

## On the Dynamics of a Rational Difference Equation, Part 2

**A.M. Amleh**

Saint Mary's University  
Department of Mathematics and Computing Science  
923 Robie Street, Halifax, NS B3H 3C3, Canada  
[aamleh@cs.smu.ca](mailto:aamleh@cs.smu.ca)

**E. Camouzis**

American College of Greece  
6 Gravias Street, Aghia Paraskevi, 15342 Athens, Greece  
[camouzis@acgmail.gr](mailto:camouzis@acgmail.gr)

**G. Ladas**

University of Rhode Island  
Department of Mathematics  
Kingston, RI 02881-0816, USA  
[gladas@math.uri.edu](mailto:gladas@math.uri.edu)

### Abstract

We continue the investigation of the global stability character, the periodic nature, and the boundedness of solutions of a rational difference equation with nonnegative parameters and with nonnegative initial conditions which we started in [1]. We also pose several open problems and conjectures which we are unable to resolve at this time.

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**Keywords:** Boundedness, global stability, invariant, periodicity, rational difference equations.

## 1 Introduction

In this paper we continue the investigation, which we started in Part 1 (see [1]), of the rational difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n x_{n-1} + \gamma x_{n-1}}{A + B x_n x_{n-1} + C x_{n-1}}, \quad n = 0, 1, \dots \quad (1.1)$$

with nonnegative parameters and with arbitrary nonnegative initial conditions such that the denominator is always positive.

Eq. (1.1), which contains some interesting and some challenging special cases of second-order rational difference equations, also arises from the rational system in the plane:

$$\left. \begin{aligned} x_{n+1} &= \frac{\alpha_1 + \gamma_1 y_n}{y_n} \\ y_{n+1} &= \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{A_2 + B_2 x_n + C_2 y_n} \end{aligned} \right\}, \quad n = 0, 1, \dots \quad (1.2)$$

when we reduce it to a single equation. See [2].

If we allow one or more of the parameters in Eq. (1.1) to be zero, then Eq. (1.1) contains

$$(2^3 - 1) \times (2^3 - 1) = 49$$

special cases of equations with positive parameters. One can see that 19 of these special cases are trivial, linear, Riccati, or reducible to linear or Riccati equations. The remaining 30 special cases are new and very interesting rational difference equations. In Part 1 [1] of this paper we investigated Equations #1 through #12. Here we look at the remaining special cases, namely, Equations #13 through #30. See Appendix A at the end of the paper and Sections 2 through 19 here.

## 2 Equation #13:

$$x_{n+1} = \frac{\alpha + x_n x_{n-1}}{(A + x_n) x_{n-1}}, \quad n = 0, 1, \dots \quad (2.1)$$

For this equation we conjecture that every solution has a finite limit but we can only confirm it for

$$\alpha \leq A.$$

Eq. (2.1) has a unique equilibrium  $\bar{x}$ , and  $\bar{x}$  is the positive solution of the cubic equation

$$\bar{x}^3 + (A - 1)\bar{x}^2 - \alpha = 0.$$

The characteristic equation of the linearized equation of Eq. (2.1) about the equilibrium  $\bar{x}$  is

$$\lambda^2 + \frac{\alpha - A\bar{x}}{(A + \bar{x})^2 \bar{x}} \lambda + \frac{\alpha}{(A + \bar{x})\bar{x}^2} = 0.$$

From this it follows by [1, Theorem 1.1] of Part 1 that  $\bar{x}$  is locally asymptotically stable for all values of the parameters  $\alpha$  and  $A$ .

**Lemma 2.1.** *The interval with end points  $\frac{\alpha}{A}$  and 1 is an invariant interval for the solutions of Eq. (2.1).*

*Proof.* When  $\alpha = A$ , the result is trivial because in this case,

$$\bar{x} = 1$$

is the equilibrium of the equation. Next we give the proof for

$$\alpha < A. \quad (2.2)$$

The proof when

$$\alpha > A \quad (2.3)$$

is similar and will be omitted. To this end assume that

$$x_{-1}, x_0 \in \left[ \frac{\alpha}{A}, 1 \right].$$

Then

$$x_1 = \frac{\alpha + x_0 x_{-1}}{(A + x_0)x_{-1}} = \frac{\alpha + x_0 x_{-1}}{Ax_{-1} + x_0 x_{-1}} \leq \frac{\alpha + x_0 x_{-1}}{\alpha + x_0 x_{-1}} = 1$$

and

$$x_1 = \frac{\alpha + x_0 x_{-1}}{(A + x_0)x_{-1}} = \frac{\alpha + x_0 x_{-1}}{Ax_{-1} + x_0 x_{-1}} \geq \frac{\alpha + x_0 x_{-1}}{A + x_0 x_{-1}} \geq \frac{\alpha}{A}.$$

Hence

$$x_1 \in \left[ \frac{\alpha}{A}, 1 \right]$$

and the proof follows by induction.  $\square$

The next result presents three identities that the solutions of Eq. (2.1) satisfy. The proof is straightforward and will be omitted.

**Lemma 2.2.** *Let  $\{x_n\}_{n=-1}^{\infty}$  be a solution of Eq. (2.1). Then the following identities are satisfied:*

$$(i) \quad x_{n+1} - 1 = \frac{A\left(\frac{\alpha}{A} - x_{n-1}\right)}{(A + x_n)x_{n-1}}, \quad \text{for all } n \geq 0;$$

$$(ii) \quad x_{n+1} - \frac{\alpha}{A} = \frac{\alpha A(1 - x_{n-1}) + (A - \alpha)x_{n-1}x_n}{A(A + x_n)x_{n-1}}, \quad \text{for all } n \geq 0;$$

$$(iii) \quad \frac{x_{n+1} - x_{n-3}}{x_{n-3}} = \frac{Ax_{n-2}\left(\frac{\alpha}{A} - x_{n-3}\right) + Ax_n\left(x_{n-1} - \frac{\alpha}{A}\right) + x_n x_{n-1}(x_{n-1} - x_{n-3})}{(A + x_n)(\alpha + x_{n-2}x_{n-3})},$$

for all  $n \geq 2$ .

We should point out that there are great similarities in the character of solutions between Eq. (2.1) and the rational difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n}{Bx_n + x_{n-1}}, \quad n = 0, 1, \dots$$

which was investigated in [5, Section 6.4].

By using the above identities and Lemma 2.2 we can establish that the interval with end points  $\frac{\alpha}{A}$  and 1 is an attracting interval for the solutions of Eq. (2.1). Finally by using [1, Theorem 1.2] of Part 1 it follows that when

$$\alpha \leq A,$$

every solution of Eq. (2.1) converges to a finite limit.

*Conjecture 2.3.* Every solution of Eq. (2.1) has a finite limit.

### 3 Equation #14:

$$x_{n+1} = \frac{\alpha + x_{n-1}}{A + x_n x_{n-1}}, \quad n = 0, 1, \dots \quad (3.1)$$

For this equation we conjecture that every solution has a finite limit but we can only confirm it for

$$\alpha \leq A.$$

Eq. (3.1) has a unique equilibrium  $\bar{x}$ , and  $\bar{x}$  is the unique positive solution of the cubic equation

$$\bar{x}^3 + (A - 1)\bar{x} - \alpha = 0.$$

The characteristic equation of the linearized equation of Eq. (3.1) about the equilibrium  $\bar{x}$  is

$$\lambda^2 + \frac{\bar{x}^2}{A + \bar{x}^2} \lambda + \frac{\alpha \bar{x} - A}{(A + \bar{x}^2)^2} = 0.$$

From this it follows by [1, Theorem 1.1] of Part 1 that  $\bar{x}$  is locally asymptotically stable for all values of the parameters  $\alpha$  and  $A$ .

