a))^{n-1} \Delta t + \int_x^b (\sigma(b) - t)^{n-1} \Delta t \right\}, \quad (3.21)$$

$\forall x \in [a, b] \cap \mathbb{T}$.

Proof. Clearly, here it holds $\|f^{\Delta^n}\|_{L_1([a,b] \cap \mathbb{T})} < \infty$.

Set $h_0(t, s) = 1$, $\forall s, t \in \mathbb{T}$; $k \in \mathbb{N}_0$, and $h_{k+1}(t, s) = \int_s^t h_k(\tau, s) \Delta \tau$, $\forall s, t \in \mathbb{T}$.

Easily, it holds $|h_n(t, s)| \leq |t - s|^n$, $\forall n \in \mathbb{N}$, $\forall s, t \in \mathbb{T}$.

By Theorem 2.13 (3) we have

$$f(t) - \sum_{k=0}^{n-1} f^{\Delta^k}(a) h_k(t, a) = \int_a^t h_{n-1}(t, \sigma(\tau)) f^{\Delta^n}(\tau) \Delta \tau,$$

and

$$f(t) - \sum_{k=0}^{n-1} f^{\Delta^k}(b) h_k(t, b) = \int_b^t h_{n-1}(t, \sigma(\tau)) f^{\Delta^n}(\tau) \Delta\tau,$$

$\forall t \in [a, b] \cap \mathbb{T}$.

Then

$$\begin{aligned} \left| f(t) - \sum_{k=0}^{n-1} f^{\Delta^k}(a) h_k(t, a) \right| &= \left| \int_a^t h_{n-1}(t, \sigma(\tau)) f^{\Delta^n}(\tau) \Delta\tau \right| \leq \\ &\int_a^t |h_{n-1}(t, \sigma(\tau))| |f^{\Delta^n}(\tau)| \Delta\tau \leq \int_a^t |t - \sigma(\tau)|^{n-1} |f^{\Delta^n}(\tau)| \Delta\tau \leq \\ &(t - \sigma(a))^{n-1} \|f^{\Delta^n}\|_{L_1([a, b] \cap \mathbb{T})}. \end{aligned}$$

Furthermore we have

$$\begin{aligned} \left| f(t) - \sum_{k=0}^{n-1} f^{\Delta^k}(b) h_k(t, b) \right| &= \left| \int_t^b h_{n-1}(t, \sigma(\tau)) f^{\Delta^n}(\tau) \Delta\tau \right| \leq \\ &\int_t^b |h_{n-1}(t, \sigma(\tau))| |f^{\Delta^n}(\tau)| \Delta\tau \leq \int_t^b |t - \sigma(\tau)|^{n-1} |f^{\Delta^n}(\tau)| \Delta\tau \leq \\ &(\sigma(b) - t)^{n-1} \|f^{\Delta^n}\|_{L_1([a, b] \cap \mathbb{T})}. \end{aligned}$$

Therefore it holds

$$\begin{aligned} - (t - \sigma(a))^{n-1} \|f^{\Delta^n}\|_{L_1([a, b] \cap \mathbb{T})} &\leq f(t) - \sum_{k=0}^{n-1} f^{\Delta^k}(a) h_k(t, a) \\ &\leq (t - \sigma(a))^{n-1} \|f^{\Delta^n}\|_{L_1([a, b] \cap \mathbb{T})} \end{aligned}$$

and

$$\begin{aligned} - (\sigma(b) - t)^{n-1} \|f^{\Delta^n}\|_{L_1([a, b] \cap \mathbb{T})} &\leq f(t) - \sum_{k=0}^{n-1} f^{\Delta^k}(b) h_k(t, b) \\ &\leq (\sigma(b) - t)^{n-1} \|f^{\Delta^n}\|_{L_1([a, b] \cap \mathbb{T})}, \end{aligned}$$

$\forall t \in [a, b] \cap \mathbb{T}$.

Consequently it holds

$$\begin{aligned} \sum_{k=0}^{n-1} f^{\Delta^k}(a) h_k(t, a) - (t - \sigma(a))^{n-1} \|f^{\Delta^n}\|_{L_1([a, b] \cap \mathbb{T})} &\leq f(t) \\ &\leq \sum_{k=0}^{n-1} f^{\Delta^k}(a) h_k(t, a) + (t - \sigma(a))^{n-1} \|f^{\Delta^n}\|_{L_1([a, b] \cap \mathbb{T})} \end{aligned}$$

and

$$\begin{aligned} & \sum_{k=0}^{n-1} f^{\Delta^k}(b) h_k(t, b) - (\sigma(b) - t)^{n-1} \|f^{\Delta^n}\|_{L_1([a, b] \cap \mathbb{T})} \leq f(t) \\ & \leq \sum_{k=0}^{n-1} f^{\Delta^k}(b) h_k(t, b) + (\sigma(b) - t)^{n-1} \|f^{\Delta^n}\|_{L_1([a, b] \cap \mathbb{T})}, \end{aligned}$$

$\forall t \in [a, b] \cap \mathbb{T}$.