**Lemma 3.1.** *Assume that*

$$\alpha = A.$$

*Then every solution of Eq. (3.1) converges to the equilibrium point*

$$\bar{x} = 1.$$

*Proof.* The result follows from the identities

$$x_{n+1} - 1 = \frac{x_{n-1}(1 - x_n)}{\alpha + x_n x_{n-1}}, \quad \text{for all } n \geq 0,$$

$$x_{n+1} - x_{n-1} = \frac{\alpha(1 - x_{n-1}) + x_{n-1}(1 - x_n x_{n-1})}{\alpha + x_n x_{n-1}}, \quad \text{for all } n \geq 0,$$

and

$$x_n x_{n-1} - 1 = \frac{\alpha(x_{n-1} - 1)}{\alpha + x_{n-1} x_{n-2}}, \quad \text{for all } n \geq 1,$$

and the fact that the equation has no prime period-two solutions. □

**Lemma 3.2.** Assume that

$$\alpha < A.$$

Let  $\{x_n\}_{n=-1}^{\infty}$  be a solution of Eq. (3.1) with initial conditions

$$x_{-1}, x_0 \in \left[ \frac{\alpha}{A}, \frac{A}{\alpha} \right].$$

Then, for all  $n \geq 1$ ,

$$x_n \in \left[ \frac{\alpha}{A}, \frac{A}{\alpha} \right].$$

That is,  $\left[ \frac{\alpha}{A}, \frac{A}{\alpha} \right]$  is an invariant interval.

*Proof.* Indeed,

$$\frac{\alpha}{A} \leq \frac{\alpha + \frac{\alpha}{A}}{A + 1} < x_1 = \frac{\alpha + x_{-1}}{A + x_0 x_{-1}} \leq \frac{\alpha + \frac{A}{\alpha}}{A + 1} < \frac{A}{\alpha}$$

and the proof follows by induction. □

**Lemma 3.3.** Assume that

$$\alpha < A.$$

Then every solution of Eq. (3.1) eventually enters the invariant interval  $\left[ \frac{\alpha}{A}, \frac{A}{\alpha} \right]$ . That

is,  $\left[ \frac{\alpha}{A}, \frac{A}{\alpha} \right]$  is an invariant and attracting interval.

*Proof.* Let  $\{x_n\}_{n=-1}^{\infty}$  be a solution of Eq. (3.1) and assume, for the sake of contradiction, that for some  $N \geq 0$

$$\text{either } x_{N+1} > \frac{A}{\alpha}$$

$$\text{or } x_{N+1} < \frac{\alpha}{A}.$$

From

$$x_{N+1} = \frac{\alpha + x_{N-1}}{A + x_N x_{N-1}} > \frac{A}{\alpha} \quad \text{or} \quad x_{N+1} = \frac{\alpha + x_{N-1}}{A + x_N x_{N-1}} < \frac{\alpha}{A}$$

we find that

$$x_N < \frac{\alpha}{A} \quad \text{or} \quad x_N > \frac{A}{\alpha}$$

and similarly

$$x_{N-1} > \frac{A}{\alpha} \quad \text{or} \quad x_{N-1} < \frac{\alpha}{A}.$$

Hence, the subsequences of the even and the odd terms of the solution  $\{x_n\}_{n=-1}^{\infty}$  are such that

$$x_{2n-1} > \frac{A}{\alpha} \quad \text{and} \quad x_{2n} < \frac{\alpha}{A}, \quad \text{for all } n \geq N,$$

or

$$x_{2n} > \frac{A}{\alpha} \quad \text{and} \quad x_{2n-1} < \frac{\alpha}{A}, \quad \text{for all } n \geq N.$$

Assume that the second set of inequalities holds. The proof when the first set of inequalities holds is similar and will be omitted. Then

$$x_{2n+2} - x_{2n} = \frac{\alpha - Ax_{2n} + x_{2n}(1 - x_{2n+1}x_{2n})}{A + x_{2n+1}x_{2n}}, \quad \text{for all } n \geq N,$$

and

$$x_{2n+1}x_{2n} - 1 = \frac{\alpha x_{2n} - A}{A + x_{2n}x_{2n-1}}, \quad \text{for all } n \geq N,$$

from which it follows that the subsequence  $\{x_{2n}\}$  decreases to a positive number  $x$ . Similarly we can show that the subsequence  $\{x_{2n+1}\}$  increases to a positive number  $y$ . The sequence

$$\dots, x, y, x, y, \dots$$

is a prime period-two solution of Eq. (3.1). This is a contradiction and the proof is complete.  $\square$

**Theorem 3.4.** *Assume that*

$$\alpha < A.$$

*Then every solution of Eq. (3.1) converges to the positive equilibrium.*

*Proof.* The proof follows from [1, Theorem 1.5] of Part 1 and the fact the equation has no prime period-two solutions.  $\square$

**Lemma 3.5.** *Assume that*

$$\alpha > A.$$

*Then every solution of Eq. (3.1) eventually enters the invariant interval  $\left[\frac{A}{\alpha}, \frac{\alpha}{A}\right]$ .*

*Proof.* The proof is along the lines of the proof of Lemma 3.3 and will be omitted.  $\square$

**Theorem 3.6.** Assume that

$$A < \alpha \leq 2A\sqrt{A+1}.$$

Then every solution of Eq. (3.1) converges to the positive equilibrium.

*Proof.* The proof follows from Lemma 3.5 and [1, Theorem 1.2] of Part 1.  $\square$

*Conjecture 3.7.* Assume that

$$\alpha > A.$$

Show that every solution of Eq. (3.1) has a finite limit.

## 4 Equation #15:

$$x_{n+1} = \frac{\alpha + x_{n-1}}{(1 + Bx_n)x_{n-1}}, \quad n = 0, 1, \dots \quad (4.1)$$

Eq. (4.1) has a unique positive equilibrium  $\bar{x}$ , and  $\bar{x}$  is the unique positive solution of the cubic equation

$$B\bar{x}^3 + \bar{x}^2 - \bar{x} - \alpha = 0.$$

The characteristic equation of the linearized equation of Eq. (4.1) about the equilibrium  $\bar{x}$  is

$$\lambda^2 + \frac{B\bar{x}}{1 + B\bar{x}}\lambda + \frac{\alpha}{\alpha + \bar{x}} = 0.$$

From this it follows by [1, Theorem 1.1] of Part 1 that  $\bar{x}$  is locally asymptotically stable for all values of the parameters  $\alpha$  and  $B$ .

**Theorem 4.1.** Every solution of Eq. (4.1) is bounded.

*Proof.* Assume for the sake of contradiction that there exists a solution of Eq. (4.1) which is unbounded. There exists a sequence of indices  $\{n_i\}$  such that

$$x_{n_i+1} \rightarrow \infty \quad \text{and} \quad x_{n_i+1} > x_j \quad \text{for} \quad j < n_i + 1. \quad (4.2)$$

From

$$x_{n_i+1} = \frac{\alpha + x_{n_i-1}}{(1 + Bx_{n_i})x_{n_i-1}}$$

it follows that

$$x_{n_i-1} \rightarrow 0.$$

From this and from

$$x_{n_i-1} = \frac{\alpha + x_{n_i-3}}{(1 + Bx_{n_i-2})x_{n_i-3}}$$

and

$$x_{n_i+1} = \frac{\alpha x_{n_i-3}}{(1 + Bx_{n_i})(\alpha + x_{n_i-3})} \cdot (1 + Bx_{n_i-2}) + \frac{1}{1 + Bx_{n_i}} \quad (4.3)$$

it follows that

$$x_{n_i-2} \rightarrow \infty \quad \text{and} \quad x_{n_i}, x_{n_i-3} \rightarrow 1.$$

But then, (4.3) implies that, eventually,

$$x_{n_i+1} < x_{n_i-2}$$

which contradicts (4.2) and completes the proof.  $\square$

**Theorem 4.2.** *Assume that*

$$\alpha B \leq 1.$$

*Then every solution of Eq. (4.1) converges to a finite limit.*

*Proof.* Let  $\{x_n\}_{n=-1}^{\infty}$  be a given solution of Eq. (4.1). Set

$$I = \liminf_{n \rightarrow \infty} x_n \quad \text{and} \quad S = \limsup_{n \rightarrow \infty} x_n.$$

We divide the proof in the following two cases:

**Case 1:**

$$\alpha B < 1.$$

Clearly,

$$\frac{\alpha + S}{1 + BS} \leq SI \leq \frac{\alpha + I}{1 + BI}$$

from which it follows that

$$S = I.$$

**Case 2:**

$$\alpha B = 1.$$

In this case the proof follows from [1, Theorem 1.4] of Part 1.  $\square$

*Conjecture 4.3.* Every solution of Eq. (4.1) has a finite limit.

## 5 Equation #16:

$$x_{n+1} = \frac{(1 + \beta x_n)x_{n-1}}{A + x_n x_{n-1}}, \quad n = 0, 1, \dots \quad (5.1)$$

Zero is always an equilibrium point of Eq. (5.1) and when

$$\beta^2 < 4(A - 1), \quad (5.2)$$



zero is the only equilibrium of the equation. When

$$A \leq 1, \quad (5.3)$$

and also when

$$A > 1 \quad \text{and} \quad \beta^2 = 4(A - 1), \quad (5.4)$$

Eq. (5.1) has the unique positive equilibrium

$$\bar{x} = \frac{\beta + \sqrt{\beta^2 - 4(A - 1)}}{2}.$$

When

$$A > 1 \quad \text{and} \quad \beta^2 > 4(A - 1), \quad (5.5)$$

Eq. (5.1) has two positive equilibrium points, namely,

$$\bar{x}_1 = \frac{\beta - \sqrt{\beta^2 - 4(A - 1)}}{2} \quad \text{and} \quad \bar{x}_2 = \frac{\beta + \sqrt{\beta^2 - 4(A - 1)}}{2}.$$

The characteristic equation of the linearized equation of Eq. (5.1) about  $\bar{x} = 0$  is

$$\lambda^2 = \frac{1}{A}.$$

From this it follows by [1, Theorem 1.1] of Part 1 that  $\bar{x} = 0$  is locally asymptotically stable when

$$A > 1.$$

When

$$A = 1,$$

the zero equilibrium is non-hyperbolic and when

$$A < 1,$$

the zero equilibrium is a repeller. Note that the linearized equation of Eq. (5.1) about a positive equilibrium point  $\bar{y}$  is

$$\lambda^2 + \frac{\bar{y}(\bar{y} - A\beta)}{(1 + \beta\bar{y})^2} \lambda - \frac{A}{1 + \beta\bar{y}} = 0.$$

It now follows, by using [1, Theorem 1.1] of Part 1 that when (5.3) holds, the positive equilibrium  $\bar{x}$  of Eq. (5.1) is locally asymptotically stable and when (5.4) holds, the positive equilibrium is a non-hyperbolic point. Also when (5.5) holds,  $\bar{x}_1$  is unstable and  $\bar{x}_2$  is locally asymptotically stable.

When one of the initial conditions of a solution of Eq. (5.1) is zero, Eq. (5.1) reduces to the linear equation

$$x_{n+1} = \frac{1}{A} x_{n-1}$$

with one initial condition equal to zero. If the other initial condition of a solution is  $\phi$ , then the solution of the equation is

$$\dots, 0, \phi, 0, \frac{1}{A} \phi, 0, \frac{1}{A^2} \phi, \dots$$

Hence the solution converges to zero when

$$A > 1.$$

When

$$A = 1,$$

the solution is the (not necessarily prime) period-two sequence:

$$\dots, 0, \phi, 0, \phi, \dots$$

and when

$$A < 1 \quad \text{and} \quad \phi > 0,$$

the solution is **unbounded**.

In the remaining part of this section we will assume that both initial conditions are positive and so we will only deal with positive solutions of Eq. (5.1).

**Lemma 5.1.** *Let  $\{x_n\}_{n=-1}^{\infty}$  be a positive solution of Eq. (5.1). Then for  $N$  sufficiently large, the following statements are true:*

$$(i) \quad A > 1 \Rightarrow x_n < \beta A, \quad \text{for all } n \geq N.$$

$$(ii) \quad A < 1 \Rightarrow x_n > \beta A, \quad \text{for all } n \geq N.$$

*Proof.* We will give the proof when  $A > 1$ . The proof when  $A < 1$  is similar and will be omitted. Assume for the sake of contradiction that for  $N$  sufficiently large

$$x_{N+1} = \frac{x_{N-1} + \beta x_N x_{N-1}}{A + x_N x_{N-1}} \geq \beta A.$$

Then

$$x_{N-1} > A (\beta A),$$

and similarly

$$x_{N-3} > A^2 (\beta A),$$

which eventually leads to a contradiction. The proof is complete.  $\square$

A very important consequence of Lemma 5.1 is that the function

$$f(x, y) = \frac{(1 + \beta x)y}{A + xy}$$

which is associated with Eq. (5.1) is increasing in both variables when

$$A > 1 \quad \text{and} \quad y < \beta A,$$

and is decreasing in  $x$  and increasing in  $y$  when

$$A < 1 \quad \text{and} \quad y > \beta A.$$

Also note that when

$$A = 1,$$

the solutions of Eq. (5.1) satisfy the following two identities:

$$x_{n+1} - \beta = \frac{x_{n-1} - \beta}{1 + x_n x_{n-1}}, \quad \text{for all } n \geq 0,$$

and

$$x_{n+1} - x_{n-1} = \frac{x_n x_{n-1} (\beta - x_{n-1})}{1 + x_n x_{n-1}}, \quad \text{for all } n \geq 0,$$

from which it follows that every solution of Eq. (5.1) converges to  $\beta$ .

In view of the above results and observations and by using the appropriate theorems from [1, Section 1] of Part 1 we obtain the following result which describes in great detail the character of all solutions of Eq. (5.1).

**Theorem 5.2.** *The following statements are true for all solutions of Eq. (5.1).*

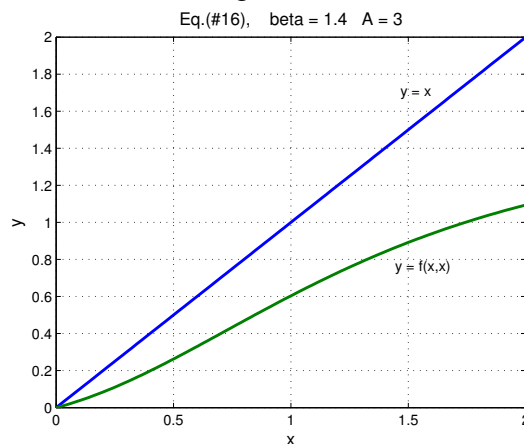
(a) *Assume*

$$A = 1.$$

*Then every positive solution of Eq. (5.1) converges to  $\beta$ .*

(b) *Assume that (5.2) holds. Then every solution of Eq. (5.1) converges to zero.*

Figure 5.1:



(c) Assume that (5.4) holds. Then the following statements are true for every non-equilibrium solution  $\{x_n\}_{n=-1}^{\infty}$  of Eq. (5.1):