Let any $x \in [a, b] \cap \mathbb{T}$, then integrating by integration we have

$$\begin{aligned} & \sum_{k=0}^{n-1} f^{\Delta^k}(a) h_{k+1}(x, a) - \left(\int_a^x (t - \sigma(a))^{n-1} \Delta t \right) \|f^{\Delta^n}\|_{L_1([a, b] \cap \mathbb{T})} \\ & \leq \int_a^x f(t) \Delta t \leq \\ & \sum_{k=0}^{n-1} f^{\Delta^k}(a) h_{k+1}(x, a) + \left(\int_a^x (t - \sigma(a))^{n-1} \Delta t \right) \|f^{\Delta^n}\|_{L_1([a, b] \cap \mathbb{T})}, \end{aligned} \quad (3.22)$$

and

$$\begin{aligned} & - \sum_{k=0}^{n-1} f^{\Delta^k}(b) h_{k+1}(x, b) - \left(\int_x^b (\sigma(b) - t)^{n-1} \Delta t \right) \|f^{\Delta^n}\|_{L_1([a, b] \cap \mathbb{T})} \\ & \leq \int_x^b f(t) \Delta t \leq \\ & - \sum_{k=0}^{n-1} f^{\Delta^k}(b) h_{k+1}(x, b) + \left(\int_x^b (\sigma(b) - t)^{n-1} \Delta t \right) \|f^{\Delta^n}\|_{L_1([a, b] \cap \mathbb{T})}, \end{aligned} \quad (3.23)$$

$\forall x \in [a, b] \cap \mathbb{T}$.

Adding (3.22) and (3.23), we obtain

$$\begin{aligned} & \sum_{k=0}^{n-1} \left(f^{\Delta^k}(a) h_{k+1}(x, a) - f^{\Delta^k}(b) h_{k+1}(x, b) \right) - \\ & \|f^{\Delta^n}\|_{L_1([a, b] \cap \mathbb{T})} \left\{ \left(\int_a^x (t - \sigma(a))^{n-1} \Delta t \right) + \left(\int_x^b (\sigma(b) - t)^{n-1} \Delta t \right) \right\} \\ & \leq \int_a^b f(t) \Delta t \leq \\ & \sum_{k=0}^{n-1} \left(f^{\Delta^k}(a) h_{k+1}(x, a) - f^{\Delta^k}(b) h_{k+1}(x, b) \right) + \end{aligned}$$

$$\|f^{\Delta^n}\|_{L_1([a,b]\cap\mathbb{T})} \left\{ \left(\int_a^x (t - \sigma(a))^{n-1} \Delta t \right) + \left(\int_x^b (\sigma(b) - t)^{n-1} \Delta t \right) \right\}, \quad (3.24)$$

$\forall x \in [a, b] \cap \mathbb{T}$.

The proof is now complete. \square

We continue with the next result.

Theorem 3.3. *Let $f \in C_{rd}^n(\mathbb{T})$, n is an odd number, $a, b \in \mathbb{T}$; $a \leq b$; $p, q > 1$; $\frac{1}{p} + \frac{1}{q} = 1$; σ is continuous and $h_{n-1}(t, s)$ is jointly continuous. Then*

1)

$$\left| \int_a^b f(t) \Delta t - \sum_{k=0}^{n-1} \left(f^{\Delta^k}(a) h_{k+1}(x, a) - f^{\Delta^k}(b) h_{k+1}(x, b) \right) \right| \leq \|f^{\Delta^n}\|_{L_q([a,b]\cap\mathbb{T})} \cdot \left[\left(\int_a^x \left(\int_a^t h_{n-1}(t, \sigma(\tau))^p \Delta \tau \right)^{\frac{1}{p}} \Delta t \right) + \left(\int_x^b \left(\int_t^b h_{n-1}(t, \sigma(\tau))^p \Delta \tau \right)^{\frac{1}{p}} \Delta t \right) \right], \quad (3.25)$$

$\forall x \in [a, b] \cap \mathbb{T}$,

2) assuming $f^{\Delta^k}(a) = f^{\Delta^k}(b) = 0$, $k = 0, 1, \dots, n-1$, by (3.25) we have that

$$\left| \int_a^b f(t) \Delta t \right| \leq \|f^{\Delta^n}\|_{L_q([a,b]\cap\mathbb{T})} \cdot \left[\left(\int_a^x \left(\int_a^t h_{n-1}(t, \sigma(\tau))^p \Delta \tau \right)^{\frac{1}{p}} \Delta t \right) + \left(\int_x^b \left(\int_t^b h_{n-1}(t, \sigma(\tau))^p \Delta \tau \right)^{\frac{1}{p}} \Delta t \right) \right], \quad (3.26)$$

$\forall x \in [a, b] \cap \mathbb{T}$,

2₁) when $x = a$ by (3.26) we get

$$\left| \int_a^b f(t) \Delta t \right| \leq \|f^{\Delta^n}\|_{L_q([a,b]\cap\mathbb{T})} \left(\int_a^b \left(\int_t^b h_{n-1}(t, \sigma(\tau))^p \Delta \tau \right)^{\frac{1}{p}} \Delta t \right), \quad (3.27)$$

2₂) when $x = b$ by (3.26) we get

$$\left| \int_a^b f(t) \Delta t \right| \leq \|f^{\Delta^n}\|_{L_q([a,b]\cap\mathbb{T})} \left(\int_a^b \left(\int_a^t h_{n-1}(t, \sigma(\tau))^p \Delta \tau \right)^{\frac{1}{p}} \Delta t \right), \quad (3.28)$$

2₃) by (3.27), (3.28) we derive that

$$\left| \int_a^b f(t) \Delta t \right| \leq \|f^{\Delta^n}\|_{L_q([a,b]\cap\mathbb{T})}. \quad (3.29)$$

$$\min \left\{ \left(\int_a^b \left(\int_t^b h_{n-1}(t, \sigma(\tau))^p \Delta\tau \right)^{\frac{1}{p}} \Delta t \right), \left(\int_a^b \left(\int_a^t h_{n-1}(t, \sigma(\tau))^p \Delta\tau \right)^{\frac{1}{p}} \Delta t \right) \right\},$$

3) assuming $f^{\Delta^k}(a) = f^{\Delta^k}(b) = 0$, $k = 1, \dots, n-1$, by (3.25) we obtain

$$\left| \int_a^b f(t) \Delta t - [f(a)(x-a) + f(b)(b-x)] \right| \leq \|f^{\Delta^n}\|_{L_q([a,b] \cap \mathbb{T})} \left[\left(\int_a^x \left(\int_a^t h_{n-1}(t, \sigma(\tau))^p \Delta\tau \right)^{\frac{1}{p}} \Delta t \right) + \left(\int_x^b \left(\int_t^b h_{n-1}(t, \sigma(\tau))^p \Delta\tau \right)^{\frac{1}{p}} \Delta t \right) \right], \quad (3.30)$$

$\forall x \in [a, b] \cap \mathbb{T}$.

Proof. As before, we have

$$K(t, a) := f(t) - \sum_{k=0}^{n-1} f^{\Delta^k}(a) h_k(t, a) = \int_a^t h_{n-1}(t, \sigma(\tau)) f^{\Delta^n}(\tau) \Delta\tau,$$

and

$$K(t, b) := f(t) - \sum_{k=0}^{n-1} f^{\Delta^k}(b) h_k(t, b) = \int_b^t h_{n-1}(t, \sigma(\tau)) f^{\Delta^n}(\tau) \Delta\tau,$$

$\forall t \in [a, b] \cap \mathbb{T}$.

We have that (by use of (2.1))

$$\begin{aligned} |K(t, a)| &\leq \left(\int_a^t h_{n-1}(t, \sigma(\tau))^p \Delta\tau \right)^{\frac{1}{p}} \left(\int_a^t |f^{\Delta^n}(\tau)|^q \Delta\tau \right)^{\frac{1}{q}} \\ &\leq \left(\int_a^t h_{n-1}(t, \sigma(\tau))^p \Delta\tau \right)^{\frac{1}{p}} \|f^{\Delta^n}\|_{L_q([a,b] \cap \mathbb{T})}, \end{aligned}$$

and

$$\begin{aligned} |K(t, b)| &= \left| \int_t^b h_{n-1}(t, \sigma(\tau)) f^{\Delta^n}(\tau) \Delta\tau \right| \leq \\ &\left(\int_t^b h_{n-1}(t, \sigma(\tau))^p \Delta\tau \right)^{\frac{1}{p}} \left(\int_t^b |f^{\Delta^n}(\tau)|^q \Delta\tau \right)^{\frac{1}{q}} \\ &\leq \left(\int_t^b h_{n-1}(t, \sigma(\tau))^p \Delta\tau \right)^{\frac{1}{p}} \|f^{\Delta^n}\|_{L_q([a,b] \cap \mathbb{T})}, \end{aligned}$$

$\forall t \in [a, b] \cap \mathbb{T}$.

Hence it holds

$$\begin{aligned} & - \left(\int_a^t h_{n-1}(t, \sigma(\tau))^p \Delta\tau \right)^{\frac{1}{p}} \|f^{\Delta^n}\|_{L_q([a,b] \cap \mathbb{T})} \leq K(t, a) \\ & \leq \left(\int_a^t h_{n-1}(t, \sigma(\tau))^p \Delta\tau \right)^{\frac{1}{p}} \|f^{\Delta^n}\|_{L_q([a,b] \cap \mathbb{T})} \end{aligned}$$

and

$$\begin{aligned} & - \left(\int_t^b h_{n-1}(t, \sigma(\tau))^p \Delta\tau \right)^{\frac{1}{p}} \|f^{\Delta^n}\|_{L_q([a,b] \cap \mathbb{T})} \leq K(t, b) \\ & \leq \left(\int_t^b h_{n-1}(t, \sigma(\tau))^p \Delta\tau \right)^{\frac{1}{p}} \|f^{\Delta^n}\|_{L_q([a,b] \cap \mathbb{T})}, \end{aligned}$$

$\forall t \in [a, b] \cap \mathbb{T}$.

That is,

$$\begin{aligned} & \sum_{k=0}^{n-1} f^{\Delta^k}(a) h_k(t, a) - \left(\int_a^t h_{n-1}(t, \sigma(\tau))^p \Delta\tau \right)^{\frac{1}{p}} \|f^{\Delta^n}\|_{L_q([a,b] \cap \mathbb{T})} \leq f(t) \\ & \leq \sum_{k=0}^{n-1} f^{\Delta^k}(a) h_k(t, a) + \left(\int_a^t h_{n-1}(t, \sigma(\tau))^p \Delta\tau \right)^{\frac{1}{p}} \|f^{\Delta^n}\|_{L_q([a,b] \cap \mathbb{T})} \end{aligned}$$

and

$$\begin{aligned} & \sum_{k=0}^{n-1} f^{\Delta^k}(b) h_k(t, b) - \left(\int_t^b h_{n-1}(t, \sigma(\tau))^p \Delta\tau \right)^{\frac{1}{p}} \|f^{\Delta^n}\|_{L_q([a,b] \cap \mathbb{T})} \leq f(t) \\ & \leq \sum_{k=0}^{n-1} f^{\Delta^k}(b) h_k(t, b) + \left(\int_t^b h_{n-1}(t, \sigma(\tau))^p \Delta\tau \right)^{\frac{1}{p}} \|f^{\Delta^n}\|_{L_q([a,b] \cap \mathbb{T})}, \end{aligned}$$

$\forall t \in [a, b] \cap \mathbb{T}$.