(i) If for some  $N \geq 0$ ,  $x_{N-1}, x_N \in [0, \frac{\beta}{2}]$ ,  
then

$$x_n \in [0, \frac{\beta}{2}], \text{ for all } n \geq N,$$

and

$$\lim_{n \rightarrow \infty} x_n = 0.$$

(ii) If for some  $N \geq 0$ ,  $x_{N-1}, x_N \in [\frac{\beta}{2}, \infty)$ , then

$$x_n \in [\frac{\beta}{2}, \infty), \text{ for all } n \geq N,$$

and

$$\lim_{n \rightarrow \infty} x_n = \frac{\beta}{2}.$$

(iii) If either

$$x_{2n} < \frac{\beta}{2} < x_{2n+1}, \text{ for all } n \geq 0$$

or

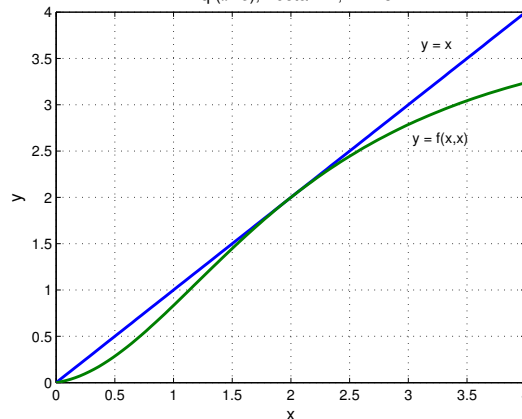
$$x_{2n+1} < \frac{\beta}{2} < x_{2n}, \text{ for all } n \geq 0$$

then

$$\lim_{n \rightarrow \infty} x_n = \frac{\beta}{2}.$$

Figure 5.2:

Eq.(#16), beta = 4, A = 5



(d) Assume that (5.5) holds. Then the following statements are true for every non-equilibrium solution  $\{x_n\}_{n=-1}^\infty$  of Eq. (5.1):

(i) If for some  $N \geq 0$ ,  $x_{N-1}, x_N \in [0, \bar{x}_1]$ , then

$$x_n \in [0, \bar{x}_1], \text{ for all } n \geq N,$$

and

$$\lim_{n \rightarrow \infty} x_n = 0.$$

(ii) If for some  $N \geq 0$ ,  $x_{N-1}, x_N \in [\bar{x}_1, \bar{x}_2]$ , then

$$x_n \in [\bar{x}_1, \bar{x}_2], \text{ for all } n \geq N,$$

and

$$\lim_{n \rightarrow \infty} x_n = \bar{x}_2.$$

(iii) If for some  $N \geq 0$ ,  $x_{N-1}, x_N \in [\bar{x}_2, \infty)$ , then

$$x_n \in [\bar{x}_2, \infty), \text{ for all } n \geq N,$$

and

$$\lim_{n \rightarrow \infty} x_n = \bar{x}_2.$$

(iv) If either

$$x_{2n} < \bar{x}_i < x_{2n+1}, \text{ for all } n \geq 0 \text{ and } i \in \{1, 2\}$$

or

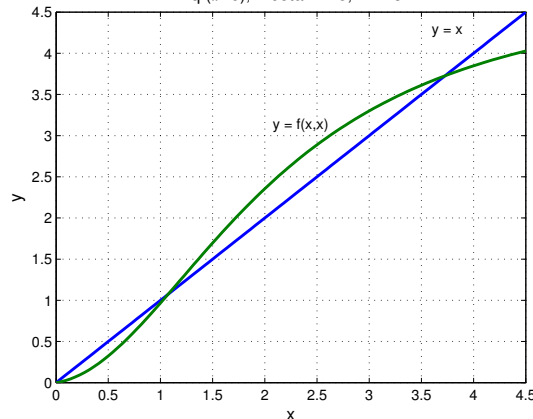
$$x_{2n+1} < \bar{x}_i < x_{2n}, \text{ for all } n \geq 0 \text{ and } i \in \{1, 2\}$$

then

$$\lim_{n \rightarrow \infty} x_n = \bar{x}_i.$$

Figure 5.3:

Eq.(#16), beta = 4.8, A = 5



(e) Assume that

$$A < 1.$$

Then every positive solution of Eq. (5.1) converges to  $\bar{x} > 0$ .

**Corollary 5.3.** Eq. (5.1) has a period-two trichotomy which can be described as follows:

(i) Every solution of Eq. (5.1) has a finite limit when

$$A > 1.$$

(ii) Every solution of Eq. (5.1) converges to a (not necessarily prime) period-two solution when

$$A = 1.$$

(iii) Eq. (5.1) has unbounded solutions when

$$A < 1.$$

## 6 Equation #17:

$$x_{n+1} = \frac{(1 + \beta x_n)x_{n-1}}{A + x_{n-1}}, \quad n = 0, 1, \dots \quad (6.1)$$

Zero is always an equilibrium solution of Eq. (6.1) and when

$$(1 - A)(1 - \beta) > 0,$$

Eq. (6.1) also has the unique positive equilibrium

$$\bar{x} = \frac{1 - A}{1 - \beta}.$$

The characteristic equation of the linearized equation of Eq. (6.1) about  $\bar{x} = 0$  is

$$\lambda^2 = \frac{1}{A}.$$

From this it follows by [1, Theorem 1.1] of Part 1 that  $\bar{x} = 0$  is locally asymptotically stable when

$$A > 1.$$

When

$$A = 1,$$

the zero equilibrium is non-hyperbolic and when

$$A < 1,$$

it is a repeller.

The characteristic equation of the linearized equation of Eq. (6.1) about the positive equilibrium is

$$\lambda^2 + \frac{\beta(1-A)}{A\beta-1}\lambda + \frac{A(1-\beta)}{A\beta-1} = 0.$$

From this and by [1, Theorem 1.1] of Part 1 we have the following:

- (i) When  $A > 1$  and  $\beta > 1$ , the positive equilibrium is unstable.
- (ii) When  $A < 1$  and  $\beta < 1$ , the positive equilibrium is locally asymptotically stable.

When

$$\beta = A = 1,$$

one can see that every non-negative number is an equilibrium point of Eq. (6.1) and so by [1, Theorem 1.4] of Part 1 every solution of Eq. (6.1) has a limit.

When one of the initial conditions of a nonzero solution  $\{x_n\}_{n=-1}^{\infty}$  of Eq. (6.1) is zero, the subsequences  $\{x_{2n}\}$  and  $\{x_{2n+1}\}$  of the solution are as follows:

One of the them is identically zero and the other satisfies the Riccati equation

$$y_{n+1} = \frac{y_n}{A + y_n}, \quad n = 0, 1, \dots$$

with a positive initial condition.

From this it follows that such a solution  $\{x_n\}_{n=-1}^{\infty}$  converges to zero when

$$A \geq 1$$

and to the prime period-two solution

$$\dots, 0, 1 - A, 0, 1 - A, \dots$$

when

$$A < 1.$$

In the remaining part of this section we will only consider positive solutions of Eq. (6.1).

It is important to note that the following identities for a solution  $\{x_n\}_{n=-1}^{\infty}$  of Eq. (6.1)

$$x_{2n+1} = \frac{x_{2n-1}}{A + x_{2n-1}}(1 + \beta x_{2n}), \quad \text{for all } n \geq 0,$$

$$x_{2n+2} = \frac{x_{2n}}{A + x_{2n}}(1 + \beta x_{2n+1}), \quad \text{for all } n \geq 0,$$

imply that if one the subsequences  $\{x_{2n}\}$  or  $\{x_{2n+1}\}$  has a finite limit, so does the other.

The following result is now a consequence of the preceding discussion and [1, Theorems 1.6–1.8] of Part 1.

**Theorem 6.1.** Let  $\{x_n\}_{n=-1}^{\infty}$  be a positive solution of Eq. (6.1). Then the following statements are true:

(i)  $\beta = 1$  and  $A = 1 \Rightarrow$  ESC.

(ii)  $\beta < 1$  and  $A > 1 \Rightarrow \lim_{n \rightarrow \infty} x_n = 0.$

(iii)  $\beta < 1$  and  $A < 1 \Rightarrow \lim_{n \rightarrow \infty} x_n = \frac{1 - A}{1 - \beta}.$

(iv)  $\beta = 1$  and  $A < 1 \Rightarrow x_n \uparrow \infty.$

(v)  $\beta > 1$  and  $A \leq 1 \Rightarrow x_n \uparrow \infty.$

(vi)  $\beta > 1$  and  $A > 1 \Rightarrow$  (depending on initial conditions) :

$$x_n \uparrow \infty, \text{ or}$$

$$\lim_{n \rightarrow \infty} x_n = 0, \text{ or}$$

$$\lim_{n \rightarrow \infty} x_n = \frac{A - 1}{\beta - 1}.$$

*Open Problem 6.2.* Assume

$$\beta = A = 1.$$

Find the limit of every solution of Eq. (6.1) in terms of its initial conditions.

## 7 Equation #18:

$$x_{n+1} = \frac{\alpha}{1 + x_n x_{n-1} + C x_{n-1}}, \quad n = 0, 1, \dots \quad (7.1)$$

For this equation we conjecture that every solution converges to the equilibrium but we can only confirm it (by using [1, Theorem 1.2] of Part 1) when

$$(\alpha - C)^2 \leq 4.$$

Note that when  $C = 0$ , Eq. (7.1) reduces to [1, Eq. (1.1)] in Part 1.

Eq. (7.1) has a unique positive equilibrium  $\bar{x}$ , and  $\bar{x}$  is the unique positive solution of the cubic equation

$$\bar{x}^3 + C\bar{x}^2 + \bar{x} - \alpha = 0.$$

The characteristic equation of the linearized equation of Eq. (7.1) about the equilibrium  $\bar{x}$  is

$$\lambda^2 + \frac{\bar{x}^3}{\alpha} \lambda + \frac{\alpha - \bar{x}}{\alpha} = 0.$$



From this it follows by [1, Theorem 1.1] of Part 1 that  $\bar{x}$  is locally asymptotically stable for all values of the parameters  $\alpha$  and  $C$ .

*Conjecture 7.1.* Every solution of Eq. (7.1) has a finite limit.

## 8 Equation #19:

$$x_{n+1} = \frac{\beta x_n x_{n-1}}{1 + B x_n x_{n-1} + x_{n-1}}, \quad n = 0, 1, \dots \quad (8.1)$$

The main result for this equation is the following:

**Theorem 8.1.** *Every solution of Eq. (8.1) has a finite limit.*

*Proof.* The proof of this theorem is a consequence of the fact that the function

$$f(x, y) = \frac{\beta xy}{1 + Bxy + y}$$

is increasing in both variables and so the character of solutions of Eq. (8.1) is easily determined by employing [1, Theorems 1.6–1.8] from Part 1, as we did for [1, Eq. (4.1)] in Part 1. □

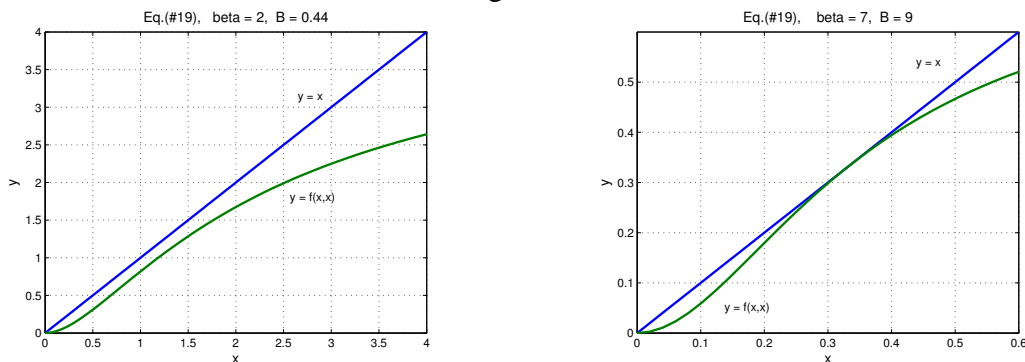
Zero is always an equilibrium solution of Eq. (8.1), and when

$$B = \left(\frac{\beta - 1}{2}\right)^2 \quad \text{and} \quad \beta > 1, \quad (8.2)$$

Eq. (8.1) also has the unique positive equilibrium

$$\bar{x} = \frac{\beta - 1}{2B} = \frac{2}{\beta - 1}.$$

Figure 8.1:

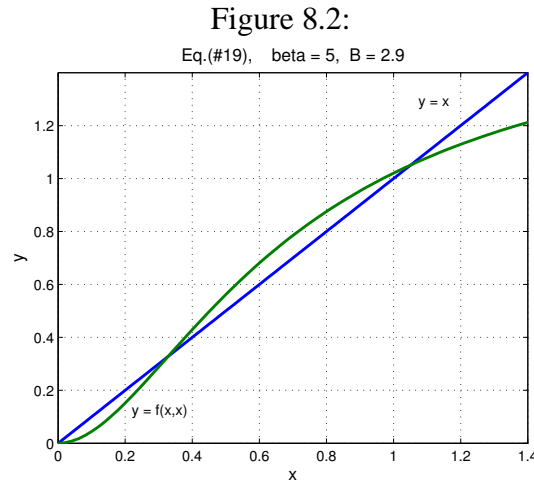


When

$$B < \left(\frac{\beta - 1}{2}\right)^2 \quad \text{and} \quad \beta > 1, \quad (8.3)$$

Eq. (8.1) has two positive equilibrium points, namely,

$$\bar{x}_1 = \frac{\beta - 1 - \sqrt{(1 - \beta)^2 - 4B}}{2B} \quad \text{and} \quad \bar{x}_2 = \frac{\beta - 1 + \sqrt{(1 - \beta)^2 - 4B}}{2B}.$$



The characteristic equation of the linearized equation of Eq. (8.1) about  $\bar{x} = 0$  is

$$\lambda^2 = 0.$$

From this it follows by [1, Theorem 1.1] of Part 1 that  $\bar{x} = 0$  is locally asymptotically stable for all values of the parameters  $\beta$  and  $B$ .

When (8.2) holds, the characteristic equation of the linearized equation of Eq. (8.1) about the positive equilibrium is

$$\lambda^2 - \left(\frac{\beta + 1}{2\beta}\right)\lambda + \frac{1 - \beta}{2\beta} = 0.$$

From this it follows that  $\bar{x} = \frac{2}{\beta - 1}$  is a non-hyperbolic equilibrium.

When (8.3) holds, the characteristic equation of the linearized equation of Eq. (8.1) about a positive equilibrium  $\bar{y}$  is

$$\lambda^2 - \frac{1 + \bar{y}}{\beta\bar{y}}\lambda - \frac{1}{\beta\bar{y}} = 0.$$

From this it follows by [1, Theorem 1.1] of Part 1 that the positive equilibrium  $\bar{x}_1$  is unstable while the positive equilibrium  $\bar{x}_2$  is locally asymptotically stable.