Let any $x \in [a, b] \cap \mathbb{T}$, then by integration we get

$$\begin{aligned} & \sum_{k=0}^{n-1} f^{\Delta^k}(a) h_{k+1}(x, a) - \|f^{\Delta^n}\|_{L_q([a,b] \cap \mathbb{T})} \left(\int_a^x \left(\int_a^t h_{n-1}(t, \sigma(\tau))^p \Delta\tau \right)^{\frac{1}{p}} \Delta t \right) \\ & \leq \int_a^x f(t) \Delta t \leq \\ & \sum_{k=0}^{n-1} f^{\Delta^k}(a) h_{k+1}(x, a) + \|f^{\Delta^n}\|_{L_q([a,b] \cap \mathbb{T})} \left(\int_a^x \left(\int_a^t h_{n-1}(t, \sigma(\tau))^p \Delta\tau \right)^{\frac{1}{p}} \Delta t \right), \end{aligned} \tag{3.31}$$

and

$$\begin{aligned}
& - \sum_{k=0}^{n-1} f^{\Delta^k}(b) h_{k+1}(x, b) - \|f^{\Delta^n}\|_{L_q([a, b] \cap \mathbb{T})} \left(\int_x^b \left(\int_t^b h_{n-1}(t, \sigma(\tau))^p \Delta\tau \right)^{\frac{1}{p}} \Delta t \right) \\
& \leq \int_x^b f(t) \Delta t \leq \\
& - \sum_{k=0}^{n-1} f^{\Delta^k}(b) h_{k+1}(x, b) + \|f^{\Delta^n}\|_{L_q([a, b] \cap \mathbb{T})} \left(\int_x^b \left(\int_t^b h_{n-1}(t, \sigma(\tau))^p \Delta\tau \right)^{\frac{1}{p}} \Delta t \right). \tag{3.32}
\end{aligned}$$

Adding (3.31) and (3.32) we obtain

$$\begin{aligned}
& \sum_{k=0}^{n-1} \left(f^{\Delta^k}(a) h_{k+1}(x, a) - f^{\Delta^k}(b) h_{k+1}(x, b) \right) - \\
& \|f^{\Delta^n}\|_{L_q([a, b] \cap \mathbb{T})} \left\{ \left(\int_a^x \left(\int_a^t h_{n-1}(t, \sigma(\tau))^p \Delta\tau \right)^{\frac{1}{p}} \Delta t \right) + \right. \\
& \left. \left(\int_x^b \left(\int_t^b h_{n-1}(t, \sigma(\tau))^p \Delta\tau \right)^{\frac{1}{p}} \Delta t \right) \right\} \\
& \leq \int_a^b f(t) \Delta t \leq \\
& \sum_{k=0}^{n-1} \left(f^{\Delta^k}(a) h_{k+1}(x, a) - f^{\Delta^k}(b) h_{k+1}(x, b) \right) + \\
& \|f^{\Delta^n}\|_{L_q([a, b] \cap \mathbb{T})} \left\{ \left(\int_a^x \left(\int_a^t h_{n-1}(t, \sigma(\tau))^p \Delta\tau \right)^{\frac{1}{p}} \Delta t \right) + \right. \\
& \left. \left(\int_x^b \left(\int_t^b h_{n-1}(t, \sigma(\tau))^p \Delta\tau \right)^{\frac{1}{p}} \Delta t \right) \right\}, \tag{3.33}
\end{aligned}$$

$\forall x \in [a, b] \cap \mathbb{T}$.

The proof is now complete. \square

We continue with the next result.

Theorem 3.4. *Let $f \in C_{rd}^n(\mathbb{T})$, $m, n \in \mathbb{N}$, $m < n$, $n - m$ is odd, $a, b \in \mathbb{T}$; $a \leq b$. Here σ is continuous and $h_{n-m-1}(t, s)$ is jointly continuous. Then*

1)

$$\left| \int_a^b f^{\Delta^m}(t) \Delta t - \left(\sum_{k=0}^{n-m-1} \left(f^{\Delta^{k+m}}(a) h_{k+1}(x, a) - f^{\Delta^{k+m}}(b) h_{k+1}(x, b) \right) \right) \right| \leq$$

$$\|f^{\Delta^n}\|_{\infty, [a, b] \cap \mathbb{T}} \left[\left(\int_a^x \left(\int_a^t h_{n-m-1}(t, \sigma(\tau)) \Delta \tau \right) \Delta t \right) + \right.$$

$$\left. \left(\int_x^b \left(\int_t^b h_{n-m-1}(t, \sigma(\tau)) \Delta \tau \right) \Delta t \right) \right], \quad (3.34)$$

 $\forall x \in [a, b] \cap \mathbb{T}$,

2) assuming $f^{\Delta^{k+m}}(a) = f^{\Delta^{k+m}}(b) = 0$, $k = 0, 1, \dots, n-m-1$, we get from (3.34) that

$$\left| \int_a^b f^{\Delta^m}(t) \Delta t \right| \leq \|f^{\Delta^n}\|_{\infty, [a, b] \cap \mathbb{T}} \cdot$$

$$\left[\left(\int_a^x \left(\int_a^t h_{n-m-1}(t, \sigma(\tau)) \Delta \tau \right) \Delta t \right) + \left(\int_x^b \left(\int_t^b h_{n-m-1}(t, \sigma(\tau)) \Delta \tau \right) \Delta t \right) \right], \quad (3.35)$$

 $\forall x \in [a, b] \cap \mathbb{T}$,2₁) when $x = a$ we get from (3.35) that

$$\left| \int_a^b f^{\Delta^m}(t) \Delta t \right| \leq \|f^{\Delta^n}\|_{\infty, [a, b] \cap \mathbb{T}} \left(\int_a^b \left(\int_t^b h_{n-m-1}(t, \sigma(\tau)) \Delta \tau \right) \Delta t \right), \quad (3.36)$$

2₂) when $x = b$ we get from (3.35) that

$$\left| \int_a^b f^{\Delta^m}(t) \Delta t \right| \leq \|f^{\Delta^n}\|_{\infty, [a, b] \cap \mathbb{T}} \left(\int_a^b \left(\int_a^t h_{n-m-1}(t, \sigma(\tau)) \Delta \tau \right) \Delta t \right), \quad (3.37)$$

2₃) by (3.36), (3.37) we get

$$\left| \int_a^b f^{\Delta^m}(t) \Delta t \right| \leq \|f^{\Delta^n}\|_{\infty, [a, b] \cap \mathbb{T}} \cdot$$

$$\min \left\{ \left(\int_a^b \left(\int_t^b h_{n-m-1}(t, \sigma(\tau)) \Delta \tau \right) \Delta t \right), \left(\int_a^b \left(\int_a^t h_{n-m-1}(t, \sigma(\tau)) \Delta \tau \right) \Delta t \right) \right\}, \quad (3.38)$$

and

3) assuming $f^{\Delta^{k+m}}(a) = f^{\Delta^{k+m}}(b) = 0$, $k = 1, \dots, n-m-1$, from (3.34) we obtain

$$\left| \int_a^b f^{\Delta^m}(t) \Delta t - [f^{\Delta^m}(a)(x-a) + f^{\Delta^m}(b)(b-x)] \right| \leq \|f^{\Delta^n}\|_{\infty, [a, b] \cap \mathbb{T}} \cdot$$

$$\left[\left(\int_a^x \left(\int_a^t h_{n-m-1}(t, \sigma(\tau)) \Delta\tau \right) \Delta t \right) + \left(\int_x^b \left(\int_t^b h_{n-m-1}(t, \sigma(\tau)) \Delta\tau \right) \Delta t \right) \right], \quad (3.39)$$

$\forall x \in [a, b] \cap \mathbb{T}$.

Proof. As in the proof of Theorem 3.1, now using Corollary 2.15 (4). \square

We give the following theorem.

Theorem 3.5. Let $f \in C_{rd}^n(\mathbb{T})$, $m, n \in \mathbb{N}$, $m < n$, $n - m$ is odd, $a, b \in \mathbb{T}$; $a \leq b$. Then

$$\left| \int_a^b f^{\Delta^m}(t) \Delta t - \sum_{k=0}^{n-m-1} \left(f^{\Delta^{k+m}}(a) h_{k+1}(x, a) - f^{\Delta^{k+m}}(b) h_{k+1}(x, b) \right) \right| \leq \|f^{\Delta^n}\|_{L_1([a, b] \cap \mathbb{T})} \left\{ \int_a^x (t - \sigma(a))^{n-m-1} \Delta t + \int_x^b (\sigma(b) - t)^{n-m-1} \Delta t \right\}, \quad (3.40)$$

$\forall x \in [a, b] \cap \mathbb{T}$,

2) assuming $f^{\Delta^{k+m}}(a) = f^{\Delta^{k+m}}(b) = 0$, $k = 0, 1, \dots, n - m - 1$, we get from (3.40) that

$$\left| \int_a^b f^{\Delta^m}(t) \Delta t \right| \leq \|f^{\Delta^n}\|_{L_1([a, b] \cap \mathbb{T})} \cdot \left\{ \int_a^x (t - \sigma(a))^{n-m-1} \Delta t + \int_x^b (\sigma(b) - t)^{n-m-1} \Delta t \right\}, \quad (3.41)$$

$\forall x \in [a, b] \cap \mathbb{T}$,

2₁) when $x = a$ by (3.41) we get

$$\left| \int_a^b f^{\Delta^m}(t) \Delta t \right| \leq \|f^{\Delta^n}\|_{L_1([a, b] \cap \mathbb{T})} \left(\int_a^b (\sigma(b) - t)^{n-m-1} \Delta t \right), \quad (3.42)$$

2₂) when $x = b$ by (3.41) we get

$$\left| \int_a^b f^{\Delta^m}(t) \Delta t \right| \leq \|f^{\Delta^n}\|_{L_1([a, b] \cap \mathbb{T})} \left(\int_a^x (t - \sigma(a))^{n-m-1} \Delta t \right), \quad (3.43)$$