## 9 Equation #20

$$x_{n+1} = \frac{\gamma x_{n-1}}{1 + Bx_n x_{n-1} + x_{n-1}}, \quad n = 0, 1, \dots \quad (9.1)$$

When one of the initial conditions of a nonzero solution  $\{x_n\}_{n=-1}^{\infty}$  of Eq. (9.1) is zero, the subsequences  $\{x_{2n}\}$  and  $\{x_{2n+1}\}$  of the solution are as follows:

One of them is identically zero and the other satisfies the Riccati equation

$$y_{n+1} = \frac{\gamma y_n}{1 + y_n}, \quad n = 0, 1, \dots$$

with a positive initial condition. From this it follows that such a solution  $\{x_n\}_{n=-1}^{\infty}$  converges to zero when

$$\gamma \leq 1$$

and to the period-two solution

$$\dots, 0, \gamma - 1, 0, \gamma - 1, \dots$$

when

$$\gamma > 1.$$

In the remaining part of this section we will only consider positive solutions of Eq. (9.1) and for such solutions the main result is the following.

**Theorem 9.1.** (a) Assume that

$$\gamma \leq 1. \quad (9.2)$$

Then every positive solution of Eq. (9.1) converges to zero.

(b) Assume that

$$\gamma > 1.$$

Then every positive solution of Eq. (9.1) converges to the unique positive equilibrium of Eq. (9.1).

*Proof.* The proof, which employs [1, Theorem 1.5] of Part 1, is similar to the proof for [3, Eq. (#109) on page 295], and the details will be omitted.  $\square$

## 10 Equation #21

$$x_{n+1} = \alpha + x_n x_{n-1} + \gamma x_{n-1}, \quad n = 0, 1, \dots \quad (10.1)$$

Here

$$f(x, y) = \alpha + xy + \gamma y$$

is increasing in both variables and we can employ [1, Theorems 1.6–1.8] of Part 1 to investigate the character of solutions of Eq. (10.1) as we did in [1, Section 7] of Part 1. The details are omitted.

## 11 Equation #22

$$x_{n+1} = \frac{\alpha + \beta x_n x_{n-1} + x_{n-1}}{x_n x_{n-1}}, \quad n = 0, 1, \dots \quad (11.1)$$

For this equation we conjecture that every solution converges to the equilibrium but we can only confirm it (by using [1, Theorem 1.2] of Part 1) when

$$\alpha \leq \beta$$

and when

$$\alpha > \beta \quad \text{and} \quad \alpha - \beta < \beta \sqrt{2\alpha\beta}.$$

For Eq. (11.1) it is easy to see that every positive solution is bounded from above and from below by positive constants. Indeed for  $n \geq 0$ ,

$$x_{n+1} = \frac{\alpha + \beta x_n x_{n-1} + x_{n-1}}{x_n x_{n-1}} > \frac{\beta x_n x_{n-1}}{x_n x_{n-1}} = \beta$$

and

$$x_{n+1} \leq \frac{\alpha}{\beta^2} + \beta + \frac{1}{\beta}.$$

Eq. (11.1) has a unique equilibrium  $\bar{x}$ , and  $\bar{x}$  is the unique positive solution of the cubic equation

$$\bar{x}^3 - \beta \bar{x}^2 - \bar{x} - \alpha = 0.$$

The characteristic equation of the linearized equation of Eq. (11.1) about the equilibrium  $\bar{x}$  is

$$\lambda^2 + \frac{\bar{x} - \beta}{\bar{x}} \lambda + \frac{\alpha}{\bar{x}^3} = 0.$$

From this it follows by [1, Theorem 1.1] of Part 1 that  $\bar{x}$  is locally asymptotically stable for all values of the parameters  $\alpha$  and  $\beta$ .

Note that the change of variables

$$x_n = \frac{1}{y_n}$$

transforms Eq. (11.1) to the difference equation

$$y_{n+1} = \frac{1}{\beta + \alpha y_n y_{n-1} + y_n}, \quad n = 0, 1, \dots \quad (11.2)$$

*Conjecture 11.1.* Every positive solution of Eq. (11.1) has a finite limit.

## 12 Equation #23

$$x_{n+1} = \frac{\alpha + \beta x_n x_{n-1} + x_{n-1}}{x_{n-1}}, \quad n = 0, 1, \dots \quad (12.1)$$

The main result for Eq. (12.1) is the following.

**Theorem 12.1.** (a) Assume that

$$\beta \geq 1.$$

Then every solution of Eq. (12.1) increases to  $\infty$ .

(b) Assume that

$$\beta < 1.$$

Then every solution of Eq. (12.1) converges to the positive equilibrium.

*Proof.* (a) Clearly

$$x_{n+1} > \beta x_n \geq x_n$$

from which the result follows.

(b) The change of variables

$$x_n = \frac{1}{y_n}$$

transforms Eq. (12.1) to

$$y_{n+1} = \frac{y_n}{\beta + y_n + \alpha y_n y_{n-1}}, \quad n = 0, 1, \dots \quad (12.2)$$

Clearly,

$$y_n < 1, \quad \text{for } n \geq 1.$$

We also claim that every positive solution of Eq. (12.2) is bounded from below by a positive constant. To see this, assume for the sake of contradiction that there exists a sequence of indices  $\{n_i\}$  such that

$$y_{n_i+1} \rightarrow 0 \quad \text{and} \quad y_{n_i+1} < y_j, \quad \text{for } j < n_i + 1.$$

From Eq. (12.2) it follows that

$$y_{n_i}, y_{n_i-1} \rightarrow 0.$$

Then eventually

$$y_{n_i+1} = \frac{y_{n_i}}{\beta + y_{n_i} + \alpha y_{n_i} y_{n_i-1}} > y_{n_i}$$

and this contradiction proves our claim.

Set

$$I = \liminf_{n \rightarrow \infty} y_n \quad \text{and} \quad S = \liminf_{n \rightarrow \infty} y_n.$$

Clearly,

$$S \leq \frac{S}{\beta + S + \alpha SI} \quad \text{and} \quad I \geq \frac{I}{\beta + I + \alpha SI}$$

from which it follows that

$$\beta + S + \alpha SI \leq 1 \leq \beta + I + \alpha SI$$

and so

$$S = I.$$

The proof is complete.  $\square$

### 13 Equation #24

$$x_{n+1} = \frac{\alpha + \beta x_n x_{n-1}}{1 + B x_n x_{n-1} + x_{n-1}}, \quad n = 0, 1, \dots \quad (13.1)$$

*Conjecture 13.1.* Every positive solution of Eq. (13.1) has a finite limit.

### 14 Equation #25

$$x_{n+1} = \frac{\alpha + \gamma x_{n-1}}{1 + B x_n x_{n-1} + x_{n-1}}, \quad n = 0, 1, \dots \quad (14.1)$$

Eq. (14.1) has a unique equilibrium  $\bar{x}$ , and  $\bar{x}$  is the positive solution of the cubic equation

$$B\bar{x}^3 + \bar{x}^2 + (1 - \gamma)\bar{x} - \alpha = 0.$$

The characteristic equation of the linearized equation of Eq. (14.1) about  $\bar{x}$  is

$$\lambda^2 + \frac{B\bar{x}^2}{1 + B\bar{x}^2 + \bar{x}}\lambda - \frac{\gamma - \alpha - \alpha B\bar{x}}{(1 + B\bar{x}^2 + \bar{x})^2} = 0.$$

From this it follows by [1, Theorem 1.1] of Part 1 that  $\bar{x}$  is locally asymptotically stable for all values of the parameters  $\alpha$ ,  $\gamma$  and  $B$ .

*Conjecture 14.1.* Every positive solution of Eq. (14.1) has a finite limit.

### 15 Equation #26

$$x_{n+1} = \frac{\beta x_n x_{n-1} + \gamma x_{n-1}}{1 + B x_n x_{n-1} + x_{n-1}}, \quad n = 0, 1, \dots \quad (15.1)$$

As in Eq. (9.1), when one of the initial conditions of a nonzero solution  $\{x_n\}_{n=-1}^{\infty}$  of Eq. (15.1) is zero, the subsequences  $\{x_{2n}\}$  and  $\{x_{2n+1}\}$  of the solution are as follows.