2₃) by (3.42), (3.43) we have

$$\left| \int_a^b f^{\Delta^m}(t) \Delta t \right| \leq \|f^{\Delta^n}\|_{L_1([a, b] \cap \mathbb{T})} \cdot \min \left\{ \left(\int_a^b (\sigma(b) - t)^{n-m-1} \Delta t \right), \left(\int_a^b (t - \sigma(a))^{n-m-1} \Delta t \right) \right\}, \quad (3.44)$$

and

3) assuming $f^{\Delta^{k+m}}(a) = f^{\Delta^{k+m}}(b) = 0$, $k = 1, \dots, n - m - 1$, from (3.40) we obtain

$$\left| \int_a^b f^{\Delta^m}(t) \Delta t - [f^{\Delta^m}(a)(x-a) + f^{\Delta^m}(b)(b-x)] \right| \leq \|f^{\Delta^n}\|_{L_1([a,b] \cap \mathbb{T})} \left\{ \int_a^x (t - \sigma(a))^{n-m-1} \Delta t + \int_x^b (\sigma(b) - t)^{n-m-1} \Delta t \right\}, \quad (3.45)$$

$\forall x \in [a, b] \cap \mathbb{T}$.

Proof. As in Theorem 3.2, now using Corollary 2.15 (4). \square

We also give the next result.

Theorem 3.6. Let $f \in C_{rd}^n(\mathbb{T})$, $m, n \in \mathbb{N}$, $m < n$, $n - m$ is odd, $a, b \in \mathbb{T}$; $a \leq b$. Here σ is continuous and $h_{n-m-1}(t, s)$ is jointly continuous. Let also $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$.

Then

1)

$$\left| \int_a^b f^{\Delta^m}(t) \Delta t - \sum_{k=0}^{n-m-1} (f^{\Delta^{k+m}}(a) h_{k+1}(x, a) - f^{\Delta^{k+m}}(b) h_{k+1}(x, b)) \right| \leq \|f^{\Delta^n}\|_{L_q([a,b] \cap \mathbb{T})} \left[\left(\int_a^x \left(\int_a^t h_{n-m-1}(t, \sigma(\tau))^p \Delta \tau \right)^{\frac{1}{p}} \Delta t \right) + \left(\int_x^b \left(\int_t^b h_{n-m-1}(t, \sigma(\tau))^p \Delta \tau \right)^{\frac{1}{p}} \Delta t \right) \right], \quad (3.46)$$

$\forall x \in [a, b] \cap \mathbb{T}$,

2) assuming $f^{\Delta^{k+m}}(a) = f^{\Delta^{k+m}}(b) = 0$, $k = 0, 1, \dots, n - m - 1$, we get from (3.46) that

$$\left| \int_a^b f^{\Delta^m}(t) \Delta t \right| \leq \|f^{\Delta^n}\|_{L_q([a,b] \cap \mathbb{T})} \left[\left(\int_a^x \left(\int_a^t h_{n-m-1}(t, \sigma(\tau))^p \Delta \tau \right)^{\frac{1}{p}} \Delta t \right) + \left(\int_x^b \left(\int_t^b h_{n-m-1}(t, \sigma(\tau))^p \Delta \tau \right)^{\frac{1}{p}} \Delta t \right) \right], \quad (3.47)$$

$\forall x \in [a, b] \cap \mathbb{T}$,

2₁) when $x = a$ we get from (3.47) that

$$\left| \int_a^b f^{\Delta^m}(t) \Delta t \right| \leq \|f^{\Delta^n}\|_{L_q([a,b] \cap \mathbb{T})} \left(\int_a^b \left(\int_t^b h_{n-m-1}(t, \sigma(\tau))^p \Delta \tau \right)^{\frac{1}{p}} \Delta t \right), \quad (3.48)$$

2₂) when $x = b$ we get from (3.47) that

$$\left| \int_a^b f^{\Delta^m}(t) \Delta t \right| \leq \|f^{\Delta^n}\|_{L_q([a,b] \cap \mathbb{T})} \left(\int_a^b \left(\int_a^t h_{n-m-1}(t, \sigma(\tau))^p \Delta \tau \right)^{\frac{1}{p}} \Delta t \right), \quad (3.49)$$

2₃) by (3.48), (3.49) we get

$$\begin{aligned} \left| \int_a^b f^{\Delta^m}(t) \Delta t \right| &\leq \|f^{\Delta^n}\|_{L_q([a,b] \cap \mathbb{T})} \cdot \\ &\min \left\{ \left(\int_a^b \left(\int_t^b h_{n-m-1}(t, \sigma(\tau))^p \Delta \tau \right)^{\frac{1}{p}} \Delta t \right), \right. \\ &\left. \left(\int_a^b \left(\int_a^t h_{n-m-1}(t, \sigma(\tau))^p \Delta \tau \right)^{\frac{1}{p}} \Delta t \right) \right\}, \end{aligned} \quad (3.50)$$

and

3) assuming $f^{\Delta^{k+m}}(a) = f^{\Delta^{k+m}}(b) = 0$, $k = 1, \dots, n - m - 1$, we get from (3.46) that

$$\begin{aligned} \left| \int_a^b f^{\Delta^m}(t) \Delta t - [f^{\Delta^m}(a)(x-a) + f^{\Delta^m}(b)(b-x)] \right| &\leq \\ \|f^{\Delta^n}\|_{L_q([a,b] \cap \mathbb{T})} &\left[\left(\int_a^x \left(\int_a^t h_{n-m-1}(t, \sigma(\tau))^p \Delta \tau \right)^{\frac{1}{p}} \Delta t \right) + \right. \\ &\left. \left(\int_x^b \left(\int_t^b h_{n-m-1}(t, \sigma(\tau))^p \Delta \tau \right)^{\frac{1}{p}} \Delta t \right) \right], \end{aligned} \quad (3.51)$$

$\forall x \in [a, b] \cap \mathbb{T}$.