One of them is identically zero and the other satisfies the Riccati equation

$$y_{n+1} = \frac{\gamma y_n}{1 + y_n}, \quad n = 0, 1, \dots$$

with a positive initial condition.

From this it follows that such a solution  $\{x_n\}_{n=-1}^{\infty}$  converges to zero when

$$\gamma \leq 1$$

and to the prime period-two solution

$$\dots, 0, \gamma - 1, 0, \gamma - 1, \dots$$

when

$$\gamma > 1.$$

In the remaining part of this section we will only consider positive solutions of Eq. (15.1). Here

$$f(x, y) = \frac{(\beta x + \gamma)y}{1 + Bxy + y}$$

increases in both variables when

$$\gamma B \leq \beta. \quad (15.2)$$

For this range of parameters one can employ [1, Theorems 1.6–1.8] of Part 1 to determine the details of the dynamics of solutions of Eq. (15.1). Note also that every solution of Eq. (15.1) is bounded and that Eq. (15.1) has no positive prime period-two solutions.

*Conjecture 15.1.* Every positive solution of Eq. (15.1) has a finite limit.

## 16 Equation #27

$$x_{n+1} = \frac{\alpha + \beta x_n x_{n-1} + x_{n-1}}{A + x_n x_{n-1}}, \quad n = 0, 1, \dots \quad (16.1)$$

For Eq. (16.1) one can see that, for  $n \geq 1$ , every positive solution is bounded from below and from above by positive constants. Indeed,

$$x_{n+1} \geq \frac{\alpha + \beta x_n x_{n-1}}{A + x_n x_{n-1}} \geq \frac{\min\{\alpha, \beta\}}{\max\{A, 1\}}$$

which shows that every solution of Eq. (16.1) is bounded from below, for  $n \geq 1$ , by the positive number

$$m = \frac{\min\{\alpha, \beta\}}{\max\{A, 1\}}.$$

Hence,

$$x_{n+1} < \frac{\alpha + \beta x_n x_{n-1} + x_{n-1}}{x_n x_{n-1}} = \frac{\alpha}{x_n x_{n-1}} + \beta + \frac{1}{x_n} \leq \frac{\alpha}{m^2} + \beta + \frac{1}{m}$$

and so every solution of Eq. (16.1) is also bounded from above, for  $n \geq 2$ , by the positive number

$$M = \frac{\alpha}{m^2} + \beta + \frac{1}{m}.$$

*Conjecture 16.1.* Every positive solution of Eq. (16.1) has a finite limit.

## 17 Equation #28

$$x_{n+1} = \frac{\alpha + \beta x_n x_{n-1} + x_{n-1}}{A + x_{n-1}}, \quad n = 0, 1, \dots \quad (17.1)$$

**Theorem 17.1.** Assume that

$$\beta > 1.$$

Let  $\{x_n\}_{n=-1}^{\infty}$  be a solution of Eq. (17.1) with initial conditions  $x_{-1}, x_0$  such that

$$x_{-1}, x_0 > \frac{A}{\beta - 1}.$$

Then the solution  $\{x_n\}_{n=-1}^{\infty}$  increases to infinity.

*Proof.* Indeed,

$$x_1 = \frac{\alpha + x_{-1}}{A + x_{-1}} + \frac{\beta x_{-1}}{A + x_{-1}} \cdot x_0 > \frac{\min\{\alpha, 1\}}{\max\{A, 1\}} + x_0 = L + x_0 > \frac{A}{\beta - 1}$$

and by induction, we find that for all  $n \geq 1$ ,

$$x_n > x_{n-1} > (n - 1)L + x_0.$$

The proof is complete. □

Note that, for  $n \geq 0$ ,

$$\begin{aligned} x_{n+1} &= \frac{\alpha + x_{n-1}}{A + x_{n-1}} + \beta \frac{x_{n-1}}{A + x_{n-1}} x_n \\ &\leq \frac{\max\{\alpha, 1\}}{\min\{A, 1\}} + \beta x_n \end{aligned}$$

from which it follows that for

$$\beta < 1$$

every solution of Eq. (17.1) is bounded.

*Open Problem 17.2.* Determine the range of the parameters for which Eq. (17.1) has unbounded solutions.



## 18 Equation #29

$$x_{n+1} = \frac{\alpha + \beta x_n x_{n-1} + x_{n-1}}{B x_n x_{n-1} + x_{n-1}}, \quad n = 0, 1, \dots \quad (18.1)$$

It is easy to see that every positive solution of Eq. (18.1) is bounded from below and from above by positive constants. Indeed for  $n \geq 1$ ,

$$x_{n+1} \geq \frac{\beta x_n x_{n-1} + x_{n-1}}{B x_n x_{n-1} + x_{n-1}} = \frac{\beta x_n + 1}{B x_n + 1} \geq \frac{\min\{\beta, 1\}}{\max\{B, 1\}}.$$

So, for  $n \geq 1$ , every positive solution is bounded from below by

$$m = \frac{\min\{\beta, 1\}}{\max\{B, 1\}}.$$

Then for  $n \geq 2$ ,

$$\begin{aligned} x_{n+1} &= \frac{\alpha}{B x_n x_{n-1} + x_{n-1}} + \frac{\beta x_n + 1}{B x_n + 1} \\ &\leq \frac{\alpha}{B m^2 + m} + \frac{\max\{\beta, 1\}}{\min\{B, 1\}} \end{aligned}$$

which proves our claim.

Eq. (18.1) has a unique equilibrium  $\bar{x}$ , and  $\bar{x}$  is the unique positive root of the cubic equation:

$$B\bar{x}^3 + (1 - \beta)\bar{x}^2 - \bar{x} - \alpha = 0.$$

The characteristic equation of the linearized equation of Eq. (18.1) about  $\bar{x}$  is

$$\lambda^2 + \left( \frac{\alpha B - (\beta - B)\bar{x}}{\bar{x}(B\bar{x} + 1)^2} \right) \lambda + \frac{\alpha}{\bar{x}^2(B\bar{x} + 1)} = 0.$$

From this it follows by [1, Theorem 1.1] of Part 1 that  $\bar{x}$  is locally asymptotically stable for all values of the parameters  $\alpha$ ,  $\beta$  and  $B$ .

*Conjecture 18.1.* Every positive solution of Eq. (18.1) has a finite limit.

## 19 Equation #30

$$x_{n+1} = \frac{\alpha + \beta x_n x_{n-1} + x_{n-1}}{A + B x_n x_{n-1} + x_{n-1}}, \quad n = 0, 1, \dots \quad (19.1)$$

*Conjecture 19.1.* Every positive solution of Eq. (19.1) has a finite limit.

## Appendix A

### Table of the Global Character of the 30 nontrivial special cases of

$$x_{n+1} = \frac{\alpha + \beta x_n x_{n-1} + \gamma x_{n-1}}{A + B x_n x_{n-1} + C x_{n-1}}, \quad n = 0, 1, \dots$$

In this table we use the following abbreviations:

- ESC** stands for "every solution has a finite limit".  
**ESC\*** stands for "we conjecture that every solution has a finite limit".  
**EPSC** stands for "every positive solution has a finite limit".  
**EPSC\*** stands for "we conjecture that every positive solution has a finite limit".  
 **$\exists$ US** stands for "there exist unbounded solutions".  
**ESB** stands for "every solution of the equation is bounded".  
**ESCP<sub>2</sub>** stands for "every solution of the equation converges to a not necessarily prime period-two solution".

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$\#1: x_{n+1} = \frac{\alpha}{1 + x_n x_{n-1}};$	<b>ESB</b> <b>ESC*</b> ; This conjecture has been confirmed for $\alpha \leq 2$ See [1]
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$\#2: x_{n+1} = \frac{\alpha}{(1 + x_n)x_{n-1}};$	This equation possesses the <b>invariant</b> : $x_{n-1} + x_n + x_{n-1}x_n + \alpha \left( \frac{1}{x_{n-1}} + \frac{1}{x_n} \right) \equiv c$ For a more general case see [4] <b>ESB</b>
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$\#3: x_{n+1} = \frac{\beta x_n x_{n-1}}{1 + x_n x_{n-1}};$	<b>ESC</b> Local Stability $\nRightarrow$ Global Stability See [1]
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$$\#4: x_{n+1} = \frac{\beta x_n x_{n-1}}{1 + x_{n-1}};$$

$\beta \leq 1 \Rightarrow \text{ESC}$   
 $\beta > 1 \Rightarrow \exists \text{US}$   
 See [1, Theorem 5.1]  
 Local Stability  $\nRightarrow$  Global Stability

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$$\#5: x_{n+1} = \frac{\gamma x_{n-1}}{1 + x_n x_{n-1}};$$

$\gamma < 1 \Rightarrow \text{ESC}$   
 $\gamma = 1 \Rightarrow \text{ESCP}_2$   
 $\gamma > 1 \Rightarrow \exists \text{US}$   
 Has Period-Two Trichotomy  
 See [1, Theorem 6.1]  
 See [6]

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$$\#6: x_{n+1} = \alpha + x_n x_{n-1};$$

$\exists \text{US}$   
 See [1, Theorem 7.1]

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$$\#7: x_{n+1} = \beta + \frac{1}{x_n x_{n-1}};$$

**ESB**  
**ESC\***; This equation can be transformed  
 to Eq.# 1  
 See [1]

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$$\#8: x_{n+1} = \beta x_n + \frac{1}{x_{n-1}};$$

**ESB**  $\Leftrightarrow \beta < 1$   
 $\beta < 1 \Rightarrow \text{ESC}$   
 See [1, Theorem 9.1]

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$$\#9: x_{n+1} = \frac{\alpha + x_{n-1}}{x_n x_{n-1}};$$

**ESC**  
 See [1, Theorem 10.1]

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#10 : $x_{n+1} = (\gamma + x_n)x_{n-1};$	<b><math>\exists</math>US</b> See [1, Theorem 11.1]
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#11 : $x_{n+1} = \frac{\alpha + x_n x_{n-1}}{A + x_n x_{n-1}};$	<b>ESB</b> <b>ESC*</b> ; This conjecture has been confirmed for $\alpha \leq A$ See [1]
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#12 : $x_{n+1} = \frac{\alpha + \beta x_n x_{n-1}}{1 + x_{n-1}};$	$\beta \leq 1 \Rightarrow$ <b>ESC</b> $\beta > 1 \Rightarrow$ <b><math>\exists</math>US</b> See [1, Theorem 13.1]
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#13 : $x_{n+1} = \frac{\alpha + x_n x_{n-1}}{(A + x_n)x_{n-1}};$	<b>ESB</b> <b>ESC*</b> ; This conjecture has been confirmed for $\alpha \leq A$ See Section 2
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#14 : $x_{n+1} = \frac{\alpha + x_{n-1}}{A + x_n x_{n-1}};$	<b>ESB</b> <b>ESC*</b> ; This conjecture has been confirmed for $\alpha \leq A$ See Section 3
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#15 : $x_{n+1} = \frac{\alpha + x_{n-1}}{(1 + Bx_n)x_{n-1}};$	<b>ESB</b> <b>ESC*</b> ; This conjecture has been confirmed for $\alpha B \leq 1$ See Section 4
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$\#16: x_{n+1} = \frac{(1 + \beta x_n)x_{n-1}}{A + x_n x_{n-1}};$	$A > 1 \Rightarrow \mathbf{ESC}$ $A = 1 \Rightarrow \mathbf{ESCP}_2$ $A < 1 \Rightarrow \exists \mathbf{US}$ <b>Has Period-Two Trichotomy</b> <b>See Section 5</b>
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$\#17: x_{n+1} = \frac{(1 + \beta x_n)x_{n-1}}{A + x_{n-1}};$	$\beta \leq 1 \Rightarrow \mathbf{EPSC}$ $\beta > 1 \text{ and } A \neq 1 \Rightarrow \exists \mathbf{US}$ <b>See Section 6</b>
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$\#18: x_{n+1} = \frac{\alpha}{1 + x_n x_{n-1} + C x_{n-1}};$	<b>ESB</b> <b>ESC*</b> ; This conjecture has been confirmed if $(\alpha - C)^2 \leq 4$ <b>See Section 7</b>
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$\#19: x_{n+1} = \frac{\beta x_n x_{n-1}}{1 + B x_n x_{n-1} + x_{n-1}};$	<b>ESC</b> <b>Local Stability <math>\not\Rightarrow</math> Global Stability</b> <b>See Section 8</b>
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$\#20: x_{n+1} = \frac{\gamma x_{n-1}}{1 + B x_n x_{n-1} + x_{n-1}};$	$\gamma \leq 1 \Rightarrow \mathbf{ESC}$ $\gamma > 1 \Rightarrow \mathbf{ESCP}_2$ <b>See Section 9</b>
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$\#21: x_{n+1} = \alpha + x_n x_{n-1} + \gamma x_{n-1};$	$\exists \mathbf{US}$ <b>See Section 10</b>
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$\#22: x_{n+1} = \frac{\alpha + \beta x_n x_{n-1} + x_{n-1}}{x_n x_{n-1}};$	<b>ESB</b> <b>ESC*</b>
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$\#23 : x_{n+1} = \frac{\alpha + \beta x_n x_{n-1} + x_{n-1}}{x_{n-1}};$	$\beta < 1 \Rightarrow \mathbf{ESC}$ $\beta \geq 1 \Rightarrow \text{Every solution increases to } \infty$ See Section 12
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$\#24 : x_{n+1} = \frac{\alpha + \beta x_n x_{n-1}}{1 + B x_n x_{n-1} + x_{n-1}};$	<b>ESB</b> <b>ESC*</b>
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$\#25 : x_{n+1} = \frac{\alpha + \gamma x_{n-1}}{1 + B x_n x_{n-1} + x_{n-1}};$	<b>ESB</b> <b>ESC*</b>
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$\#26 : x_{n+1} = \frac{\beta x_n x_{n-1} + \gamma x_{n-1}}{1 + B x_n x_{n-1} + x_{n-1}};$	<b>ESB</b> <b>EPSC*</b> ; This conjecture has been confirmed for $\gamma B \leq \beta$ See Section 15
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$\#27 : x_{n+1} = \frac{\alpha + \beta x_n x_{n-1} + x_{n-1}}{A + x_n x_{n-1}};$	<b>ESB</b> <b>ESC*</b>
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$\#28 : x_{n+1} = \frac{\alpha + \beta x_n x_{n-1} + x_{n-1}}{A + x_{n-1}};$	$\exists \mathbf{US}$ See Section 17
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$\#29 : x_{n+1} = \frac{\alpha + \beta x_n x_{n-1} + x_{n-1}}{B x_n x_{n-1} + x_{n-1}};$	<b>ESB</b> <b>ESC*</b>
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$\#30 : x_{n+1} = \frac{\alpha + \beta x_n x_{n-1} + x_{n-1}}{A + B x_n x_{n-1} + x_{n-1}};$	<b>ESB</b> <b>ESC*</b>
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