Proof. As in Theorem 3.3, by using Corollary 2.15 (4). \square

4 Applications

We need the following.

Remark 4.1 (See [8]). i) When $\mathbb{T} = \mathbb{R}$, then $h_k(t, s) = \frac{(t-s)^k}{k!}$, $\forall k \in \mathbb{N}_0$, $\forall t, s \in \mathbb{R}$, $\sigma(t) = t$, $\int_a^b f(t) \Delta t = \int_a^b f(t) dt$, $f^\Delta(t) = f'(t)$, $f^{\Delta^k} = f^{(k)}$; rd-continuous corresponds to f continuous.

ii) When $\mathbb{T} = \mathbb{Z}$, $h_k(t, s) = \frac{(t-s)^{(k)}}{k!}$, $\forall k \in \mathbb{N}_0$, $\forall t, s \in \mathbb{Z}$, where $t^{(0)} = 1$,
 $t^{(k)} = \prod_{i=0}^{k-1} (t-i)$ for $k \in \mathbb{N}$, $\sigma(t) = t+1$,

$$\int_a^b f(t) \Delta t = \sum_{t=a}^{b-1} f(t), \quad a < b,$$

$$f^\Delta(t) = f(t+1) - f(t) = \Delta f(t),$$

$$f^{\Delta^k}(t) = \Delta^k f(t) = \sum_{l=0}^k \binom{k}{l} (-1)^{k-l} f(t+l),$$

rd-continuous f corresponds to any f .

We also need the following.

Remark 4.2 (See [1, 8]). Consider $q > 1$, $q^{\mathbb{Z}} = \{q^k : k \in \mathbb{Z}\}$, and the time scale $\mathbb{T} = q^{\mathbb{Z}} = q^{\mathbb{Z}} \cup \{0\}$, which is very important in q -difference equations.

It holds that

$$h_k(t, s) = \prod_{\nu=0}^{k-1} \frac{t - q^\nu s}{\sum_{\mu=0}^{\nu} q^\mu}, \quad \forall s, t \in \mathbb{T};$$

$$\sigma(t) = qt, \quad \rho(t) = \frac{t}{q}, \quad \forall t \in \mathbb{T},$$

$$f^\Delta(t) = \frac{f(qt) - f(t)}{(q-1)t}, \quad \forall t \in \mathbb{T} - \{0\},$$

$$f^\Delta(0) = \lim_{s \rightarrow 0} \frac{f(s) - f(0)}{s}.$$

Next we give applications of our initial main results.

Theorem 4.3. *Let $f \in C^n([a, b])$, $n \in \mathbb{N}$ is odd and $[a, b] \subset \mathbb{R}$. Then*

$$\left| \int_a^b f(t) dt - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(f^{(k)}(a) (x-a)^{k+1} + (-1)^k f^{(k)}(b) (b-x)^{k+1} \right) \right| \leq \frac{\|f^{(n)}\|_{\infty, [a, b]}}{(n+1)!} [(x-a)^{n+1} + (b-x)^{n+1}], \quad (4.1)$$

$\forall x \in [a, b]$.

Proof. By Theorem 3.1, (3.1). □

We continue with the following.

Theorem 4.4. Let $f \in C^n([a, b])$, $n \in \mathbb{N}$ is odd, $[a, b] \subset \mathbb{R}$. Then

$$\left| \int_a^b f(t) dt - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(f^{(k)}(a) (x-a)^{k+1} + (-1)^k f^{(k)}(b) (b-x)^{k+1} \right) \right| \leq \frac{\|f^{(n)}\|_{L_1([a,b])}}{n} [(x-a)^n + (b-x)^n], \quad (4.2)$$

$\forall x \in [a, b]$.

Proof. By Theorem 3.2, (3.16). \square

We also give the following.

Theorem 4.5. Let $f \in C^n([a, b])$, $n \in \mathbb{N}$ is odd and $[a, b] \subset \mathbb{R}$. Let also $p, q > 1$: $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\left| \int_a^b f(t) dt - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(f^{(k)}(a) (x-a)^{k+1} + (-1)^k f^{(k)}(b) (b-x)^{k+1} \right) \right| \leq \frac{\|f^{(n)}\|_{L_q([a,b])}}{(n-1)! (p(n-1)+1)^{\frac{1}{p}} \left(n + \frac{1}{p}\right)} \left[(x-a)^{n+\frac{1}{p}} + (b-x)^{n+\frac{1}{p}} \right], \quad (4.3)$$

$\forall x \in [a, b]$.

Proof. By Theorem 3.3, (3.25). \square

We continue with the following.

Theorem 4.6. Let $f : \mathbb{Z} \rightarrow \mathbb{R}$, n is an odd number, $a, b \in \mathbb{Z}$; $a \leq b$. Then

$$\left| \sum_{t=a}^{b-1} f(t) - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\Delta^k f(a) (x-a)^{(k+1)} - \Delta^k f(b) (x-b)^{(k+1)} \right) \right| \leq \frac{\|\Delta^n f\|_{\infty, [a,b] \cap \mathbb{Z}}}{(n-1)!} \left[\left(\sum_{t=a}^{x-1} \left(\sum_{\tau=t}^{t-1} (t-\tau-1)^{(n-1)} \right) \right) + \left(\sum_{t=x}^{b-1} \left(\sum_{\tau=t}^{b-1} (t-\tau-1)^{(n-1)} \right) \right) \right], \quad (4.4)$$

$\forall x \in [a, b] \cap \mathbb{Z}$.

Proof. By Theorem 3.1, (3.1), see also Remark 4.1 (ii). \square

We give the next result.

Theorem 4.7. Let $f : \mathbb{Z} \rightarrow \mathbb{R}$, $n \in \mathbb{N}$ is odd, $a, b \in \mathbb{Z}$; $a \leq b$. Then

$$\left| \sum_{t=a}^{b-1} f(t) - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\Delta^k f(a) (x-a)^{(k+1)} - \Delta^k f(b) (x-b)^{(k+1)} \right) \right| \leq \left(\sum_{t=a}^{b-1} |\Delta^n f(t)| \right) \left\{ \sum_{t=a}^{x-1} (t-a-1)^{n-1} + \sum_{t=x}^{b-1} (b+1-t)^{n-1} \right\}, \quad (4.5)$$

$\forall x \in [a, b] \cap \mathbb{Z}$.

Proof. By Theorem 3.2, (3.16) and Remark 4.1 (ii). \square

We give the following result.

Theorem 4.8. Let $f : \mathbb{Z} \rightarrow \mathbb{R}$, n is an odd number, $a, b \in \mathbb{Z}$; $a \leq b$, let also $p, q > 1$: $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\left| \sum_{t=a}^{b-1} f(t) - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left(\Delta^k f(a) (x-a)^{(k+1)} - \Delta^k f(b) (x-b)^{(k+1)} \right) \right| \leq \frac{\left(\sum_{t=a}^{b-1} |\Delta^n f(t)|^q \right)^{\frac{1}{q}}}{(n-1)!} \left[\left(\sum_{t=a}^{x-1} \left(\sum_{\tau=a}^{t-1} \left((t-\tau-1)^{(n-1)} \right)^p \right)^{\frac{1}{p}} \right) + \left(\sum_{t=x}^{b-1} \left(\sum_{\tau=t}^{b-1} \left((t-\tau-1)^{(n-1)} \right)^p \right)^{\frac{1}{p}} \right) \right], \quad (4.6)$$

$\forall x \in [a, b] \cap \mathbb{Z}$.

Proof. By Theorem 3.3, (3.25) and Remark 4.1 (ii). \square

We continue with the following theorem.

Theorem 4.9. Let $f \in C_{rd}^n(q^{\overline{\mathbb{Z}}})$, $n \in \mathbb{N}$ is odd, $a, b \in q^{\overline{\mathbb{Z}}}$; $a \leq b$. Then

$$\left| \int_a^b f(t) \Delta t - \sum_{k=0}^{n-1} \left(f^{\Delta^k}(a) \prod_{\nu=0}^k \frac{x - q^\nu a}{\sum_{\mu=0}^{\nu} q^\mu} - f^{\Delta^k}(b) \prod_{\nu=0}^k \frac{x - q^\nu b}{\sum_{\mu=0}^{\nu} q^\mu} \right) \right| \leq \|f^{\Delta^n}\|_{L_1([a, b] \cap q^{\overline{\mathbb{Z}}})} \left\{ \int_a^x (t - qa)^{n-1} \Delta t + \int_x^b (qb - t)^{n-1} \Delta t \right\}, \quad (4.7)$$

$\forall x \in [a, b] \cap q^{\overline{\mathbb{Z}}}$.

Proof. By Theorem 3.2, (3.16), and Remark 4.2. \square

We finish with the next result.

Theorem 4.10. Let $f \in C_{rd}^n(\overline{q^{\mathbb{Z}}})$, $m, n \in \mathbb{N}$; $m < n$, $n - m$ is odd, $a, b \in \overline{q^{\mathbb{Z}}}$; $a \leq b$. Then

$$\left| \int_a^b f^{\Delta^m}(t) \Delta t - \sum_{k=0}^{n-m-1} \left(f^{\Delta^{k+m}}(a) \prod_{\nu=0}^k \frac{x - q^\nu a}{\sum_{\mu=0}^{\nu} q^\mu} - f^{\Delta^{k+m}}(b) \prod_{\nu=0}^k \frac{x - q^\nu b}{\sum_{\mu=0}^{\nu} q^\mu} \right) \right| \leq$$

$$\|f^{\Delta^n}\|_{L_1([a,b] \cap \overline{q^{\mathbb{Z}}})} \left\{ \int_a^x (t - qa)^{n-m-1} \Delta t + \int_x^b (qb - t)^{n-m-1} \Delta t \right\}, \quad (4.8)$$

$\forall x \in [a, b] \cap \overline{q^{\mathbb{Z}}}$.

Proof. By Theorem 3.5, (3.40), and Remark 4.2. \square

One can give many similar applications for other time scales.

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